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EQUATIONAL LOGIC AND THEORIES IN SENTENTIAL LANGUAGES

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Equational logic based on identity predicate is well known in abstract algebra (see [5], p. 275-288, [2] and [1], chapter 4). One may try to build and study equational logic based on identity connective, introduced in [4]. There is an analogy between both kinds of equational logics, since both formators, the identity predicate and identity connective, formalize the genuine notion of identity. However, there must also be some differences between them which follow the difference in the syntactic category (predicate, connective).

The idea of this note is to compare the identity connective with the identity predicate, using but one sentential language $\mathfrak{L}$ described in section 1. The binary formator $\equiv$ behaves like the identity predicate (with respect to E_0) in section 2 and becomes the identity connective (with respect to E) in section 3. Obviously, the identity predicate relates to congruences of algebras. From section 4 it can be seen that the identity connective relates to *regular* congruences which are not totally unknown in abstract algebra.

The restriction to regular congruences is forced by the rule of inference $\alpha, \alpha \equiv \beta \ / \ \beta$ which is characteristic for the identity connective. It modifies the usual notion of an equational theory (i.e., E_0-theory) and calls for a certain change of other notions (model, variety, equational class). Section 3 contains two theorems for E-theories which parallel classical theorems of G. Birkhoff (for E_0-theories), stated in section 1, that is, the completeness theorem and the equational class theorem. An intermediary notion of an n-variety of algebras is introduced in section 4. Again, an equational class theorem relates n-varieties to equational E-theories.

1. Let $\mathfrak{L}$ be a sentential language with the binary connective $\equiv$ and other sentential connectives. $FM_{\mathfrak{L}}$ is the set of all formulas of $\mathfrak{L}$, built up from sentential variables p_j, with $j \in J$, where J is an infinite set, by means of connectives, including $\equiv$. If $X \subseteq FM_{\mathfrak{L}}$, then $Sb(X)$ is the set of all substitutions in formulas in X. Formulas $\alpha \equiv \beta$ are called *equations* and

$\mathrm{EQ}_{\mathfrak{L}}$ is the set of them. If $\mathrm{T} \subseteq \mathrm{FM}_{\mathfrak{L}}$, then $\alpha \underset{\mathrm{T}}{\approx} \beta$ stands for $(\alpha \equiv \beta) \in \mathrm{T}$.

2. If $\mathrm{X} \subset \mathrm{EQ}_{\mathfrak{L}}$, then $\mathrm{E}_0(\mathrm{X})$ is the smallest set $\mathrm{Y} \subseteq \mathrm{EQ}_{\mathfrak{L}}$ over X such that (1) all trivial equations $\alpha \equiv \alpha$ are in Y, (2) for each (!) connective ω of $\mathfrak{L}$, Y is closed under the ω-invariance rule

$$\alpha_i \equiv \beta_i,\ i = 1, \ldots, k \ /\ \omega(\alpha_1, \ldots, \alpha_k) \equiv \omega(\beta_1, \ldots, \beta_k)$$

and, (3) Y is closed under both rules

$$\alpha \equiv \beta \ /\ \beta \equiv \alpha \quad \text{and} \quad \alpha \equiv \gamma,\ \beta \equiv \gamma \ /\ \alpha \equiv \beta.$$

Subsets of $\mathrm{EQ}_{\mathfrak{L}}$ of the form $\mathrm{E}_0(\mathrm{X})$ for $\mathrm{X} \subseteq \mathrm{EQ}_{\mathfrak{L}}$ are called E_0-*theories.*

The language $\mathfrak{L}$ is an absolutely free algebra in the class $\mathscr{K}_{\mathfrak{L}}$ of all algebras similar to $\mathfrak{L}$. The map

$$\mathrm{T} \to \underset{\mathrm{T}}{\approx}$$

is a one-one correspondence between E_0-theories and congruences of $\mathfrak{L}$. If $\mathfrak{A} \in \mathscr{K}_{\mathfrak{L}}$ and $\mathrm{h} \in \mathrm{Hom}(\mathfrak{L}, \mathfrak{A})$, then $\mathrm{EQ}(\mathrm{h}, \mathfrak{A})$ is the set of all equations $\alpha \equiv \beta$ such that $\mathrm{h}(\alpha) = \mathrm{h}(\beta)$. It follows that $\mathrm{E}_0(\mathrm{X}) \subseteq \mathrm{EQ}(\mathrm{h}, \mathfrak{A})$ whenever $\mathrm{X} \subseteq \mathrm{EQ}(\mathrm{h}, \mathfrak{A})$. Let

$$\mathrm{EQ}(\mathfrak{A}) = \bigcap \{\mathrm{EQ}(\mathrm{h}, \mathfrak{A}) \colon \mathrm{h} \in \mathrm{Hom}(\mathfrak{L}, \mathfrak{A})\}.$$

Then $\mathrm{EQ}(\mathfrak{A}) = \mathrm{E}_0\big(\mathrm{Sb}(\mathrm{EQ}(\mathfrak{A}))\big)$.

COMPLETENESS THEOREM (G. Birkhoff). $\mathrm{T} = \mathrm{E}_0\big(\mathrm{Sb}(\mathrm{X})\big)$ *for some* $\mathrm{X} \subseteq \mathrm{EQ}_{\mathfrak{L}}$ *iff* $\mathrm{T} = \mathrm{EQ}(\mathfrak{A})$ *for some* $\mathfrak{A} \in \mathscr{K}_{\mathfrak{L}}$.

A class $\mathscr{K} \subseteq \mathscr{K}_{\mathfrak{L}}$ is called a *variety* iff $\mathscr{K}$ is closed under operations I, S, P and Q, of taking isomorphic images, subalgebras, product algebras and quotient algebras (modulo congruences), respectively.

EQUATIONAL CLASS THEOREM (G. Birkhoff). *A class* $\mathscr{K} \subset \mathscr{K}_{\mathfrak{L}}$ *is a variety iff there exists an* $\mathrm{X} \subseteq \mathrm{EQ}_{\mathfrak{L}}$ *such that the class* $\{\mathfrak{A} \in \mathscr{K}_{\mathfrak{L}} \colon \mathrm{X} \subseteq \mathrm{EQ}(\mathfrak{A})\}$ *is just* $\mathscr{K}$.

3. If $\mathrm{X} \subseteq \mathrm{FM}_{\mathfrak{L}}$, then $\mathrm{E}(\mathrm{X})$ is the smallest set $\mathrm{Y} \subseteq \mathrm{FM}_{\mathfrak{L}}$ over X such that (1) all trivial equations are in Y, (2) for each connective ω of $\mathfrak{L}$, Y is closed under the ω-invariance rule and, (3) Y is closed under the special identity rule $\alpha, \alpha \equiv \beta \ /\ \beta$.

Consequently, $(\alpha \equiv \beta) \in \mathrm{E}(\mathrm{X})$ whenever $(\beta \equiv \alpha) \in \mathrm{E}(\mathrm{X})$ or $(\alpha \equiv \gamma)$, $(\beta \equiv \gamma) \in \mathrm{E}(\mathrm{X})$. Subsets of $\mathrm{FM}_{\mathfrak{L}}$ of the form $\mathrm{E}(\mathrm{X})$ for $\mathrm{X} \subseteq \mathrm{FM}_{\mathfrak{L}}$ (or $\mathrm{X} \subseteq \mathrm{EQ}_{\mathfrak{L}}$) are called E-*theories* (or *equational* E-*theories*).

If $\mathfrak{A} \in \mathscr{K}_{\mathfrak{L}}$, the binary operation $\circ$ of $\mathfrak{A}$ shall correspond to the identity connective $\equiv$. For any proper subset D of A, the carrier of $\mathfrak{A}$, the pair $\mathfrak{M} = \langle \mathfrak{A}, \mathrm{D} \rangle$ is called an m-*structure* (*matrix structure*), the algebra $\mathfrak{A}$ is also denoted by $\mathrm{alg}(\mathfrak{M})$ and the relation $\mathrm{a} \underset{\mathrm{D}}{\sim} \mathrm{b}$ on A is defined as $(\mathrm{a} \circ \mathrm{b}) \in \mathrm{D}$.

A congruence $\sim$ of $\mathfrak{A}$ is said to be an m-*congruence of* $\mathfrak{M}$ if, for all $a, b \in A$,

$$b \in D \text{ whenever } a \sim b \text{ and } a \in D.$$

We write A° for $\{a \circ a\colon a \in A\}$.

Let $\mathfrak{M}_i = \langle \mathfrak{A}_i, D_i \rangle$ for $i = 1, 2$. We say that $\mathfrak{M}_2$ is an m-*expansion of* $\mathfrak{M}_1$ if $A_1 = A_2$ and $D_1 \subseteq D_2$. If $h \in \mathrm{Hom}(\mathfrak{A}_1, \mathfrak{A}_2)$, then h is called m-*homomorphism of* $\mathfrak{M}_1$ to $\mathfrak{M}_2$ iff $D_1 = \check{h}(D_2)$, where $\check{h}$ is the h-counter-image operation. If $\sim$ is an m-congruence of $M = \langle A, D \rangle$ and $\mathfrak{M} / \sim\, = \langle \mathfrak{A} / \sim, D / \sim \rangle$, where $\mathfrak{A} / \sim$ is the quotient of $\mathfrak{A}$ modulo $\sim$ and $D / \sim\, = \{a / \sim\colon a \in D\}$, then the canonical map $a \to a / \sim$ is an m-epimorphism of $\mathfrak{M}$ to $\mathfrak{M} / \sim$ (the quotient of $\mathfrak{M}$ modulo $\sim$). The operations I, S and P are also well defined for m-structures.

An m-structure $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ is said to be a *model*, $\mathfrak{M} \in \mathscr{M}_{\mathfrak{L}}$, if $\underset{D}{\sim}$ is m-congruence of $\mathfrak{M}$. If $\underset{D}{\sim}$ is the identity relation on A, then the model $\mathfrak{M}$ is called *normal*, $\mathfrak{M} \in \mathscr{N}\mathscr{M}_{\mathfrak{L}}$. The m-structure $\langle \mathfrak{L}, T \rangle$ is a model for every E-theory T. If $h \in \mathrm{Hom}(\mathfrak{L}, \mathfrak{A})$, then, clearly,

$$E(X) \subseteq \check{h}(D) \quad \text{whenever } X \subseteq \check{h}(D).$$

For any model $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$, the set $\mathrm{TR}_{\mathfrak{L}}(\mathfrak{M}) \subseteq \mathrm{FM}_{\mathfrak{L}}$ is defined as

$$\bigcap \{\check{h}(D)\colon h \in \mathrm{Hom}(\mathfrak{L}, \mathfrak{A})\}.$$

It follows that $\mathrm{TR}_{\mathfrak{L}}(\mathfrak{M}) = E(\mathrm{Sb}(\mathrm{TR}_{\mathfrak{L}}(\mathfrak{M})))$.

m-EPIMORPHISM LEMMA. *If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are models and there exists an* m-*epimorphism of $\mathfrak{M}_1$ to $\mathfrak{M}_2$, then* $\mathrm{TR}_{\mathfrak{L}}(\mathfrak{M}_1) = \mathrm{TR}_{\mathfrak{L}}(\mathfrak{M}_2)$.

CONTRACTION LEMMA. *If $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ is a model, then $\mathfrak{M} / \underset{D}{\sim}$ is a normal model, the contraction of $\mathfrak{M}$.*

COROLLARY. *A congruence $\sim$ of $\mathfrak{A}$ is* m-*congruence of the model $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ iff $\sim \leqslant \underset{D}{\sim}$.*

ISOMORPHISM THEOREM. *If $\mathfrak{M} \in \mathscr{N}\mathscr{M}_{\mathfrak{L}}$, and* J *and* $\mathrm{alg}(\mathfrak{M})$ *are of equal power, then there exists an* E-*theory* T *such that $\mathfrak{M}$ and the contraction of $\langle \mathfrak{L}, T \rangle$ are* m-*isomorphic.*

COMPLETENESS THEOREM. $T = E(\mathrm{Sb}(X))$ *for some* $X \subseteq \mathrm{FM}_{\mathfrak{L}}$ *iff there exists an* $\mathfrak{M} \in \mathscr{N}\mathscr{M}_{\mathfrak{L}}$ *such that* $T = \mathrm{TR}_{\mathfrak{L}}(\mathfrak{M})$.

Let E, C and CE be, correspondingly, the operation of m-expanding models to models, that of contracting models and the composition of C and E. Then, a class $\mathscr{K} \subseteq \mathscr{N}\mathscr{M}_{\mathfrak{L}}$ is said to be an m-*variety* if $\mathscr{K}$ is closed under I, S, P and CE.

EQUATIONAL CLASS THEOREM. *A class* $\mathscr{K} \subseteq \mathscr{N}\mathscr{M}_{\mathfrak{L}}$ *is an* m-*variety iff there exists an* $\mathrm{X} \subseteq \mathrm{EQ}_{\mathfrak{L}}$ *such that the class*

$$\{\mathfrak{M} \in \mathscr{N}\mathscr{M}_{\mathfrak{L}} \colon \mathrm{X} \subset \mathrm{TR}_{\mathfrak{L}}(\mathfrak{M})\}$$

is just $\mathscr{K}$.

4. An algebra $\mathfrak{A} \in \mathscr{K}_{\mathfrak{L}}$ is *regular* (or *normal*), $\mathfrak{A} \in \mathscr{K}^{\circ}_{\mathfrak{L}}$ (or $\mathfrak{A} \in \mathscr{N}\mathscr{K}^{\circ}_{\mathfrak{L}}$), if $\mathfrak{A} = \mathrm{alg}(\mathfrak{M})$ for some $\mathfrak{M}$ in $\mathscr{M}_{\mathfrak{L}}$ (or in $\mathscr{N}\mathscr{M}_{\mathfrak{L}}$). A congruence $\sim$ of $\mathfrak{A}$ in $\mathscr{K}_{\mathfrak{L}}$ is said to be *regular* if, for all a, b, c, d in A,

$$\mathrm{a} \sim \mathrm{b} \text{ if } (\mathrm{a} \circ \mathrm{b}) \sim (\mathrm{c} \circ \mathrm{c}) \text{ for some } \mathrm{c}.$$

(1) $\mathfrak{A} \in \mathscr{K}_{\mathfrak{L}}$ *is regular iff there exists a proper regular congruence of* $\mathfrak{A}$.

(2) *Let* $\sim$ *be a congruence of* $\mathfrak{A} \in \mathscr{K}_{\mathfrak{L}}$. *Then* $\sim$ *is regular iff, for all* $\mathrm{a}, \mathrm{b} \in \mathrm{A}$,

$$\mathrm{a} \sim \mathrm{b} \textit{ iff } (\mathrm{a} \circ \mathrm{b}) \in \mathrm{D},$$

where $\mathrm{D} = \{\mathrm{d} \in \mathrm{A} \mid \mathrm{d} \sim \mathrm{c} \textit{ for some } \mathrm{c} \in \mathrm{A}^{\circ}\}$.

(3) $\mathfrak{A} \in \mathscr{K}_{\mathfrak{L}}$ *is normal iff the identity relation on* A *is regular, i.e., for all* $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathrm{A}$,

$$\mathrm{a} \circ \mathrm{b} = \mathrm{c} \circ \mathrm{c} \textit{ implies } \mathrm{a} = \mathrm{b}.$$

(4) *If* $\mathfrak{A}, \mathfrak{B} \in \mathscr{K}_{\mathfrak{L}}$, $\mathrm{h} \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$ *and* $\mathfrak{B}$ *is normal, then the kernel of* h *is regular and, hence,* $\mathfrak{A}$ *is regular. If* $\mathfrak{A} \in \mathscr{K}_{\mathfrak{L}}$ *and* $\sim$ *is a regular congruence of* $\mathfrak{A}$, *then* $\mathfrak{A} \,/\, \sim$ *is normal.*

(5) *If* $\mathfrak{A} \in \mathscr{K}_{\mathfrak{L}}$ *is normal and* $\mathfrak{M} = \langle \mathfrak{A}, \mathrm{A}^{\circ} \rangle$, *then*

$$\mathrm{EQ}(\mathfrak{A}) = \mathrm{EQ}_{\mathfrak{L}} \cap \mathrm{TR}_{\mathfrak{L}}(\mathfrak{M}).$$

(6) *Every normal algebra is isomorphic to* $\mathfrak{L} \,/\, \mathrm{T}$, *i.e., the quotient of* $\mathfrak{L}$ *modulo* $\underset{\mathrm{T}}{\approx}$, *for some* E-*theory* T.

A class $\mathscr{K} \subseteq \mathscr{K}_{\mathfrak{L}}$ is called an n-*variety* if every algebra in $\mathscr{K}$ is normal and $\mathscr{K}$ is closed under I, S, P, and Q°, the operation of taking quotients modulo regular congruences.

EQUATIONAL CLASS THEOREM. *A class* $\mathscr{K} \subseteq \mathscr{K}_{\mathfrak{L}}$ *is an* n-*variety iff there exists an* $\mathrm{X} \subseteq \mathrm{EQ}_{\mathfrak{L}}$ *such that the class*

$$\{\mathfrak{A} \in \mathscr{N}\mathscr{K}^{\circ}_{\mathfrak{L}} \colon \mathrm{X} \subseteq \mathrm{EQ}(\mathfrak{A})\}$$

is just $\mathscr{K}$.

Proof. Let $\mathscr{K}$ be an n-variety and let $\mathscr{T}$ be the class of all E_0-theories $\mathrm{EQ}(\mathfrak{A})$ for $\mathfrak{A} \in \mathrm{K}$. Set $\mathrm{X} = \bigcap \mathscr{T}$ and form the product A^* of all $\mathfrak{L} \,/\, \mathrm{T}$ for $\mathrm{T} \in \mathscr{T}$. Observe that $\mathfrak{L} / \mathrm{X}$ is isomorphic to a subalgebra of $\mathfrak{A}^*$. Thus $\mathfrak{L} / \mathrm{X} \in \mathscr{K}$. Let $\mathfrak{A}$ be any algebra in $\mathscr{N}\mathscr{K}^{\circ}_{\mathfrak{L}}$ such that $\mathrm{X} \subseteq \mathrm{EQ}(\mathfrak{A})$. Set $\mathrm{S} = \mathrm{EQ}(\mathrm{g}, \mathfrak{A})$, where g is an epimorphism of $\mathfrak{L}$ to $\mathfrak{A}$. Clearly, $\mathfrak{L} \,/\, \mathrm{S}$ and $\mathfrak{A}$ are isomorphic. Since $X \subseteq S$, there is an epimorphism h of $\mathfrak{L} / X$ to $\mathfrak{L} / S$. But $\mathfrak{A}$ is normal. Hence, the kernel of h is regular. Therefore, $\mathfrak{L} / S$, i.e., $\mathfrak{A}$ is in $\mathscr{K}$.

Remark. Regular congruences $\sim$ satisfying the condition of uniformity

$$(a \circ a) \sim (b \circ b) \quad \text{for all } a, b$$

have been examined by Słomiński in [3].

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