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SEPARATE APPROXIMATE CONTINUITY AND STRONG APPROXIMATE CONTINUITY

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A real valued function of two real variables, $f(x, y)$, is said to be *separately approximately continuous* if it is approximately continuous in y for each x fixed and approximately continuous in x for each y fixed. In [1], Davies proved all such functions are Baire class 2 though not necessarily Baire class 1. Here we define a topology for which the continuous functions are precisely the separately approximately continuous functions and then examine the relation between this type of continuity and strong approximate continuity as introduced by Goffman, Neugebauer and Nishiura in [2]. Also we note how these concepts connect with recent work by Grande ([4]) and the author ([5]).

Notation. We will use $|\ |_1$ and $|\ |_2$ to denote Lebesgue one and two dimensional measure. Further d, D, and D_s will be the symbols representing metric density conditions in R^1, ordinary metric density in R^2, and strong metric density in R^2, respectively. It is known ([3]) that the collection of measurable sets $A \subset R^1$ having $d(A, x) = 1$ for all x in A form a topology. Such sets are designated d-open.

Definition. Let d_{xy} be the collection of all subsets U of R^2 satisfying

1) U is a measurable subset of R^2;

2) $U(x, y_0) = \{x\colon (x, y_0) \text{ belongs to } U\}$ and $U(x_0, y) = \{y\colon (x_0, y) \text{ belongs to } U\}$ are measurable in R^1 for each x_0, y_0 in R^1. These are called the y and x *sections of* U respectively;

3) each y section and x section is d-open.

THEOREM 1. *The collection d_{xy} forms a topology on R^2. Moreover, the continuous functions relative to this topology are precisely the separately approximately continuous functions.*

Proof. It is clear that only one condition is to be verified. Namely, for some index set A, if U_α belongs to d_{xy} for each $\alpha \in A$, then $U = \bigcup_\alpha U_\alpha$ is measurable in R^2. To show this we first introduce two operations on sets B satisfying conditions 1) and 2) above.

Let

$$B_1 = \bigcup_{y_0} \{(x, y_0)\colon B(x, y_0) \text{ has density } 1 \text{ at } (x, y_0)\}$$

and

$$B_2 = \bigcup_{x_0} \{(x_0, y)\colon B(x_0, y) \text{ has density } 1 \text{ at } (x_0, y)\}.$$

The sets B_1 and B_2 will also satisfy conditions 1) and 2). Moreover, $|B_1 \Delta B|_2 = 0$ where $B_1 \Delta B$ denotes the symmetric difference of B_1 and B.

For the set U, we may assume that every U_α is a subset of the unit square, and set

$$\beta = \sup_{A^*} \left[\Big| \bigcup_{\alpha \in A^*} U_\alpha \Big|_2 \colon A^* \text{ a countable subfamily of } A \right].$$

Then $0 < \beta \leqslant 1$ and there is a countable $A^* \subset A$ such that:

$$\Big| \bigcup_{\alpha \in A^*} U_\alpha \Big|_2 = \beta.$$

Setting $B = \bigcup_{\alpha \in A^*} U_\alpha$ we note that B belongs to d_{xy}, and we will show $B \subset U \subset (B_1)_2$. Then since $|(B_1)_2 \setminus B|_2 = 0$ we will be finished. Let α be fixed. Then $|U_\alpha \setminus B|_2 = 0$ since $|B|_2 = \beta$. By Fubini's theorem we have

$$|U_\alpha(x, y_0) \setminus B(x, y_0)|_1 = 0 \quad \text{for a.e. } y_0.$$

We pick any $p_1 = (x_1, y_1) \in U_\alpha$. The set $U(x_1, y)$ has density 1 at y_1 and for y_0 belonging to $U(x_1, y)$, $U(x, y_0)$ has density 1 at x_1. Therefore, for those y_0 belonging to $U(x_1, y)$ for which

$$|U_\alpha(x, y_0) \setminus B(x, y_0)|_1 = 0$$

$B(x, y_0)$ has density 1 at x, and (x_1, y_0) belongs to B_1. This means that

$$|U_\alpha(x_1, y) \setminus B_1(x_1, y)|_1 = 0,$$

and $B_1(x_1, y)$ has density 1 at y_1 so that (x_1, y_1) belongs to $(B_1)_2$. This completes the proof of Theorem 1. □

We next consider the relation between separate approximate continuity and strong approximate continuity.

Already the following is known ([2]):

THEOREM. *If f is strongly approximately continuous, then f is separately approximately continuous.*

Further, it is easy to show the converse statement is false. In one sense, this can be thought of as saying that if U is d_{xy} open, then U need not have strong density 1 at all its points. However, we do have:

THEOREM 2. *If U is d_{xy} open and (x_0, y_0) belongs to U, then U has strong upper density equal to 1 at (x_0, y_0).*

Proof. We may assume $(x_0, y_0) = (0, 0)$. Let $\varepsilon > 0$ be given. Since $U(x, 0)$ has density 1 at $(0, 0)$ there is a $\delta_1 > 0$ such that:

$$|U(x, 0) \cap [-\delta, \delta]|_1 > (1-\varepsilon)(2\delta) \quad \text{for all} \quad \delta \leqslant \delta_1.$$

Pick any such $\delta > 0$. Let $I_n = [-\delta, \delta] \times [0, 1/n]$. Then

$$\frac{|U \cap I_n|_2}{|I_n|_2} = \frac{|U \cap I_n|_2}{(2\delta)/n} = \frac{1}{2\delta} \int_{-\delta}^{\delta} \frac{|l(x, 1/n) \cap U|_1}{1/n} dx$$

where $l(x, 1/n) = \{(x, y)\colon 0 \leqslant y \leqslant 1/n\}$. By Fatou's lemma,

$$\liminf_{n\to\infty} \frac{|U \cap I_n|_2}{|I_n|_2} \geqslant \frac{1}{2\delta} \int_{-\delta}^{\delta} \liminf_{n\to\infty} \frac{|l(x, 1/n) \cap U|_1}{1/n} dx.$$

However, for x belonging to $U(x, 0)$

$$\liminf_{n\to\infty} \frac{|l(x, 1/n) \cap U|_1}{1/n} = 1.$$

That is,

$$\liminf_{n\to\infty} \frac{|U \cap I_n|_2}{|I_n|_2} \geqslant \frac{1}{2\delta} |U(x, 0) \cap [-\delta, \delta]|_1 > 1-\varepsilon,$$

and for some n, $|U \cap I_n|_2 > (1-\varepsilon)|I_n|_2$. From this it follows that U has strong upper metric density 1 at $(0, 0)$. □

COROLLARY. *If f is separately approximately continuous and (x_0, y_0) and $\varepsilon > 0$ are fixed, then $\{(x, y)\colon |f(x, y) - f(x_0, y_0)| < \varepsilon\}$ has strong upper density 1 at (x_0, y_0).*

We now examine the relation in another direction. We will establish the following theorem:

THEOREM 3. *If f is separately approximately continuous, then for each y_0 the function f is strongly approximately continuous at (x, y_0) except for a set of x of Lebesgue one dimensional measure zero.*

(This will follow immediately from a lemma based on results in [6].)

LEMMA 1. *If U is d_{xy} open and y_0 is fixed then U has strong density 1 at a.e. x in $U(x, y_0)$.*

Proof. Consider $B = R^2/U$. At every x belonging to $U(x, y_0)$ the upper density of B in the y direction is 0. By Lemma 2 in [6], the strong upper density of B is also 0 at a.e. x in $U(x, y_0)$. Therefore U has strong density 1 at a.e. x in $U(x, y_0)$. □

Note. In an interesting paper [4], Grandé discusses 8 topologies in R^2 deriving from 2 topologies defined in [5]. In that paper ([4]) topologies R_4 and T_4 will have the same properties if the additional condition of measurability as subsets of R^2 is included.

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