COLLOQUIUM MATHEMATICUM

VOL. XLVI 1982 FASC. 2

A PROBLEM ABOUT SET TRANSLATIONS ON THE REAL LINE

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Let R be the set of the real numbers with the usual topology. For any $t \in R$ and $F \subset R$ we define the translation $T_t F$ of F as

$$T_t F = \{x \in R: x = t + z \text{ for some } z \in F\}.$$

In the paper we consider the following

PROBLEM. Are there two closed sets A and B contained in R such that for any $t_i, s_j \in R$ ($i = 1, \ldots, p$; $j = 1, \ldots, q$) the relation

$$T_{t_1}A \cap \ldots \cap T_{t_p}A \cap T_{s_1}B \cap \ldots \cap T_{s_q}B = \emptyset \tag{1}$$

is satisfied if and only if $t_i = s_j$ for some i and j?

The affirmative answer to this problem is obtained by constructing A and B with the above-mentioned property.

Suppose ε_μ ($\mu = 1, 2, \ldots$) is a monotone nonincreasing sequence of positive numbers such that

$$\sum_{\mu=1}^{\infty} \varepsilon_\mu = 1.$$

Suppose $I_\mu \subset (0, 1)$ ($\mu = 1, 2, \ldots$) are closed nonintersecting intervals of lengths $|I_\mu| = \varepsilon_\mu$. Given a positive integer n, we define the intervals I_μ^n ($\mu = 1, \ldots, n^2$) as $I_\mu^n = T_k I_r$, where k and r are integers such that $1 \leqslant r \leqslant n$ and $\mu = kn + r$.

Let $\Omega^n = \{\omega_\mu^n\}$ ($\mu = 0, 1, \ldots, 2^{n^2} - 1$) be the set of all ordered finite sequences of length n^2 consisting of zeros and units. For an integer r ($1 \leqslant r \leqslant n^2$), $\omega_\mu^n(r)$ denotes the number at the r-th place of ω_μ^n.

Given any integer N and a positive integer μ, we put

$$A_\mu(N) = \left\{x \in [N, N + \mu \cdot 2^{\mu^2}]: x - N - \left[\frac{x-N}{\mu}\right]\mu \in I_r^\mu \right.$$

$$\left. \text{for some } r \text{ and } \omega_{[(x-N)/\mu]}^\mu(r) = 0\right\},$$

$$B_\mu(N) = \left\{x \in [N, N+\mu\cdot 2^{\mu^2}]:\ x-N-\left[\frac{x-N}{\mu}\right]\mu \in I_r^\mu \right.$$

$$\left. \text{for some } r \text{ and } \omega^\mu_{[(x-N)/\mu]}(r) = 1\right\},$$

where the brackets $[\cdot]$ denote the integer part of the comprised quantity. Put now

$$A' = \bigcup_{\mu=1}^{\infty} A_\mu(N_\mu) \quad \text{and} \quad B' = \bigcup_{\mu=1}^{\infty} B_\mu(N_\mu),$$

where the integers N_μ are chosen so that

$$N_\mu > \max\{\sup A_\nu(N_\nu)\cup B_\nu(N_\nu):\ 1 \leqslant \nu \leqslant \mu-1\}.$$

LEMMA. *The sets A' and B' are closed and disjoint, and for any $t_i, s_j \in R$ $(i = 1, \ldots, p;\ j = 1, \ldots, q)$ such that*

$$\min\{|t_i - s_j| : 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q\} > \varepsilon_1$$

the following relation holds true:

$$T_{t_1}A' \cap \ldots \cap T_{t_p}A' \cap T_{s_1}B' \cap \ldots \cap T_{s_q}B' \neq \emptyset. \tag{2}$$

Proof. We write A_μ and B_μ instead of $A_\mu(N_\mu)$ and $B_\mu(N_\mu)$, since no misunderstanding can occur.

The sets A_μ and B_μ $(\mu = 1, 2, \ldots)$ are closed as unions of finite number of closed intervals, hence each of the sets A' and B', being a union of locally finite family of closed sets, is closed (1). Further, A_μ and B_ν are disjoint, which follows from the defining formulas for $\mu = \nu$ and is evidently true for $\mu \neq \nu$ by construction. Hence

$$A' \cap B' = \bigcup_{\mu,\nu} A_\mu \cap B_\nu = \emptyset.$$

Let t_i, s_j $(i = 1, \ldots, p;\ j = 1, \ldots, q)$ be chosen accordingly to the assumption above. We define a mapping f of R onto the unit circle K in the complex plane by $f(x) = e^{2\pi i x}$. Let $S' = A' \cup B'$. We see that

$$f(S') = \bigcup_{\mu=1}^{\infty} f(I_\mu)$$

is the union of nonintersecting arcs $f(I_\mu)$ on K and its Lebesgue measure on K is

$$|f(S')| = \sum_{\mu=1}^{\infty} |f(I_\mu)| = 2\pi \sum_{\mu=1}^{\infty} \varepsilon_\mu = 2\pi,$$

(1) See p. 33 in: R. Engelking, *General topology*, Warszawa 1977.

whence $|K\setminus f(S')| = 0$. Since the translations on R preserve the measure of the image, we have

$$\begin{aligned}|K\setminus f(T_{t_1}S'\cap \dots \cap T_{t_p}S'\cap T_{s_1}S'\cap \dots \cap T_{s_q}S')| \\ &= \Big|\bigcup_{i=1}^{p}(K\setminus f(T_{t_i}S'))\cup\bigcup_{j=1}^{q}(K\setminus f(T_{s_j}S'))\Big| \\ &\leqslant \sum_{i=1}^{p}|K\setminus f(T_{t_i}S')|+\sum_{j=1}^{q}|K\setminus f(T_{s_j}S')| = 0.\end{aligned}$$

Thus

$$|f(T_{t_1}S'\cap \dots \cap T_{t_p}S'\cap T_{s_1}S'\cap \dots \cap T_{s_q}S')| = 2\pi$$

and, in particular, this intersection is nonvoid. Suppose it contains a point z'. Since

$$f(T_tS') = \bigcup_{\mu=1}^{\infty} f(T_tI_\mu), \qquad t\in R,$$

there are integers k_i, l_j $(i = 1, \dots, p;\ j = 1, \dots, q)$ such that

$$z' \in \bigcap \{f(T_{t_i}I_{k_i})\cap f(T_{s_j}I_{l_j}) : 1\leqslant i\leqslant p,\ 1\leqslant j\leqslant q\}.$$

Consequently, there is a real number z such that

$$z-t_i-[z-t_i] = \alpha_i \in I_{k_i} \quad (i = 1, \dots, p),$$
$$z-s_j-[z-s_j] = \beta_j \in I_{l_j} \quad (j = 1, \dots, q).$$

Choose now an integer $n > \max\{m_1, m_2\}$, where

$$m_1 = \max\{k_i, l_j : 1\leqslant i\leqslant p,\ 1\leqslant j\leqslant q\},$$
$$m_2 = 1+2\max\{|t_i|,\ |s_j| : 1\leqslant i\leqslant p,\ 1\leqslant j\leqslant q\}.$$

Let $z_1 \in (0, n)$ be such that $z_1 \equiv z \pmod 1$ and $m_2 < z_1 < n-m_2$. The existence of z_1 follows from the choice of n.

It is easy to check that each of z_1-t_i and z_1-s_j belongs to some interval I_μ^n but that they cannot both belong to the same interval I_μ^n. Indeed, z_1-t_i and z_1-s_j belong to $(0, n)$. Further,

$$z_1-t_i \equiv z-t_i \equiv \alpha_i \pmod 1, \qquad z_1-s_j \equiv z-s_j \equiv \beta_j \pmod 1.$$

Since $\alpha_i \in I_{k_i}$, $\beta_j \in I_{l_j}$, and $n > m_1$, we see that the first statement holds true. The second statement follows from $|z_1-t_i-(z_1-s_j)| = |t_i-s_j| > \varepsilon_1$; hence it is impossible that both z_1-t_i and z_1-s_j belong to the same I_μ^n, whose length by assumption is not greater than ε_1.

Consider now such ω_μ^n for which $\omega_\mu^n(r) = 0$ if I_r^n contains some number z_1-t_i and $\omega_\mu^n(r) = 1$ otherwise. By construction of A_n and B_n it follows that one can find an integer N such that

$$A' \cap (N, N+n) = \{x \in (N, N+n)\colon\ x - N \in I_r^n \text{ for some } r \text{ and } \omega_\mu^n(r) = 0\},$$

$$B' \cap (N, N+n) = \{x \in (N, N+n)\colon\ x - N \in I_r^n \text{ for some } r \text{ and } \omega_\mu^n(r) = 1\}.$$

Let now $z_2 \in (N, N+n)$ be such that $z_2 \equiv z_1 \pmod n$. The choice of z_1 yields that $z_2 - t_i$ and $z_2 - s_j$ belong to $(N, N+n)$. Since each of $z_2 - t_i - N = z_1 - t_i$ and $z_2 - s_j - N = z_1 - s_j$ belongs to some I_r^n satisfying $\omega_\mu^n(r) = 0$ if $z_1 - t_i \in I_r^n$ and $\omega_\mu^n(r) = 1$ otherwise, we see that $z_2 - t_i \in A'$ and $z_2 - s_j \in B'$. Thus

$$z_2 \in T_{t_1}A' \cap T_{t_2}A' \cap \ldots \cap T_{t_p}A' \cap T_{s_1}B' \cap \ldots \cap T_{s_q}B'.$$

Let now $\{\varepsilon_\mu^\nu\}$ $(\mu = 1, 2, \ldots;\ \nu = 1, 2, \ldots)$ be a set of positive numbers ε_μ^ν such that

$$\lim_{\nu \to \infty} \varepsilon_1^\nu = 0$$

and ε_μ^ν $(\mu = 1, 2, \ldots)$ for each $\nu = 1, 2, \ldots$ is a monotone nonincreasing sequence with

$$\sum_{\mu=1}^{\infty} \varepsilon_\mu^\nu = 1.$$

Let $\{I_\mu^\nu\}$ be a set of closed intervals, each contained in $(0, 1)$, such that for each $\nu = 1, 2, \ldots$ the intervals $\{I_\mu^\nu\}$ $(\mu = 1, 2, \ldots)$ do not intersect and the length of I_μ^ν is ε_μ^ν.

For an integer N and positive integers μ and ν we put $A_\mu^\nu(N) = A_\mu(N)$, where $A_\mu(N)$ has been constructed with the use of the intervals $\{I_\mu^\nu\}$ instead of $\{I_\mu\}$. In a similar way we define $B_\mu^\nu(N)$.

We put

$$A = \bigcup_{\mu,\nu} A_\mu^\nu(N_\mu^\nu) \quad \text{and} \quad B = \bigcup_{\mu,\nu} B_\mu^\nu(N_\mu^\nu),$$

where the integers N_μ^ν are chosen so that

$$N_\mu^\nu > \max\{\sup A_{\mu'}^{\nu'} \cup B_{\mu'}^{\nu'}\colon\ \mu' + \nu' < \mu + \nu \text{ or } \mu' + \nu' = \mu + \nu \text{ but } \mu' < \mu\}.$$

THEOREM. *The sets A and B are closed. For any $t_i, s_j \in R$ $(i = 1, \ldots, p;$ $j = 1, \ldots, q)$ relation* (1) *holds if and only if $t_i = s_j$ for some i and j.*

Proof. We write A_μ^ν and B_μ^ν instead of $A_\mu^\nu(N_\mu^\nu)$ and $B_\mu^\nu(N_\mu^\nu)$. By construction, $A_\mu^\nu \cap B_{\mu'}^{\nu'} = \emptyset$ for all positive integers μ, μ', ν, ν', hence $A \cap B = \emptyset$. Being unions of locally finite families of closed sets, both A and B are closed.

If $t_i = s_j$ for some i and j, then (1) is satisfied since $A \cap B = \emptyset$. Suppose now that

$$m = \min\{|t_i - s_j| : 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q\} > 0$$

and choose ν such that $\varepsilon_1^\nu < m$. We put

$$A^\nu = \bigcup_{\mu=1}^{\infty} A_\mu^\nu \quad \text{and} \quad B^\nu = \bigcup_{\mu=1}^{\infty} B_\mu^\nu .$$

Applying now the Lemma we see that

$$T_{t_1}A^\nu \cap \ldots \cap T_{t_p}A^\nu \cap T_{s_1}B^\nu \cap \ldots \cap T_{s_q}B^\nu \neq \emptyset,$$

and since $A^\nu \subset A$ and $B^\nu \subset B$, relation (1) does not hold.

In the sequel we give an application of the previous results.

If the domain of a function x is R and $t \in R$, then the translation x_t of x is defined as the function $x_t(\theta) = x(t+\theta)$. If x is a bounded function on R we write

$$\|x\| = \sup_{t \in R} |x(t)|.$$

Corollary. *There exists a bounded continuous function $\hat{x}$ on R such that*

$$\Big\| \sum_{k=1}^{n} a_k \hat{x}_{t_k} \Big\| = \sum_{k=1}^{n} |a_k| \tag{3}$$

for arbitrary n, a_k, and t_k such that $t_i \neq t_j$ for $i \neq j$.

Proof. Let A and B be closed subsets of R such that for every $t_i, s_j \in R$ ($i = 1, \ldots, p$; $j = 1, \ldots, q$) relation (1) holds if and only if $t_i = s_j$ for some i and j. In particular, A and B are disjoint.

Let $\hat{x}$ be a continuous function on R such that $\hat{x}(t) = 1$ for $t \in A$, $\hat{x}(t) = -1$ for $t \in B$, and $\|\hat{x}\| = 1$. Given a linear combination $\sum_{k=1}^{n} a_k \hat{x}_{t_k}$ let us divide the numbers t_k into two groups $\{t_i'\}$ ($i = 1, \ldots, p$) and $\{t_j''\}$ ($j = 1, \ldots, q$) according to whether $a_k \geqslant 0$ or $a_k < 0$. Let

$$t \in T_{-t_1'}A \cap \ldots \cap T_{-t_p'}A \cap T_{-t_1''}B \cap \ldots \cap T_{-t_q''}B.$$

Then $\hat{x}_{t_i'}(t) = 1$ ($i = 1, \ldots, p$) and $\hat{x}_{t_j''}(t) = -1$ ($j = 1, \ldots, q$). Therefore

$$\Big\| \sum_{k=1}^{n} a_k \hat{x}_{t_k} \Big\| \geqslant \Big| \sum_{k=1}^{n} a_k \hat{x}_{t_k}(t) \Big| = \sum_{k=1}^{n} |a_k|.$$

The inverse inequality is obvious, since $\|\hat{x}_{t_k}\| = 1$.

Remark. We consider the following Banach spaces of functions on R supplied with the sup-norm:

the space M of bounded functions;

the space C of continuous and bounded functions;

the subspace C_x of M ($x \in M$) generated by $\{x_t\colon t \in R\}$.

Given a Banach space X we denote its conjugate by X'. Using the Corollary we can show that

there exist functions $x \in C$ *such that*

(i) C'_x *is isomorphic to* M;

(ii) *each function* $y \in M$ *is of the form* $y(t) = \langle f, x_t \rangle$ *for some* $f \in C'$.

Indeed, let $x = \hat{x} \in C$, where $\hat{x}$ satisfies (3). To prove (i) we observe that $C_{\hat{x}}$ is isomorphic to $l^1(R)$ under the mapping

$$\sum_{k=1}^{\infty} a_k \hat{x}_{t_k} \to \sum_{k=1}^{\infty} a_k \chi_{\{t_k\}},$$

where χ denotes the set indicator. Since $(l^1(R))'$ is isomorphic to M under the mapping $(l^1(R))' \ni f \to \langle f, \chi_{\{t\}} \rangle$, the space $(C_{\hat{x}})'$ is isomorphic to M under the mapping $(C_{\hat{x}})' \ni f \to \langle f, \hat{x}_t \rangle \in M$. Using (i) and the Hahn-Banach theorem for $C_{\hat{x}} \subset C$ we obtain (ii).

If we had $C_{\hat{x}} = C$ for some $\hat{x}$ satisfying (3), we could find a representation for C', however a simple argument shows that this is impossible. In fact, no function x in C has the property $C_x = C$. This can be seen by the following proof due to Professor C. Ryll-Nardzewski. Take a Banach mean $\mathfrak{M}$ on C, that is a translation-invariant linear functional normed by $\mathfrak{M}(\mathit{1}) = 1$, where $\mathit{1}$ stands for the function identically equal to 1. It is known that for every function $\exp(i\lambda\cdot)$ with $\lambda \neq 0$ we have $\mathfrak{M}(\exp(i\lambda\cdot))$ $= 0$, so the functions $\exp(i\lambda\cdot)$ are orthogonal with respect to $\mathfrak{M}$, i.e.

$$\mathfrak{M}\big(\exp(i\lambda\cdot)\overline{\exp(i\lambda'\cdot)}\big) = 0 \quad \text{if } \lambda \neq \lambda';$$

besides,

$$\mathfrak{M}\big(\exp(i\lambda\cdot)\,\overline{\exp(i\lambda\cdot)}\big) = 1.$$

Thus the "Fourier coefficients" with respect to $\mathfrak{M}$, $a_\lambda = \mathfrak{M}(x\exp(-i\lambda\cdot))$, satisfy the Bessel inequality

$$\sum_{\lambda} |a_\lambda|^2 \leqslant \mathfrak{M}(|x|^2) < \infty.$$

Hence the set spec $x = \{\exp(i\lambda\cdot)\colon a_\lambda \neq 0\}$ is countable, and so there exists a function $\exp(i\lambda_0\cdot)$ orthogonal to x, therefore to all x_t. Obviously, $\exp(i\lambda_0\cdot)$ is not in the closure of the linear span of x_t with respect to the norm $\mathfrak{M}(|\cdot|^2)^{1/2}$ and all the more with respect to the sup-norm.

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Reçu par la Rédaction le 13. 2. 1979;
en version modifiée le 15. 2. 1980