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REGARDING A QUESTION ABOUT THE LEAST ELEMENT MAP

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In his paper [1] Bednarek raised the question as to whether or not there is any loss of generality (P 671) when "partial order" is used to replace "relation" in the theorem of Franklin and Wallace [3], which may be stated as follows:

THEOREM. *If a topological space X is provided with a closed relation, the function which maps each closed compact set with least element into its least element is continuous (i. e., continuous with respect to the Vietoris topology).*

This theorem is an improvement on an earlier result, by Capel and Strother [2], which required X to be a compact Hausdorff space and the relation to be a partial order.

In [1] Bednarek proved that there is no loss of generality in this theorem if X is assumed to be compact.

Recall that, for a relation R on X, $A \subseteq X$ has an R-least element a_0 if and only if

(i) $a_0 \in A$,

(ii) $\{a_0\} \times A \subseteq R$,

(iii) $(A \times \{a_0\}) \cap R = \{(a_0, a_0)\}$.

The following example shows that there is a loss of generality when "partial order" replaces "relation" in the theorem:

Example. Let $X = \{x_1, x_2, x_3, x_4\}$, and let the topology on X be discrete. X is clearly a compact Hausdorff space, as is $X \times X$. Define a relation R on X by the table

	x_1	x_2	x_3	x_4
x_1	1	1	0	0
x_2	0	1	1	0
x_3	1	0	1	0
x_4	1	1	1	1

where a "1" in the ij-th position indicates that $x_i R x_j$, and a zero entry indicates no relation. R is clearly a closed reflexive relation on X.

We list the compact (closed) subsets of X having R-least element,

$$\Sigma_R = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_1; x_2\}, \{x_2, x_3\}, \{x_3, x_1\}, \\ \{x_4, x_1\}, \{x_4, x_2\}, \{x_4, x_3\}, \{x_4, x_1, x_2\}, \{x_4, x_1, x_3\}, \\ \{x_4, x_2, x_3\}, X\},$$

where the element first-listed in each set is the R-least element of that set.

Note that R acts as a partial order on each element of Σ_R with the exception of X itself.

Suppose there exists a partial order P on X such that $\Sigma_P \supseteq \Sigma_R$. Now $\{x_1, x_2\} \in \Sigma_R \subseteq \Sigma_P$ with least element x_1, and $\{x_2, x_3\} \in \Sigma_R \subseteq \Sigma_P$ with least element x_2, together with the transitivity of P requires x_1 to be the P-least element of $\{x_1, x_3\}$. However, x_3 is the R-least element of $\{x_1, x_3\}$. Thus $\Sigma_R \nsubseteq \Sigma_P$ — a contradiction. Hence there does not exist a partial order P on X having the desired property.

REFERENCES

[1] A. R. Bednarek, *A note on the least element map*, Colloquium Mathematicum 20 (1969), p. 227-228.

[2] C. E. Capel and W. L. Strother, *Multi-valued functions and partial order*, Portugaliae Mathematica 17 (1958), p. 41-47.

[3] S. P. Franklin and A. D. Wallace, *The least element map*, Colloquium Mathematicum 15 (1966), p. 217-221.

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