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ON THE SHAPE OF THE SUSPENSION

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1. Introduction. Two compacta X, Y lying in the Hilbert cube Q are said to be *fundamentally equivalent* ([1], p. 233) if there exist two sequences $\{f_k\}$, $\{g_k\}$ of (continuous) maps of Q into itself such that for every neighborhood U of X and for every neighborhood V of Y there are neighborhoods U_0 of X and V_0 of Y such that for almost all k the homotopies

(1.1) $$f_k/U_0 \simeq f_{k+1}/U_0 \quad \text{in } V,$$

(1.2) $$g_k/V_0 \simeq g_{k+1}/V_0 \quad \text{in } U,$$

(1.3) $$g_k f_k/U_0 \simeq i/U_0 \quad \text{in } U,$$

(1.4) $$f_k g_k/V_0 \simeq i/V_0 \quad \text{in } V$$

hold true. If we omit the last condition (1.4), then instead of the fundamental equivalence of X and Y we get a weaker relation of the *fundamental domination of X by Y* ([1], p. 233).

If X, Y are ANR-sets, then the relation of the fundamental equivalence is the same ([1], p. 234) as the relation of the homotopy equivalence (in the sense of Hurewicz [4], p. 125), and the relation of the fundamental domination is the same as the relation of the homotopy domination (in the sense of Whitehead [5], p. 1133).

By the *shape* $\mathrm{Sh}(X)$ of a compactum X we understand (cf. [2], p. 221) the class of all compacta Y such that X and Y are homeomorphic to two fundamentally equivalent compacta lying in Q. The relation $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ means that X and Y are homeomorphic with two compacta X', Y' lying in Q and such that X' is fundamentally dominated by Y'.

The aim of this note is to study how the shape of the suspension $\sum(X)$ of a compactum X depends on the shape of X.

2. Preliminary constructions. It is convenient for our purposes to regard Q as the subset of the Hilbert space H consisting of all points $x = (0, x_2, x_3, \ldots)$ with $0 \leqslant x_k \leqslant 1/(k-1)$ for $k = 2, 3, \ldots$ Consider the points $a = (1, 0, 0, \ldots)$ and $b = (-1, 0, 0, \ldots)$ of H and let R denote the union of all segments (in H) of the form $\overline{ax}$ and $\overline{bx}$, with $x \in Q$. One easily sees that R is a convex subset of H homeomorphic with Q.

Assign to every positive number $\varepsilon < \frac{1}{2}$ and to every $t \in \langle 0, 1\rangle$ the map $a_t^\varepsilon\colon R \to R$ given by the following formulas:

If $\varrho(z, a) \leqslant \varepsilon$, then

$$a_t^\varepsilon(z) = t\cdot a + (1-t)\cdot z.$$

If $\varrho(z, b) \leqslant \varepsilon$, then

$$a_t^\varepsilon(z) = t\cdot b + (1-t)\cdot z.$$

If $\varepsilon \leqslant \varrho(z, a) \leqslant 2\varepsilon$, then

$$a_t^\varepsilon(z) = z + \left[2 - \frac{\varrho(z, a)}{\varepsilon}\right]\cdot t\cdot (a - z).$$

If $\varepsilon \leqslant \varrho(z, b) \leqslant 2\varepsilon$, then

$$a_t^\varepsilon(z) = z + \left[2 - \frac{\varrho(z, b)}{\varepsilon}\right]\cdot t\cdot (b - z).$$

If $\varrho(z, a) \geqslant 2\varepsilon$ and $\varrho(z, b) \geqslant 2\varepsilon$, then

$$a_t^\varepsilon(z) = z.$$

It is easy to see that a_t^ε is continuous and it depends continuously on t. Moreover, a_0 is the identity map.

Now let us assign to every set $Z \subset Q$ the set $\hat{Z}$ being the union of all segments $\overline{ax}$ and $\overline{bx}$ with $x \in Z$. It is clear that $\hat{Z}$ is a subset of R homeomorphic with the suspension $\Sigma(Z)$.

Let ε be a positive number $< \frac{1}{2}$. Denote by $Z^{(\varepsilon)}$ the union of the set Z and of two balls in the space R with centers a and b and with radius ε. Observe, that if W is a neighborhood (in Q) of the set Z, then the set $W^{(\varepsilon)}$ is a neighborhood (in R) of the set $\hat{Z}$. One easily sees that if t runs through the interval $\langle 0, 1\rangle$, then the restriction $a_t^\varepsilon / Z^{(\varepsilon)}$ is a continuous deformation of the set $Z^{(\varepsilon)}$ in itself, joining the identity map $i/Z^{(\varepsilon)}$ with the map $a_1^\varepsilon / Z^{(\varepsilon)}$ having values in the set $\hat{Z}$.

If A and B are subsets of Q, then to every map $f\colon A \to B$ we can assign a map $\hat{f}\colon \hat{A} \to \hat{B}$, called the *suspension of the map f*, given by the formulas:

If z belongs to a segment $\overline{ax}$ with $x \in A$, then

$$\hat{f}(z) = \frac{\varrho(a, z)}{\varrho(a, x)}\cdot f(x) + \left[1 - \frac{\varrho(a, z)}{\varrho(a, x)}\right]\cdot a.$$

If z belongs to a segment $\overline{bx}$ with $x \in A$, then

$$\hat{f}(z) = \frac{\varrho(b, z)}{\varrho(b, x)} \cdot f(x) + \left[1 - \frac{\varrho(b, z)}{\varrho(b, x)}\right] \cdot b.$$

In particular, if $f\colon Q \to Q$, then $\hat{f}\colon R \to R$. Let us observe that if $f, g\colon Q \to Q$, then the suspension $\hat{h}$ of the composition $h = fg\colon Q \to Q$ is the same as the composition $\hat{f}\hat{g}$ of the suspensions of f and of g.

Moreover, if $\varphi\colon A \times \langle 0, 1\rangle \to B$ is a homotopy, then for every $t \in \langle 0, 1\rangle$ the function $f_t\colon A \to B$ given by the formula $f_t(x) = \varphi(x, t)$ is a map depending continuously on t. Setting

$$\hat{\varphi}(z, t) = \hat{f}_t(z) \quad \text{for every } (z, t) \in \hat{A} \times \langle 0, 1\rangle,$$

we get a homotopy $\hat{\varphi}\colon \hat{A} \times \langle 0, 1\rangle \to \hat{B}$ called the *suspension of the homotopy* φ.

3. Shape of the suspension. Now let us consider two compacta $X, Y \subset Q$ with $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ and let us prove that $\mathrm{Sh}(\hat{X}) = \mathrm{Sh}(\hat{Y})$.

Since $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, there exist two sequences $\{f_k\}, \{g_k\}$ of maps of Q into itself such that for every neighborhood (in Q) U of X and for every neighborhood (in Q) V of Y there is a neighborhood (in Q) U_0 of X and a neighborhood (in Q) V_0 of Y such that the relations (1.1)-(1.4) hold for almost all k.

Let N be a neighborhood (in R) of the set $\hat{Y}$. Then we may select a neighborhood (in Q) V of Y and a positive number $\varepsilon_1 < \frac{1}{2}$ so that $V^{(\varepsilon_1)} \subset N$. Let U_0 be a neighborhood of X (in Q) and k_1 an index such that for every $k \geqslant k_1$ relation (1.1) holds true. It means that there is a homotopy $\varphi_k\colon U_0 \times \langle 0, 1\rangle \to V$ such that $\varphi_k(x, 0) = f_k(x)$ and $\varphi_k(x, 1) = f_{k+1}(x)$ for every point $x \in U_0$. Using the homotopy $\alpha_t^{\varepsilon_1}$, the suspension $\hat{f}_k$ of the map f_k and the suspension $\hat{\varphi}_k$ of the homotopy φ_k, let us define a homotopy $\varphi_k^{\varepsilon_1}\colon U_0^{(\varepsilon_1)} \times \langle 0, 1\rangle \to V_0^{(\varepsilon_1)} \subset N$ by the formulas:

$$\varphi_k^{\varepsilon_1}(z, t) = \begin{cases} \hat{f}_k \alpha_{3t}^{\varepsilon_1}(z) & \text{for } (z, t) \in U_0^{(\varepsilon_1)} \times \langle 0, \frac{1}{3}\rangle, \\ \hat{\varphi}_k[\alpha_1^{\varepsilon_1}(z), 3t-1] & \text{for } (z, t) \in U_0^{(\varepsilon_1)} \times \langle \frac{1}{3}, \frac{2}{3}\rangle, \\ \hat{f}_{k+1} \alpha_{3-3t}^{\varepsilon_1}(z) & \text{for } (z, t) \in U_0^{(\varepsilon_1)} \times \langle \frac{2}{3}, 1\rangle. \end{cases}$$

It is clear that this homotopy joins the restriction $\hat{f}_k / U_0^{(\varepsilon_1)}$ with the restriction $\hat{f}_{k+1} / U_0^{(\varepsilon_1)}$. Hence

$$\hat{f}_k / U_0^{(\varepsilon_1)} \simeq \hat{f}_{k+1} / U_0^{(\varepsilon_1)} \quad \text{in } N \text{ for every } k \geqslant k_1. \tag{3.1}$$

By an analogous argument, we infer by (1.2) that for every neighborhood M of $\hat{X}$ (in R) there is a neighborhood V_0 of Y (in Q), a positive number $\varepsilon_2 < \frac{1}{2}$ and an index k_2 such that

$$(3.2) \qquad \hat{g}_k / V_0^{(\varepsilon_2)} \simeq \hat{g}_{k+1} / V_0^{(\varepsilon_2)} \quad \text{in } M \text{ for every } k \geqslant k_2.$$

Moreover, there exists a neighborhood U of X (in Q) and a positive number $\varepsilon_3 < \frac{1}{2}$ such that $U^{(\varepsilon_3)} \subset M$. Since U_0 may be replaced by any smaller neighborhood of X (in Q), we infer by (1.3) that U_0 can be selected so that there is an index k_3 with the property that relation (1.3) holds true for every $k \geqslant k_3$. It means that there is a homotopy ψ_k: $U_0 \times \langle 0, 1\rangle \to U$ such that $\psi_k(x, 0) = g_k f_k(x)$ and $\psi_k(x, 1) = x$ for every point $x \in U_0$. Using the operation of the suspension for the map $h_k = g_k f_k$: $Q \to Q$ and for the homotopy ψ_k, let us define a homotopy $\psi_k^{\varepsilon_3}$: $U_0^{(\varepsilon_3)} \times \langle 0, 1\rangle \to U^{(\varepsilon_3)} \subset M$ by the following formulas:

$$\psi_k^{\varepsilon_3}(z,t) = \begin{cases} \hat{h}_k a_{3t}^{\varepsilon_3}(z) & \text{for } (z, t) \in U_0^{(\varepsilon_3)} \times \langle 0, \frac{1}{3}\rangle, \\ \hat{\psi}_k[a_1^{\varepsilon_3}(z), 3t-1] & \text{for } (z, t) \in U_0^{(\varepsilon_3)} \times \langle \frac{1}{3}, \frac{2}{3}\rangle, \\ a_{3-3t}^{\varepsilon_3}(z) & \text{for } (z, t) \in U_0^{(\varepsilon_3)} \times \langle \frac{2}{3}, 1\rangle. \end{cases}$$

One easily sees that this homotopy joins in M the restriction $\hat{h}_k / U_0^{(\varepsilon_3)} = \hat{g}_k \hat{f}_k / U_0^{(\varepsilon_3)}$ with the map $i / U_0^{(\varepsilon_3)}$. Hence

$$(3.3) \qquad \hat{g}_k \hat{f}_k / U_0^{(\varepsilon_3)} \simeq i / U_0^{(\varepsilon_3)} \quad \text{in } M \text{ for every } k \geqslant k_3.$$

Thus we have shown that the fundamental domination of X by Y implies the fundamental domination of $\hat{X}$ by $\hat{Y}$.

If relation (1.4) holds true, then by an analogous argument, we infer that for every neighborhood N of Y (in R) there is a neighborhood V_0 of Y (in Q), a positive number $\varepsilon_4 < \frac{1}{2}$ and an index k_4 such that

$$(3.4) \qquad \hat{f}_k \hat{g}_k / V_0^{(\varepsilon_4)} \simeq i / V_0^{(\varepsilon_4)} \quad \text{in } N \text{ for every } k \geqslant k_4.$$

It follows that the fundamental equivalence of X and Y implies the fundamental equivalence of $\hat{X}$ and $\hat{Y}$.

The obtained results can be formulated as the following

(3.5) THEOREM. *The shape of the suspension* $\sum(X)$ *of a compactum* X *depends only on the shape of* X. *Moreover, if* $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$, *then* $\mathrm{Sh}(\sum(X)) \leqslant \mathrm{Sh}(\sum(Y))$.

Thus the operation of the suspension may be regarded as an operation on the shapes. In fact, we can define the suspension $\sum(\mathrm{Sh}(X))$ of the shape $\mathrm{Sh}(X)$ as the shape of the suspension $\sum(X)$ of X.

Remark. Let us observe that the shape of the suspension $\sum(X)$ does not determine the shape of X, even for polyhedra. In fact, consider a triangulation T of a Poincaré's 3-sphere, that is of a closed manifold M for which all homology groups are isomorphic with the corresponding groups of the Euclidean 3-sphere, but the fundamental group is not trivial. If one removes from M the interior of a 3-dimensional simplex belonging to T, then one gets an acyclic polyhedron X with non-trivial shape, because its fundamental group is not trivial. However, one easily sees that the suspension $\Sigma(X)$ is an AR-set, hence its shape is trivial.

4. Suspension of movable compacta. The property of movability ([3], p. 137) belongs to important shape-invariants. A compactum Y is said to be *movable*, if there exists in Q a set X homeomorphic with Y satisfying the following condition:

(4.1) *For every neighborhood U of X (in Q) there is a neighborhood U_0 of X (in Q) which by a continuous deformation in U can be carried onto a subset of every neighborhood of X (in Q).*

Let us prove the following

(4.2) THEOREM. *If X is a movable compactum, then the suspension $\Sigma(X)$ of X is also movable.*

Proof. We can assume that $X \subset Q$. Keeping the notations of sections 2 and 3, consider a neighborhood W of the set $\hat{X}$ in the space R. It is clear that there exists a neighborhood U of X (in Q) and a positive number $\varepsilon < \frac{1}{2}$ such that $U^{(\varepsilon)} \subset W$. Since X is movable, there exists a neighborhood U_0 of X (in Q) satisfying (4.1). Then the set $U_0^{(\varepsilon)} \subset U^{(\varepsilon)} \subset W$ is a neighborhood of $\hat{X}$ (in R). In order to prove that $\hat{X}$ is movable, it suffices to show that $U_0^{(\varepsilon)}$ can be carried by a continuous deformation in W onto a subset of an arbitrarily given neighborhood W_0 of the set $\hat{X}$ (in R). Consider a neighborhood V of X (in Q) such that $\hat{V} \subset W_0$. It follows by (4.1) that there is a homotopy $\varphi\colon U_0 \times \langle 0, 1\rangle \to U$ such that $\varphi(x, 0) = x$ and $\varphi(x, 1) \in V$ for every point $x \in U_0$.

Setting

$$\psi(z, t) = \begin{cases} a_{2t}^{\varepsilon}(z) & \text{for every } (z, t) \in U_0^{(\varepsilon)} \times \langle 0, \frac{1}{2}\rangle, \\ \hat{\varphi}\,[a_1^{\varepsilon}(z), 2t-1] & \text{for every } (z, t) \in U_0^{(\varepsilon)} \times \langle \frac{1}{2}, 1\rangle, \end{cases}$$

one gets a homotopy $\psi\colon U_0^{(\varepsilon)} \times \langle 0, 1\rangle \to U^{(\varepsilon)} \subset W$ joining the map $a_0^{\varepsilon}/U_0^{(\varepsilon)} = i/U_0^{(\varepsilon)}$ with the map, the values of which $\psi(z, 1) = \hat{\varphi}\,[a_1^{\varepsilon}(z), 1]$ belong to the set $\hat{V} \subset W_0$. Thus $\hat{X} = \sum(X)$ is movable and the proof of theorem (4.2) is finished.

(4.3) **Problem.** *Does there exist a non-movable compactum X such that its suspension is movable?* (**P 690**)

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