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ON THE ORDER OF LINEAR DEPENDENCE OF REAL FUNCTIONS

BY

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Let us consider a set

$$f_1(x), f_2(x), \ldots, f_n(x) \tag{1}$$

of n real functions of a real variable which have at least $n-1$ derivatives on an interval Δ.

Definition. We call functions (1) *w^m-dependent* ($1 \leqslant m \leqslant n$) on the interval Δ if the rank of the Wronski matrix

$$W_n(f_1, f_2, \ldots, f_n) = \begin{pmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \cdot & \cdot & \cdot & \cdot \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{pmatrix}$$

is not greater than $n-m$ at every point of the interval Δ.

The term *w-dependent functions* (i.e., w^1-dependent in the sense of the above-given definition) was introduced for the first time by Moszner [7] in 1966 although similar sets of functions were known much earlier because of their frequent applications in many branches of mathematics and classical mechanics ([5], [1], p. 86, [2], p. 109 and others).

The problem of finding the necessary and sufficient conditions for w^1-dependence of the set (1) in Δ was attacked (see [6] and [4]) from the very beginning but it was only in 1961 that Moszner [6] proved the following theorem:

THEOREM M. *The set of functions* (1), *differentiable up to the order $n-1$ on the interval Δ, is on this interval w^1-dependent if and only if there exists a sequence* (*finite or not*) *of open subintervals $\delta_1, \delta_2, \ldots$ of the interval Δ such that*

1. *the set* (1) *is linearly dependent on every δ_i* ($i = 1, 2, \ldots$);
2. *if*

$$A = \Delta \setminus \bigcup_{i=1}^{\infty} \delta_i$$

and

$A^* = \{x \in \Delta\colon \det W_{n-1}(f_1, \ldots, f_{l-1}, f_{l+1}, \ldots, f_n) = 0$ *for* $l = 1, 2, \ldots, n\}$,

then $A \subset A^*$;

3. *the set* A *is nowhere dense.*

The necessary and sufficient condition for w^m-dependence $(1 \leqslant m \leqslant n)$ on Δ is given in this paper. It is a generalization of Theorem M. Before we formulate the condition we need to reintroduce an important definition and to prove an auxiliary lemma.

Definition (see [6]). The set of functions (1) is called *linearly dependent up to the order* k on the interval Δ if there exist constants c_{ij} $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, k)$ such that

$$\sum_{i=1}^{n} c_{ij} f_i(x) \underset{\Delta}{\equiv} 0 \quad \text{for each } j = 1, \ldots, k \tag{2}$$

and

$$\operatorname{rank}[c_{ij}] = k. \tag{3}$$

Remark. It is clear that the set (1) which is linearly dependent up to the order k $(k > 1)$ on the interval Δ is also linearly dependent up to the order $k-1$ on the same interval.

LEMMA. *Suppose that functions* (1) *are*

1. *differentiable up to the order* $n-1$ *on the interval* Δ;

2. w^m*-dependent on* Δ *where* $1 \leqslant m \leqslant n$.

Then there exists an open subinterval Δ^* *of the interval* Δ *such that the set of functions* (1) *is linearly dependent at least up to the order* m *on* Δ^*.

Proof. Let us denote by r the maximal rank of the matrix

$$W_{n-1}(f_1, f_2, \ldots, f_n) = \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(n-2)} & f_2^{(n-2)} & \cdots & f_n^{(n-2)} \end{pmatrix}$$

on the interval Δ, and by x_0 $(x_0 \in \Delta)$ the point at which this maximal value is realized. From the hypothesis of the lemma and from the definition of r, it follows that $r \leqslant n-m \leqslant n-1$.

Let us consider two cases: $r = n-1$ and $r < n-1$.

In the first case, where

$$\operatorname{rank}\{W_{n+1}(f_1, f_2, \ldots, f_n)\}_{x=x_0} = n-1,$$

there exists a minor $M(x)$ of order $n-1$ extracted from the matrix $W_{n-1}(f_1, f_2, \ldots, f_n)$ which is different from zero at x_0. Fréchet has proved (see [3], p. 122 and 123) that if functions (1) are w^1-dependent on the interval Δ and at the point $x_0 \in \Delta$ at least one of all determinants $\det W_{n-1}(f_1, \ldots, f_{l-1}, f_{l+1}, \ldots, f_n)$ is not equal to zero, then there exists a neighbourhood of the point x_0 such that the set (1) is in it linearly dependent up to the order one. Hence, it follows the conclusion of the lemma for the case $r = n-1$.

If

$$\operatorname{rank}\{W_{n-1}(f_1, f_2, \ldots, f_n)\}_{x=x_0} = r \leqslant n-m < n-1,$$

then from assumption 1 we can conclude that there exists a neighbourhood Δ^* of the point x_0 for which

$$\operatorname{rank} W_{n-1}(f_1, f_2, \ldots, f_{n-1}) = r \quad \text{for every } x \in \Delta^*.$$

Therefore, the hypothesis of Moszner's theorem (see [6], Theorem T, p. 177) concerning the linear dependence up to the order $n-r$ is fulfilled. Hence our set (1) is linearly dependent on Δ^* up to the order $n-r \geqslant m$. This completes the proof.

We can now formulate the following theorem:

THEOREM. *The set of real functions* (1), *differentiable up to the order* $n-1$ *on the interval* Δ, *is on this interval* w^m-*dependent* $(1 \leqslant m \leqslant n)$ *if and only if there exists a sequence (finite or not) of open subintervals* $\delta_1, \delta_2, \ldots$ *of the interval* Δ *for which the following conditions are fulfilled*:

1. *the set* (1) *is linearly dependent up to the order* $l_i \geqslant m$ *on every interval* δ_i $(i = 1, 2, \ldots)$;

2. *if*

$$X = \Delta \setminus \bigcup_{i=1}^{\infty} \delta_i \quad \text{and} \quad X^* = \{x \in \Delta : \operatorname{rank} W_{n-1}(f_1, f_2, \ldots, f_n) < n-m\},$$

then $X \subset X^*$;

3. *the set* X *is nowhere dense on* Δ.

Proof. We start with the proof of the necessity.

Let us call an element $\bar{x} \in \Delta$ a *point of local linear dependence* of the set (1) up to the order m if there exists a neighbourhood $\delta_{\bar{x}}$ of the point $\bar{x}$ such that the set (1) is on it linearly dependent up to the order $l(\bar{x}) \geqslant m$. We consider now the set K of all points $\bar{x} \in \Delta$ which are points of local linear dependence at least up to the order m. From the definition of the set K, we can conclude that it is an open set and that $K \subset \bigcup_{\bar{x} \in K} \delta_{\bar{x}}$. On the other hand, by the lemma we know that it is not empty. We can now use Lindelöf's theorem and take out from the set $\{\delta_{\bar{x}}\}_{\bar{x} \in K}$ a sequence $\{\delta_{\bar{x}_i}\}$ such that

$$\bigcup_{\bar{x} \in K} \delta_{\bar{x}} = \bigcup_{i=1}^{\infty} \delta_{\bar{x}_i}.$$

On the set $\{\delta_{\bar{x}_i}\}$ $(i = 1, 2, \ldots)$, functions (1) are linearly dependent up to the order $l_i = l(\bar{x}_i) \geqslant m$.

Moszner has shown (see [6], p. 182 and ff.) that if the set of functions (1) is w^m-dependent on the interval Δ and if at a point $a \in \Delta$ there exists a minor of rank $n-m$ extracted from the matrix $W_{n-1}(f_1, f_2, \ldots, f_n)$ and different from zero, then a is a point of local linear dependence up to the order m for the set (1). Hence, it follows that

$$\Delta \setminus X^* \subset K \subset \bigcup_{i=1}^{\infty} \delta_{\bar{x}_i}$$

and, therefore,

$$\Delta \setminus \bigcup_{i=1}^{\infty} \delta_{\bar{x}_i} \subset X^*.$$

Let us now assume that the set X is not nowhere dense on Δ. There must then exist an open interval δ such that $\delta \subset \bar{X}$ and $\delta \subset \Delta$. The set of functions (1) is w^m-dependent on δ, and thus, by our lemma, there follows the existence of an open subinterval δ^* of the interval δ on which functions (1) are linearly dependent up to the order m. Since $\delta^* \subset \bar{X}$ and δ^* is an open set, there must exist an element ξ such that

$$\xi \in \delta^*, \tag{4}$$

$$\xi \in X. \tag{5}$$

We conclude from (4) that ξ is a point of local linear dependence up to the order m. Hence $\xi \in \bigcup_{i=1}^{\infty} \delta_{\bar{x}_i}$, which is contrary to (5). Thus the proof of the necessity is complete.

The proof of the sufficiency is almost immediate. If $x \in \bigcup_{i=1}^{\infty} \delta_{\bar{x}_i}$, then the w^m-dependence of functions (1) at the point x follows from the linear dependence of these functions at least up to the order m on every interval $\delta_{\bar{x}_i}$. If, however, $x \in X^*$, then the rank of the matrix $W_n(f_1, f_2, \ldots, f_n)$ is not greater than $n-m$, because the addition of one row to the matrix $W_n(f_1, f_2, \ldots, f_{n-1})$ may increase its rank at the most on one.

Remark. If the interval Δ is open, then the set X is not only nowhere dense but also closed with respect to the interval Δ.

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