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TOEPLITZ AND HANKEL FORMS RELATED TO UNITARY REPRESENTATIONS OF THE SYMPLECTIC PLANE

BY

MISCHA COTLAR (CARACAS)
AND CORA SADOSKY (WASHINGTON, D.C.)

TO ANTONI ZYGMUND, OUR BELOVED TEACHER,
A CONSTANT SOURCE OF INSPIRATION AND ENCOURAGEMENT,
AND A PERMANENT EXAMPLE OF INTEGRITY

Introduction. In previous work we proved a Generalized Bochner Theorem (GBT) for bounded Hankel forms, which allowed their integral expression in terms of measures, thus relating the classical Bochner theorem to both continuity problems for the Hilbert transform, and interpolation problems addressed by the Nehari theorem. Since the Bochner theorem was extended by I. Segal to unitary representations of symplectic spaces, it is natural to seek symplectic versions of the GBT, and the corresponding applications to singular integrals and interpolation problems.

We have already given in [CS3] a version of the GBT in the special case where the space of the representation is that of the Hilbert–Schmidt operators acting in $L^2(\mathbf{R})$.

In this note we give a version of the GBT for arbitrary unitary representations of the symplectic plane that is of a different nature. It is based on Segal's theorem and on the fact that every unitary representation of the symplectic plane has a cyclic element (see the Appendix). This fact adds interest to the present version of the GBT, in view of the possibility of generalizations in directions developed by M. Livshitz, M. G. Krein and H. Langer.

Here we describe the motivation underlying this approach, as well as the changes made necessary by the crucial difference between the representations of the symplectic plane and those of the group $\mathbf{Z}$ (or other commutative groups like $\mathbf{Z}^n$ or $\mathbf{R}^n$). The representations of the Heisenberg group, closely related to those of the symplectic plane, will be discussed elsewhere.

The relevant aspects of both the Bochner theorem and the GBT are recalled in Section 1, in the simplest case of the circle $\mathbf{T}$. Those results are stated in the case of unitary representations of the group $\mathbf{Z}$ in terms of cyclic elements, in a way suitable to their translation to corresponding representations of the symplectic plane, which are given in Section 2.

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1. Toeplitz forms related to unitary representations of $\mathbf{Z}$. Every unitary operator $U : H \to H$, in the Hilbert space H, defines a unitary representation $n \mapsto U_n = U^n$ of the group $\mathbf{Z}$, denoted by $[H, U_n = U^n]$. If $\mathbf{T} \sim [0, 2\pi)$ is the unit circle, then to each point $z = e^{it} \in \mathbf{T}$ there corresponds an irreducible unitary representation $[H_z, U_n(z)]$, where $H_z = \mathbf{C}$ and $U_n(z)\lambda = z^n\lambda$, $\forall\lambda \in \mathbf{C}$, and all the irreducible representations of $\mathbf{Z}$ are of this type. If $[H, U_n = U^n]$ is a representation of $\mathbf{Z}$ then every function $a : \mathbf{Z} \to \mathbf{C}$ of finite support (i.e., every finite sequence $a(n)$) gives rise to a bounded operator $a(U) = \sum_n a(n)U^n \in \mathcal{L}(H)$, and these finite sequences $a(n)$ are in 1-1 correspondence with the trigonometric polynomials $f(z) = \sum a(n)z^n = \sum f^\wedge(n)z^n$. Let V be the vector space of all such polynomials, $\tau : V \to V$ the shift operator $\tau : f(t) \mapsto zf(z)$, and write $f(U) = \sum a(n)U^n = \sum f^\wedge(n)U^n$.

A sesquilinear form $B : V \times V \to \mathbf{C}$ is called *Toeplitz* if

$$B(\tau f, \tau g) = B(f, g),$$

and *positive* if $B(f, f) \geq 0$. Every form B gives rise to a kernel $K : \mathbf{Z}\times\mathbf{Z} \to \mathbf{C}$ defined by $K(m, n) = B(z^m, z^n)$, and since (z^n) is a basis of V there is a 1-1 correspondence $B \leftrightarrow K = K_B$, and $K = K_B$ is called *positive definite* (respectively, *Toeplitz*) if B is positive (resp., Toeplitz).

1.1 (The Bochner–Herglotz theorem). *There is a* 1-1 *correspondence* $B \leftrightarrow \mu$ *between positive Toeplitz forms* $B : V \times V \to \mathbf{C}$ *and finite measures* $\mu \geq 0$ *in* $\mathbf{T}$, *given by*

$$B(f, g) = \int f\bar{g}\, d\mu, \qquad \forall f, g \in V, \tag{1}$$

so that

$$K(m, n) = K_B(m, n) = \mu^\wedge(m - n) = \int e^{i(m-n)t}\, d\mu. \tag{1a}$$

Given a unitary representation $[H, U^n]$ of $\mathbf{Z}$ and an element $\omega \in H$, for each $f \in V$ we write $\rho f = f(U)\omega$, set $H_\omega = \{\rho f : f \in V\}$ and say that ω is a *cyclic element* (or a *vacuum*) if H_ω is dense in H. If ω is cyclic then, setting $B(f, g) = \langle\rho f, \rho g\rangle$, we get a positive Toeplitz form in V and Theorem 1.1 gives:

1.2 (Spectral theorem for cyclic representations of $\mathbf{Z}$). *If* $[H, U^n]$ *is a unitary representation of* $\mathbf{Z}$ *with a cyclic element* ω, *then there exists a* *finite measure* $\mu \geq 0$ *in* $\mathbf{T}$ *such that the above map* $\rho : V \to H_\omega$ *extends to a unitary isomorphism of* $L^2(\mu)$ *onto* H, *under which the shift* τ *passes into* $U = U_1$ *so that*

$$\langle \rho f, \rho g \rangle = \int f\bar{g}\, d\mu, \qquad \forall f, g \in L^2(\mu). \tag{2}$$

Moreover, if E *is the spectral measure of* U, *then*

$$\mu(\Delta) = \langle E(\Delta)\omega, \omega \rangle. \tag{2a}$$

Conversely, if $B : V \times V \to \mathbf{C}$ is a positive Toeplitz form, then $\langle f, f \rangle_B = B(f, f)$ defines a Hilbertian seminorm giving rise to a Hilbert space H_B, while since $\langle \tau f, \tau g \rangle = \langle f, g \rangle$, τ gives rise to a unitary operator, and $z^0 = 1$ to a cyclic element in H_B. Thus Theorem 1.2 gives 1.1 and both theorems are logically equivalent.

Similar properties hold for unitary representations of the group $\mathbf{R}, t \mapsto U_t = e^{itA}$, where A is selfadjoint or symmetric. and we can consider A-cyclic elements. M. Livshitz generalized the notion of A-cyclic element so as to provide integral representations for A-symmetric forms, even in cases when A has no cyclic element. M. G. Krein developed Livshitz' idea into a powerful method of *directing functionals*, but with his notion it is not clear whether a cyclic element gives rise to such a functional. However, following Livshitz' original idea, we get a notion of directing functional, naturally associated to each cyclic element as follows.

Observe that in Theorem 1.2, V is considered as a set of elements of $L^2(\mu)$ and, if $f \in V$ and $\rho f = 0$, then $\rho f = 0$ as an element of $L^2(\mu)$, but there may exist non-zero functions $f \in V$ such that $\rho f = 0$. Thus if we consider V as a space of continuous functions then ρ^{-1} is not defined as a map from H_ω to V. But if $V_\omega = \{f \in V : \rho f = 0\}$ and Q is a projection of V onto V_ω, then ρ^{-1} can be defined as an injective map Γ of H_ω onto $V' = (I - Q)V_\omega$, so that each $\Gamma\xi$ is a function and $\Gamma\xi(z)$ is defined for all $z \in \mathbf{T}$, and we can speak of the value $\Gamma\xi(z) = \Gamma_z\xi$. In this way we get

1.3. *If* $[H, U^n]$, *and* ω, μ *and* ρ *are as in* 1.2, *then there is a linear map* $\Gamma : \mathbf{T} \times H_\omega \to \mathbf{C}$ *such that, setting* $\Gamma_z\xi = \Gamma(z, \xi) - \xi^\wedge(z)$, *the following properties hold:*

(a) $\forall z \in \mathbf{T}$, Γ_z *is a linear functional in* H_ω, *and* $\forall \xi \in H_\omega$, *the function* $\Gamma_z\xi = \xi^\wedge(z) \in V$;

(b) $\omega^\wedge(z) \neq 0, \forall z \in \mathbf{T}$;

(c) *if* $\xi \in H_\omega$ *and* $\xi^\wedge(z_0) = 0$ *then* $\exists \eta \in H_\omega$ *such that* $\xi = (U - z_0)\eta$.
Moreover, for every $\xi \in H_\omega$, $\xi^\wedge \in V$ *and* $\rho\xi^\wedge = \xi$, $\xi^\wedge = \rho^{-1}\xi$, *so that by* (2),

$$\langle \xi, \eta \rangle = \int \xi^\wedge(z)\overline{\eta^\wedge(z)}\, d\mu = \int (\Gamma_z\xi)(\overline{\Gamma_z\eta})\, d\mu, \qquad \forall \xi, \eta \in H_\omega. \tag{3}$$

Since the space $H_z = \mathbf{C} = \{\lambda 1\}$, each number $\Gamma_z\xi$ can be considered as an operator in H_z, and (3) can be written as

(3a) $$\langle \xi, \eta \rangle = \int \mathrm{Tr}\, 1(\Gamma_z \eta)^*(\Gamma_z \xi)\, d\mu(z), \qquad \forall \xi, \eta \in H_\omega.$$

If $[H, U_n]$ is an arbitrary unitary representation of $\mathbf{Z}$, H_1 a dense subspace of H, and if to each $\xi \in H_1$ there is assigned a smooth function $\xi^\wedge : \mathbf{T} \to \mathbf{C}$, $\xi^\wedge(z) = \Gamma_z \xi$, satisfying conditions (a), (b), (c) of 1.3 (with H_ω replaced by H_1), then we say that the map $\xi \mapsto \xi^\wedge(z) = \Gamma_z \xi$ is a *weak directing functional* of $[H, U_n]$(1). For a clear exposition of the ideas of Livshitz and Krein, see [A]; the theory was furhter elaborated by H. Langer [L].

From an argument of Livshitz and Krein (cf. [A]) follows

1.4. *If $\xi \mapsto \xi^\wedge$ is a weak directing functional of $[H, U_n]$, then there exists a measure $\mu \geq 0$ in $\mathbf{T}$ such that* (3) *and* (3a) *hold. Thus the map extends to a unitary isomorphism between $L^2(\mu)$ and H.*

Finally, let $[H, U_n]$ be a unitary representation of $\mathbf{Z}$ and let $B : H \times H \to \mathbf{C}$ be a positive form which is *U-invariant* or *Toeplitz*, $B(U\xi, U\eta) = B(\xi, \eta)$, and *bounded*, $|B(\xi, \eta)| \leq c\|\xi\|\,\|\eta\|$. For simplicity assume that B is *strictly positive*, $B(\xi, \xi) > 0$ whenever $\xi \neq 0$, so that the metric $\langle \xi, \eta \rangle_B = B(\xi, \eta)$ gives rise to a Hilbert space $[H_B, \langle \xi, \eta \rangle_B]$, in which $H \subset H_B$ is a dense subspace, and to a unitary representation $n \mapsto U^n$ in H_B, since $\langle U\xi, U\eta \rangle_B = B(U\xi, U\eta) = \langle \xi, \eta \rangle_B$. Moreover, from the continuity of B it follows that every dense subspace in H is dense in H_B, and every cyclic element (respectively, directing functional) of $[H, U^n]$ is also a cyclic element (resp. directing functional) of $[H_B, U^n]$, hence:

1.5. *If $[H, U_n]$ has a cyclic element ω (respectively, a weak directing functional in $H_1 \subset H$) then for every positive Toeplitz form $B : H \times H \to \mathbf{C}$ there exists a measure $\mu \geq 0$ in $\mathbf{T}$ such that*

(4) $$B(f(U)\omega, g(U)\omega) = \int f\bar{g}\, d\mu, \qquad \forall f, g \in V$$

(*respectively,*

(4a) $$B(\xi, \eta) = \int \xi^\wedge \overline{\eta^\wedge}\, d\mu, \qquad \forall \xi, \eta \in H_1).$$

Since, as shown above, every cyclic representation has an associate weak directing functional, 1.5 can be considered as a generalization of 1.2, as well as of Bochner's theorem 1.1. While Bochner's theorem gives an integral expression of positive forms $B : V \times V \to \mathbf{C}$ satisfying $B(\tau f, \tau g) = B(f, g)$, 1.5 gives a similar expression of the positive forms $B : H \times H \to \mathbf{C}$ satisfying $B(U\xi, U\eta) = B(\xi, \eta)$. Theorem 1.5, as extended by Krein and Langer

(1) In Krein's definition, condition (b) is somewhat relaxed and another condition, $(U\xi)^\wedge = \tau\xi^\wedge$, is required. A similar notion can be formulated for vector-valued functionals $\xi^\wedge : \mathbf{T} \to N$, N a Hilbert space, by suitably modifying condition (b) (see Section 2).

to symmetric operators and to vector-valued ξ, provided many important applications in Analysis and is closely related to a general eigenexpansion method for positive definite kernels due to Krein and Berezanskii (cf. [B], [M]).

The previous considerations extend to the space $\mathcal{V} = V \times V = \{(f_1, f_2) : f_1, f_2 \in V\}$, and the shift operator $\tau : \mathcal{V} \to \mathcal{V}$ defined by $\tau(f_1, f_2) = (\tau f_1, \tau f_2)$. Here a sesquilinear form $\mathcal{B} : \mathcal{V} \times \mathcal{V} \to \mathbf{C}$ is called *Toeplitz* if $\mathcal{B}(\tau(f_1, f_2), \tau(g_1, g_2)) = \mathcal{B}((f_1, f_2), (g_1, g_2))$. For instance, the following analogue of Bochner's theorem 1.1 gives integral expressions for such forms in terms of positive matrices of measures, i.e. $(\mu_{ij}) \geq 0$ if the scalar matrix $(\mu_{ij}(\Delta))$ is positive definite for every Borel set Δ.

1.1a. *If $\mathcal{B} : \mathcal{V} \times \mathcal{V} \to \mathbf{C}$ is a positive Toeplitz form then there exist four measures μ_{ij}, $i, j = 1, 2$, in $\mathbf{T}$ such that the matrix measure $(\mu_{ij}) \geq 0$, and*

$$\mathcal{B}((f_1, f_2), (g_1, g_2)) = \sum_{i,j=1,2} \int f_i \overline{g_j}\, d\mu_{ij}. \tag{5}$$

This gives the following analogue of 1.2:

1.2a. *Let $[H, U_n = U^n]$ be a unitary representation of $\mathbf{Z}$, and ω_1, ω_2 two elements of H such that the subspace $H_{\omega_1,\omega_2} = \rho\mathcal{V}$ is dense in H, where for $(f_1, f_2) \in \mathcal{V}$ we set $\rho(f_1, f_2) = f_1(U)\omega_1 + f_2(U)\omega_2$. Then there exist four measures μ_{ij}, $i, j = 1, 2$, in $\mathbf{T}$ such that the matrix measure $(\mu_{ij}) \geq 0$, and such that ρ extends to a unitary isomorphism of $L^2(\mu)$ onto H under which the shift τ of $\mathcal{V}$ passes into U, and so that*

$$\langle \rho(f_1, f_2), \rho(g_1, g_2) \rangle_H = \int \sum_{i,j=1,2} f_i \overline{g_j}\, d\mu_{ij}, \quad \forall (f_1, f_2),\ (g_1, g_2) \in \mathcal{V}. \tag{5a}$$

Conversely (5) can be deduced from (5a).

As above, every representation $[H, U^n]$ with a pair ω_1, ω_2 such that H_{ω_1,ω_2} is dense in H gives rise to a "directing functional" $\Gamma : (H_{\omega_1,\omega_2}, \mathbf{T}) \to \mathbf{C}^2$, $(\xi, t) \mapsto \Gamma_t \xi = \xi^\wedge(t) \in \mathbf{C}^2$, $\xi^\wedge \in \mathcal{V}$, which has properties similar to the $\mathbf{C}^2$-valued Krein functionals, for which an analogue of Proposition 1.4 holds. Finally, if $[H, U^n]$ is a unitary representation of $\mathbf{Z}$ which either has a cyclic pair ω_1, ω_2 or a $\mathbf{C}^2$-valued functional Γ, then an analogue of 1.5 holds for every continuous positive U-Toeplitz form $\mathcal{B} : H \times H \to \mathbf{C}$.

In all these propositions we have a 2×2 matrix measure $(\mu_{ij}) \geq 0$ in $\mathbf{T}$ which gives the desired integral representation.

We shall not go into details, but consider now Hankel forms and the GBT for unitary representations of $\mathbf{Z}$. Let $\mathbf{Z}_+ = \{n \in \mathbf{Z} : n \geq 0\}$, $\mathbf{Z}_- = \{n \in \mathbf{Z} : n < 0\}$ and set

$$V_1 = \{f = \textstyle\sum f^\wedge(n) z^n \in V : \operatorname{supp} f^\wedge \subset \mathbf{Z}_+\}, \quad V_2 = \{f \in V : \operatorname{supp} f^\wedge \subset \mathbf{Z}_-\}$$

so that

$$\tau V_1 \subset V_1, \qquad \tau^{-1} V_2 \subset V_2. \tag{6}$$

A sesquilinear form $B_0 : V_1 \times V_2 \to \mathbf{C}$ is called *Hankel* if there is a Toeplitz form $B : V \times V \to \mathbf{C}$ such that $B_0 = B$ in $V_1 \times V_2$, so that

$$B_0(\tau f, g) = B_0(f, \tau^{-1} g), \qquad \forall (f,g) \in V_1 \times V_2. \tag{7}$$

Let us fix two positive Toeplitz forms $B_i : V \times V \to \mathbf{C}$, $i = 1,2$ and let $\|f\|_{B_i} = \langle f, f\rangle_{B_i}^{1/2}$ be the corresponding quadratic τ-invariant seminorms, where $\langle f, g\rangle_{B_i} = B_i(f,g)$, $\langle \tau f, \tau g\rangle_{B_i} = \langle f, g\rangle_{B_i}$. If $B_0 : V_1 \times V_2 \to \mathbf{C}$ (respectively, $B : V \times V \to \mathbf{C}$) is a Hankel (resp., Toeplitz) form, then we say that B_0 (resp., B) is *bounded*, and write

$$B_0 \leq (B_1, B_2) \text{ in } V_1 \times V_2 \quad (\text{respectively, } B \leq (B_1, B_2) \text{ in } V \times V) \tag{8}$$

if $|B_0(f,g)| \leq \|f\|_{B_1} \, \|g\|_{B_2}$ (respectively, $|B(f,g)| \leq \|f\|_{B_1} \, \|g\|_{B_2}$) holds for all $(f,g) \in V_1 \times V_2$ (respectively, $(f,g) \in V \times V$).

With each Toeplitz form B we associate a form $\mathcal{B} : \mathcal{V} \times \mathcal{V} \to \mathbf{C}$ defined by

$$\mathcal{B}((f_1, f_2), (g_1, g_2)) = B_1(f_1, g_1) + B_0(f_1, g_2) + \overline{B_0(g_1, f_2)} + B_2(f_2, g_2). \tag{9}$$

Then $B \leq (B_1, B_2)$ in $V \times V$ iff $\mathcal{B}$ is a positive Toeplitz form.

We have then the following two theorems (see [CS1], [CS2]).

1.6 (Lifting property of bounded Hankel forms). *If the Hankel form B_0 satisfies $B_0 \leq (B_1, B_2)$ in $V_1 \times V_2$, then there exists a Toeplitz form $B : V \times V \to \mathbf{C}$ such that $B \leq (B_1, B_2)$ in $V \times V$ and $B_0 = B|_{V_1 \times V_2}$.*

From 1.6 and (5) we get

1.7 (The GBT for Hankel forms in $\mathbf{T}$). *If B_0 is a Hankel form satisfying $B_0 \leq (B_1, B_2)$ in $V_1 \times V_2$, then there exist four measures μ_{ij}, $i,j = 1,2$, $\mu_{21} = \overline{\mu}_{12}$ in $\mathbf{T}$, such that $(\mu_{ij}) \geq 0$ and $B_1(f,g) = \int f\overline{g}\, d\mu_{11}$ in $V_1 \times V_1$, $B_2(f,g) = \int f\overline{g}\, d\mu_{22}$ in $V_2 \times V_2$, and $B_0(f,g) = \int f\overline{g}\, d\mu_{12}$ in $V_1 \times V_2$.*

Let now $[H, U^n]$ be a unitary representation of $\mathbf{Z}$ and H_1, H_2 two subspaces of H satisfying

$$UH_1 \subset H_1, \qquad U^{-1} H_2 \subset H_2. \tag{10}$$

A form $B : H \times H \to \mathbf{C}$ (respectively, $B_0 : H_1 \times H_2 \to \mathbf{C}$) is *$U$-Toeplitz* (or *$U$-Hankel*) if $B(U\xi, U\eta) = B(\xi, \eta)$ (or $B_0(U\xi, \eta) = B_0(\xi, U^{-1}\eta)$) for all $(\xi, \eta) \in H \times H$ (or for all $(\xi, \eta) \in H_1 \times H_2$). From the 1-parametric lifting theorem in [CS2], [CS3] follows

1.8 (Lifting theorem in $[H, U^n]$). *If B_1, B_2 are positive U-Toeplitz forms, and B_0 is a U-Hankel form satisfying $B_0 \leq (B_1, B_2)$ in $H_1 \times H_2$, then there exists a U-Toeplitz form $B : H \times H \to \mathbf{C}$ such that $B \leq (B_1, B_2)$ in $H \times H$ and $B_0 = B$ in $H_1 \times H_2$.*

From 1.8 and the integral representation of continuous positive U-Toeplitz forms follows

1.9 (The GBT for unitary representations of $\mathbf{Z}$). *Let* $[H, U^n]$, H_1, H_2 *satisfy* (10), *let* B_1 *and* $B_2 : H \times H \to \mathbf{C}$ *be two positive U-Toeplitz forms, and assume that* $[H \times H, U^n]$, *where* $U(\xi_1, \xi_2) = (U\xi_1, U\xi_2)$ *for* $(\xi_1, \xi_2) \in H \times H$, *has either a cyclic pair* (ω_1, ω_2), $\omega_1 \in H \times \{0\}$, $\omega_2 \in \{0\} \times H$, *or a* $\mathbf{C}^2$*-valued weak directing functional* Γ. *Then, for every U-Hankel form* B_0 *satisfying* $B_0 \leq (B_1, B_2)$ *in* $H_1 \times H_2$, *there exist four measures* μ_{ij}, $i, j = 1, 2$, $\mu_{21} = \overline{\mu}_{12}$ *in* $\mathbf{T}$ *such that* $(\mu_{ij}) \geq 0$, *and* B_1, B_2 *and* B_0 *are given in* $H_1 \times H_1$, $H_2 \times H_2$ *and* $H_1 \times H_2$, *respectively, by the measures* μ_{11}, μ_{22} *and* μ_{12}, *as in* (4) *under the cyclic hypothesis, or as in* (4a) *under the directing functional assumption.*

Other abstract versions of the GBT for representations of $\mathbf{Z}$ are given in [CS3] and [CS4].

Let us remark again that from the GBT follow the results on the continuity of the Hilbert transform in weighted L^p spaces in the one-dimensional cases as well as in product spaces as given in [CS2] and [CS3]. For the significance of these problems, see [Z].

2. Toeplitz forms related to representations of $[\mathbf{C}, [\ ,\ ]]$. We now translate Theorems 1.1–1.5 of Section 1 by replacing unitary representations of $\mathbf{Z}$ by unitary representations of the symplectic plane. Let us identify $\mathbf{C}$ with $\mathbf{R}^2$, denote their points by $z = x + iy = (x, y)$, $x, y \in \mathbf{R}$, and set

$$[z, z'] = -\operatorname{Im} z\overline{z'} = xy' - yx'.$$

$\mathbf{C}$ with the symplectic form $[\ ,\ ]$ is called the *symplectic plane* $[\mathbf{C}, [\ ,\ ]]$. $[H, W(z)]$ is a *unitary representation* of $[C, [\ ,\ ]]$ if H is a Hilbert space and $W : \mathbf{C} \to \mathcal{L}(H)$ is a function assigning to each $z \in \mathbf{C}$ a unitary operator $W(z)$ in H satisfying

$$W(z)W(u) = \exp(\pi i[z, u])W(z + u), \tag{11}$$

and continuous in the strong topology of $\mathcal{L}(H)$. With each representation $[H, W(z)]$ there is associated a bounded linear map $\widetilde{W} : L^1(\mathbf{R}^2) \to \mathcal{L}(H)$ which assigns to each $F \in L^1(\mathbf{R}^2)$ the operator $\widetilde{W}(F)$ given by

$$\widetilde{W}(F) = \int\int F(x, y)W(x + iy)\, dx\, dy. \tag{11a}$$

The representation is said to be *irreducible* if there is no proper subspace of H which is $W(z)$-invariant for all z. The basic example of such an irreducible representation is the Schrödinger representation $[L^2(\mathbf{R}), \Phi(z)]$ where

$$(\Phi(x + iy)\psi)(t) = \exp(2\pi iyt + \pi ixy)\psi(t + x), \qquad \forall\psi \in L^2(\mathbf{R}). \tag{12}$$

The unitary symplectic representation $[H, W(z)]$ is said to be a direct sum of Schrödinger's representations if

(a) $H = \bigoplus_{n=1}^{N} H_n$, where $N \leq \infty$ and $H_1 = H_2 = \ldots = L^2(\mathbf{R})$;

(b) for each $k \leq N$ and for all $z \in \mathbf{C}$, the subspace $H_k = L^2(\mathbf{R})$ is $W(z)$-invariant and $W(z)|_{H_k} = \Phi(z)$.

In this case let π_k be the orthogonal projection of H onto H_k, so that the elements $\xi \in H$ are denoted by $\xi = (\pi_1\xi, \pi_2\xi, \ldots)$, $W(z)\xi = (\Phi(z)\pi_1\xi, \Phi(z)\pi_2\xi, \ldots)$, $\langle \xi, \eta \rangle = \sum_k \langle \pi_k\xi, \pi_k\eta \rangle_{L^2(\mathbf{R})}$, and we write $H = \Omega_N$, $H_k = \pi_k\Omega_N$, $[H, W(z)] = [\Omega_N, \Psi(z)] = [\Omega_N, (\pi_k), \Psi(z)]$, $\Psi(z) = \Psi_N(z)$.

The spectral theorem for symplectic representations, due to von Neumann and Stone, says that

(1) all irreducible unitary representations of the symplectic plane are unitary equivalent, so that there is essentially only one such irreducible representation, $[L^2(\mathbf{R}), \Phi(z)]$;

(2) for every unitary representation $[H, W(z)]$ there is an $N \leq \infty$ and a unitary isomorphism U of H onto Ω_N under which the operators $W(z)$ pass into the $\Phi(z)$, $\forall z$, i.e., $[H, W(z)]$ is unitarily equivalent to $[\Omega_N, \Psi(z)]$.

Moreover, there is an explicit canonical procedure (see [F]) for constructing the unitary map $U : H \to \Omega_N$ in (2), as follows. There is a fixed Gaussian function $\gamma : \mathbf{R}^2 \to \mathbf{C}$ (the same for all the representations) such that the operator $P = \widetilde{W}\gamma : H \to H$ is an orthogonal projection of H onto a subspace H_γ of dimension N, and such that if $\{h_1, h_2, \ldots\}$ is a fixed orthonormal basis of H_γ and H_n is the closed subspace spanned by the elements $\{W(z)h_n : z \in \mathbf{R}^2\}$, then $H = H_1 \oplus H_2 \oplus \ldots$ where all $H_k \sim L^2(\mathbf{R})$ and the desired operator U is given by

$$U(W(z)h_n) = (\delta_{n1}\gamma, \delta_{n2}\gamma, \ldots) \in L^2(\mathbf{R}) \oplus L^2(\mathbf{R}) \oplus \ldots = \Omega_N, \tag{13}$$

where $\delta_{nk} = 0$ if $k \neq n$ and $\delta_{nn} = 1$.

In the case of $\mathbf{Z}$ there is an irreducible representation $[H_z, U_n(z)]$ for each $z \in \mathbf{T}$, $H_z = \mathbf{C}$, $U_n(z) \cong z^n = e^{int} = e_n(t)$, while the symplectic plane has essentially only one irreducible representation $z \mapsto \Phi(z)$, so that in passing from $\mathbf{Z}$ to $[\mathbf{C}, [\ ,\ ]]$ we replace $n \in \mathbf{Z}$ by $z \in \mathbf{C}$, and the functions $e_n(t)$ by the operators $\Phi(z)$. Then the space $V \subset L^2(\mathbf{T})$ of trigonometric polynomials $\sum_n a_n e_n(t)$ is replaced now by the subspace $\Pi \subset \mathcal{L}(L^2(\mathbf{R}))$ of the operators $A = \sum_z a(z)\Phi(z)$, where $a : \mathbf{C} \to \mathbf{C}$ has finite support. A sesquilinear form $B : \Pi \times \Pi \to \mathbf{C}$ is called *Toeplitz* if

$$B(\Phi(z)A_1, \Phi(z)A_2) = B(A_1, A_2), \qquad \forall A_1, A_2 \in \Pi \text{ and } \forall z \in \mathbf{C}. \tag{14}$$

It is easy to see that $\{\Phi(z) : z \in \mathbf{C}\}$ is a basis in Π so that B is determined by the associated kernel $K_B : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, defined by $K_B(z, u) = B(\Phi(z), \Phi(u))$, and $B(\sum a(z)\Phi(z), \sum b(u)\Phi(u)) = \sum_z \sum_u a(z)\overline{b(u)} K_B(z, u)$.

If B is Toeplitz then K_B will be called *symplectic Toeplitz*, and in this case we have

(14a) $$K_B(z,u) = e^{i\pi[z,u]}K_1(z,u),$$ where $K_1(z,u) = K_1(z-u)$ is ordinary Toeplitz.

Given a representation $[H, W(z)]$, an element $\omega \in H$ and an operator $A = \sum a(z)\Phi(z) \in \Pi$, we write $A(W) = \sum a(z)W(z) \sim a(W)$, $H_\omega = \{A(W)\omega : A \in \Pi\}$ and say that ω is a *cyclic element* or a *vacuum* if H_ω is dense in H. In particular $[\Omega_N, \Psi(z)]$ has a cyclic element $\omega \in \Omega_N$ if $\Omega_\omega = \{A(\Psi)\omega : A \in \Pi\}$ is dense in Ω_N. In this case writing $\omega = \{\pi_1\omega, \pi_2\omega, \ldots\}$, $\pi_k\omega \in L^2(\mathbf{R})$, we have

$$\begin{aligned}\langle A_1(\Psi)\omega, A_2(\Psi)\omega\rangle &= \sum_z\sum_u a_1(z)\overline{a_2(u)}\langle\Psi(z)\omega, \Psi(u)\omega\rangle \\ &= \sum_z\sum_u a_1(z)\overline{a_2(u)}\sum_{k=1}^{N}\langle\Phi(z)\pi_k\omega, \Phi(u)\pi_k\omega\rangle,\end{aligned}$$

so that

(15) $$\langle A_1(\Psi)\omega, A_2(\Psi)\omega\rangle = \operatorname{Tr} SA_2^*A_1,$$

where S is a positive trace class operator in $L^2(\mathbf{R})$, given by

(15a) $$S = \sum_k(\pi_k\omega)\otimes(\pi_k\omega).$$

Thus we have the following analogue of Theorem 1.2:

2.1. *If ω is a vacuum of $[\Omega_N, \Psi(z)]$ then there exists a positive trace class operator S in $L^2(\mathbf{R})$ such that* (15) *holds for all $A_1, A_2 \in \Pi$. Moreover, S is given explicitly through ω by* (15a).

If $B : \Pi \times \Pi \to \mathbf{C}$ is positive Toeplitz form then, as in the case of V, B gives rise to a cyclic representation $[H, W(z)] \sim [\Omega_N, \Psi(z)]$, and Theorem 2.1 gives

2.2 (I. Segal's theorem [S]). *If $B : \Pi \times \Pi \to \mathbf{C}$ (respectively, if $K : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$) is a positive Toeplitz form (respectively, a positive definite symplectic Toeplitz kernel) then there exists a positive trace class operator S in $L^2(\mathbf{R})$ such that*

(16) $$B(A_1, A_2) = \operatorname{Tr} SA_2^*A_1, \qquad K(z,u) = \operatorname{Tr} S\Phi(-u)\Phi(z).$$

Conversely, Theorem 2.1 can be obtained from Theorem 2.2 by setting

$$B(A_1, A_2) = \langle A_1(\Psi)\omega, A_2(\Psi)\omega\rangle,$$

hence Theorems 2.1 and 2.2 are logically equivalent, where 2.2 is the analogue of the Bochner theorem 1.1.

The analogue of Theorem 1.3 is as follows. Let ω be a cyclic element of $[\Omega_N, \Psi(z)]$, $\Omega_\omega = \{A(\Psi)\omega : A \in \Pi\}$ and $\Pi_\omega = \{A \in \Pi : A(\Psi)\omega = 0\}$. Since $\Phi(0)\omega = \omega$, $\Phi(0) \notin \Pi_\omega$, there is an algebraic linear projection Q of Π onto Π_ω such that $(I-Q)\Phi(0) = (0)$. The mapping $A \mapsto A(\Psi)\omega$ maps Π onto Ω_ω and Π_ω onto 0, hence its restriction to $(I-Q)\Pi$ is a bijection onto Ω_ω which takes $\Phi(0)$ onto ω, and if Γ is the inverse of this restriction, then Γ is a bijection of Ω_ω onto $(I-Q)\Pi$. Hence

2.3. *If $[\Omega_N, \Psi(z)], \omega$ and S are as in 2.1, and $\Gamma : \Omega_\omega \to \Pi$ is a linear injection such that writing $\Gamma\xi = \xi^\wedge$ we have $\omega^\wedge = \Phi(0) = I$, then $A = \xi^\wedge$ implies $\xi = A(\Psi)\omega$, so that*

$$\langle \xi, \eta \rangle = \operatorname{Tr} S(\eta^\wedge)^* \xi^\wedge, \qquad \forall \xi, \eta \in \Omega_\omega. \tag{17}$$

Setting $\Pi'_\omega = (I-Q)\Pi = \Gamma(\Omega_\omega)$, we have

(a_1) $\xi \mapsto \xi^\wedge = \Gamma\xi$ is a bijection of Ω_ω onto Π'_ω;

(b_1) for every integer $k \le N$ there is an element $\varepsilon_k \in L^2(\mathbf{R})$, with $\sum_k \|\varepsilon_k\|^2 < \infty$, and a linear map $\gamma_k : \Pi'_\omega \to \Omega_\omega$ such that every $A \in \Pi'_\omega$ satisfies $A\varepsilon_k = (\gamma_k A)^\wedge \varepsilon_k = \pi_k(\gamma_k A)$;

(c_1) for every $\xi \in \Omega_\omega$, $\xi^\wedge \varepsilon_k = 0 \Rightarrow \pi_k \xi = 0$.

Condition (b_1) corresponds to the modified condition (b) mentioned in the footnote (1). Furthermore, here we have Γ instead of Γ_z as in Section 1, since in the case of the group $\mathbf{Z}$ there is an irreducible representation for each $z \in \mathbf{T}$, while now there is only one irreducible representation repeated N times, so that conditions (a_1)–(c_1) involve $k \le N$, but not z.

Setting $\varepsilon_k = \pi_k \omega$ and $\gamma_k A = A(\Psi)\omega$ for $A \in \Pi'_\omega$, we have $(\gamma_k A)^\wedge = (I-Q)A = A$ (since $QA \in \Pi_\omega$, $\gamma_k(I-Q)A = \gamma_k A$ and γ_k is injective on Π'_ω) and in particular $A\varepsilon_k = (\gamma_k A)^\wedge \varepsilon_k$. Moreover,

$$\begin{aligned}\pi_k(\gamma_k A) = \pi_k(A(\Psi)\omega) &= \pi_k(A(\Psi)\pi_1\omega, A(\Psi)\pi_2\omega, \ldots) \\ &= A(\Psi)\pi_k\omega = A(\Psi)\varepsilon_k,\end{aligned}$$

so that (b_1) holds. Finally, if $\xi^\wedge \varepsilon_k = 0$ then by definition of γ_k there exists $A = (I-Q)A \in \Pi'_\omega$ such that $\xi = A(\Psi)\omega$, and $\xi^\wedge = A$, hence

$$\pi_k \xi = \pi_k A(\Psi)\omega = \pi_k(A\pi_1\omega, \ldots, A\pi_k\omega, \ldots) = A\pi_k\omega = A\varepsilon_k = 0,$$

so that also (c_1) holds.

If Ω_1 is an arbitrary dense subspace of Ω_N, Π' a subspace of Π and $\Gamma : \Omega_1 \to \Pi'$ a map satisfying conditions (a_1), (b_1), (c_1) (with Π'_ω replaced by Π'), then we say that Γ is a *weak directing functional* of $[\Omega_N, \Psi(z)]$, and write $\xi^\wedge = \Gamma\xi$.

2.4. *If $\Gamma : \Omega_1 \to \Pi'$ is an arbitrary weak directional functional of $[\Omega_N, \Psi(z)]$, then there exists a positive trace class operator S in $L^2(\mathbf{R})$ such* *that*

$$(18) \qquad \langle \xi, \eta \rangle = \operatorname{Tr} S(\Gamma\eta)^*(\Gamma\xi) = \operatorname{Tr} S(\eta^\wedge)^*\xi^\wedge, \qquad \forall \xi, \eta \in \Omega_1,$$

and $S = (\varepsilon_1 \otimes \varepsilon_1) + (\varepsilon_2 \otimes \varepsilon_2) + \dots$

Proof. Setting, for a given $\xi \in \Omega_1$, $\xi_1 = \xi - \gamma_k(\xi^\wedge)$, we have, by (b_1), $\xi_1^\wedge \varepsilon_k = \xi^\wedge \varepsilon_k - (\gamma_k(\xi^\wedge))^\wedge \varepsilon_k = \xi^\wedge \varepsilon_k - \xi^\wedge \varepsilon_k = 0$, so that $\xi_1^\wedge \varepsilon_k = 0$ and by (c_1), $\pi_k \xi_k = 0$. Hence $\pi_k \xi = \pi_k \gamma_k(\xi^\wedge) = \xi^\wedge \varepsilon_k$ by (b_1). Thus $\pi_k \xi = \xi^\wedge \varepsilon_k$, $\forall \xi \in \Omega_1$ and $\forall k$, hence

$$\langle \xi, \eta \rangle = \sum_k \langle \pi_k \xi, \pi_k \eta \rangle = \sum_k \langle \xi^\wedge \varepsilon_k, \eta^\wedge \varepsilon_k \rangle = \sum_k \langle (\eta^\wedge)^* \xi^\wedge \varepsilon_k, \varepsilon_k \rangle,$$

and setting $S = (\varepsilon_1 \otimes \varepsilon_1) + (\varepsilon_2 \otimes \varepsilon_2) + \dots$ we get the desired equality (18). ∎

Since, as shown above, each cyclic element ω has an associated directional functional, Proposition 2.4 contains 2.2 as a special case.

Let now $[H, W(z)]$ be an arbitrary unitary representation of the symplectic plane and $B : H \times H \to \mathbf{C}$ a continuous positive form which is $W(z)$-Toeplitz: $B(W(z)\xi, W(z)\eta) = B(\xi, \eta)$, $\forall z$. Assume for simplicity that B is strongly positive: $B(\xi, \xi) > 0$ if $\xi \neq 0$. Then $\langle \xi, \eta \rangle_B = B(\xi, \eta)$ is a scalar product in H giving rise to a Hilbert space H^B such that H is a dense subspace in H^B. Since B is $W(z)$-Toeplitz, all $W(z)$ extend to unitary operators in H^B, so that $[H^B, W(z)]$ becomes a unitary representation of the symplectic plane. Since B is continuous, the convergence in the norm of H implies that in the norm of H^B, and every subspace $H_1 \subset H$, dense in H, is dense in H^B. Since the integral $(\int W(z)\gamma(z)\,dz)\xi$ (where γ is the Gaussian function used in (13)) converges in the norm of H, it converges to the same limit in the norm of H^B, thus if P^B is the orthogonal projector corresponding to $[H^B, W(z)]$ (see the construction preceding (13)) then $P^B = P$ on H. The above orthonormal system (h_n) in H_γ may not be orthogonal in H^B, but since (h_n) is complete in $H_\gamma \subset H$, it also generates the closure $\overline{H}_\gamma$ of H_γ in H^B, and orthogonalizing (h_n) we get an orthonormal basis (h_n^B) for $\overline{H}_\gamma = H_\gamma^B$ where the h_n^B are expressed through the h_n by explicit formula of the Schmidt procedure. Thus the elements $W(z)h_n^B$ span in H^B the subspace H_n^B such that $H^B = H_1^B \oplus H_2^B \oplus \dots$ and U^B, defined by $U^B(W(z)h_n^B) = (\delta_{n1}\gamma, \delta_{n2}\gamma, \dots) = U(W(z)h_n)$ gives as in (13) the canonical unitary map of $[H^B, W(z)]$ onto $[\Omega_N, \Phi(z)]$. This shows in the first place that Ω_N is the same for $[H^B, W(z)]$ and for $[H, W(z)]$, so that $[H^B, W(z)] \cong [\Omega_N, \Phi(z)] \cong [H, W(z)]$ through the isomorphism U and U^B. Moreover, knowing the spectral decomposition of $[H, W(z)]$, i.e. knowing U through the h_n, we can write by explicit formulae the spectral decomposition of $[H^B, W(z)] \cong [\Omega_N, \Phi(z)]$ through the h_n^B and U^B.

In particular, if ω is a cyclic element of $[\Omega_N, \Phi(z)]$ then $\rho^B = (U^B)^{-1}\rho$ gives a linear map of Π onto a dense subspace of H^B, and $(A_1, A_2) \mapsto \langle \rho^B A_1, \rho^B A_2 \rangle_{H^B}$ gives a positive Toeplitz form in Π. Therefore,

2.5. *There is a positive trace class operator S in $L^2(\mathbf{R})$ such that*

$$\langle \rho^B A_1, \rho^B A_2 \rangle_{H^B} = \operatorname{Tr} S A_2^* A_1, \qquad \forall A_1, A_2 \in \Pi. \tag{19}$$

In particular, from the explicit formula relating (h_n^B) to (h_n), we obtain a representation of $B(A_1 h_m^B, A_2 h_n^B)$ through S. Without going into details, we may state the following proposition:

2.6. *Let $[H, W(z)]$ be a symplectic representation with spectral decomposition $[H, W(z)] \cong [\Omega_N, \Phi(z)]$, and let ω be a cyclic element in Ω_N and $B : H \times H \to \mathbf{C}$ a continuous positive $W(z)$-Toeplitz form. Then B can be expressed explicitly by a formula of type* (16) *through a trace class operator S in $L^2(\mathbf{R})$. Similarly, any weak directing functional defined in the linear span of the elements $\{W(z)h_n\}$ can be transferred to the elements $\{W(z)h_n^B\}$, providing a formula of type* (18) *for $B(\xi, \eta) = \langle \xi, \eta \rangle_B$.*

Now the crucial difference between the representations of the symplectic plane and those of the group $\mathbf{Z}$ (or other commutative groups $\mathbf{Z}^n$, $\mathbf{R}^n$) is the following property which follows from 2.6 and the results proved in the Appendix.

2.7. *Every unitary representation $[\Omega_N, \Phi(z)]$ has a cyclic element ω, and in particular also directing functionals. Thus if $[H, W(z)]$ is an arbitrary symplectic representation and $B : H \times H \to \mathbf{C}$ a continuous positive $W(z)$-Toeplitz form, then B can be given explicit representations of type* (16) *or* (18) *through a trace class operator S in $L^2(\mathbf{R})$.*

As in (5) and (5a), it follows then that Segal's theorem extends to positive Toeplitz forms $\mathcal{B} : \Pi^2 \times \Pi^2 \to \mathbf{C}$, $\Pi^2 = \Pi \times \Pi$, through four operators (S_{ij}) in $L^2(\mathbf{R})$ such that $(S_{ij}) \geq 0$ in an obvious sense, and 2.2 and 2.4 extend to forms $\mathcal{B} : H \times H \to \mathbf{C}$, since $[H, W(z)]$ has always a cyclic pair (ω_1, ω_2) and Π^2-valued weak directing functionals.

Let us pass now to the GBT for symplectic representations. Let

$$\begin{aligned}
\Pi_1 &= \{A = \textstyle\sum a(z)\Phi(z) \in \Pi : \operatorname{supp} a(z) \subset \{z = (x,y) \in \mathbf{R}^2 : x \geq 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y \geq 0\}\}, \\
\Pi_2 &= \{\textstyle\sum a(z)\Phi(z) : \operatorname{supp} a(z) \subset \{(x,y) : x < 0\} \cup \{(x,y) : y < 0\}\}, \qquad (20) \\
\Pi_{21} &= \{\textstyle\sum a(z)\Phi(z) : \operatorname{supp} a(z) \subset \{x < 0\}\}, \\
\Pi_{22} &= \{\textstyle\sum a(z)\Phi(z) : \operatorname{supp} a(z) \subset \{y < 0\}\},
\end{aligned}$$

so that $\Pi_2 = \Pi_{21} + \Pi_{22}$.

A sesquilinear form $B_0 : \Pi_1 \times \Pi_2 \to \mathbf{C}$ is said to be *Hankel* if there exists a Toeplitz form $B : \Pi \times \Pi \to \mathbf{C}$ such that $B_0 = B$ in $\Pi_1 \times \Pi_2$.

As in Section 1, fix two positive Toeplitz forms and define the relations $B_0 \leq (B_1, B_2)$ in $\Pi_1 \times \Pi_2$, or in $\Pi_1 \times \Pi_{21}$, and $B_0 \leq (B_1, B_2)$ in $\Pi \times \Pi$. Then from the general 2-parametric lifting theorem given in [CS3], we obtain

2.8 (Lifting theorem for bounded Hankel forms in $\Pi_1 \times \Pi_2$). *If $B_0 : \Pi_1 \times \Pi_2 \to \mathbf{C}$ is a Hankel form satisfying $B_0 \leq (B_1, B_2)$ in $\Pi_1 \times \Pi_2$, then there exist two Toeplitz forms $B' : \Pi \times \Pi \to \mathbf{C}$, $B'' : \Pi \times \Pi \to \mathbf{C}$ satisfying $B' \leq (B_1, B_2)$ and $B'' \leq (B_1, B_2)$ in $\Pi \times \Pi$, and such that $B_0 = B'$ in $\Pi_1 \times \Pi_{21}$, $B_0 = B''$ in $\Pi_1 \times \Pi_{22}$.*

From 2.8, and the preceding discussion we get

2.9 (GBT for Hankel forms in $\Pi_1 \times \Pi_2$). *If $B_0 : \Pi_1 \times \Pi_2 \to \mathbf{C}$ is Hankel and satisfies $B_0 \leq (B_1, B_2)$ in $\Pi_1 \times \Pi_2$, then there exist four trace class operators (S'_{ij}), and four trace class operators (S''_{ij}), $i, j = 1, 2$, satisfying $(S'_{ij}) \geq 0$, $(S''_{ij}) \geq 0$, such that*

$$\begin{aligned} &B_0 \leq (A_1, A_2) = \operatorname{Tr} S'_{12} A_2^* A_1 \quad \textit{for } (A_1, A_2) \in \Pi_1 \times \Pi_{21}, \\ &B_0 \leq (A_1, A_2) = \operatorname{Tr} S''_{12} A_2^* A_1 \quad \textit{for } (A_1, A_2) \in \Pi_1 \times \Pi_{22}, \\ &B_i \leq (A_1, A_2) = \operatorname{Tr} S'_{ii} A_2^* A_1 = \operatorname{Tr} S''_{ii} A_2^* A_1, \ \textit{in } \Pi_i \times \Pi_i, \\ &\hspace{20em} i = 1, 2. \end{aligned} \tag{21}$$

Let $[H, W(z)]$ be an arbitrary unitary representation of the symplectic plane and H_1, H_2 two subspaces of H such that

$$\begin{aligned} W(z)H_1 \subset H_1 \quad &\text{if} \quad z = (x, y) \text{ with } x \geq 0, y \geq 0, \\ W(z)H_2 \subset H_2 \quad &\text{if} \quad z = (x, y) \text{ with } x < 0, y < 0. \end{aligned}$$

Set $H_{21} = \{\xi \in H_2 : W(x + iy)\xi \in H_2 \text{ if } x < 0\}$, $H_{22} = \{\xi \in H_2 : W(x + iy)\xi \in H_2 \text{ if } y < 0\}$. $B_0 : H_1 \times H_2 \to \mathbf{C}$ is said to be *Hankel* if there exists an $W(z)$-Toeplitz form $B : H \times H \to \mathbf{C}$ such that $B_0 = B$ in $H_1 \times H_2$. Fixing two positive $W(z)$-Toeplitz forms $B_1, B_2 : H \times H \to \mathbf{C}$ we shall have the *lifting theorem*:

(22) *if the Hankel form B_0 satisfies $B_0 \leq (B_1, B_2)$ in $H_1 \times H_2$ then there exist two Toeplitz forms B', B'' such that B' and B'' are $\leq (B_1, B_2)$ in $H \times H$ and $B_0 = B'$ in H_{12}, $B_0 = B''$ in H_{21}.*

From (22) it follows that one can write explicit formulae, similar to those of 2.9 for bounded Hankel forms in $H_1 \times H_2$, for arbitrary symplectic representations $[H, W(z)]$, where we can always fix a cyclic pair or a weak directing functional. We shall not go into explicit formulae here, and only add the following remark. Of special interest is the representation $[H, W(z)]$ where $H = \mathcal{L}^2(L^2(\mathbf{R}))$ = the space of all Hilbert–Schmidt operators $X : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ with scalar product $\langle X, Y \rangle = \operatorname{Tr} Y^* X$, and $W(z)X = \Phi(z)X$. The GBT for this representation $[\mathcal{L}^2(L^2(\mathbf{R})), W(z) = \Phi(z)]$ was studied in [CS3] through the following substitute of Segal's theorem: if B :

$\mathcal{L}^2(L^2(\mathbf{R})) \times \mathcal{L}^2(L^2(\mathbf{R})) \to \mathbf{C}$ is a continuous $W(z)$-Toeplitz form, then there exists a trace class operator S_1 in $L^2(\mathbf{R})$ such that $B(X,Y) = \operatorname{Tr} S_1 Y^* X$. Instead, in the present treatment of this case, we have $B(X,Y) = \operatorname{Tr} S A_2^* A_1$ for certain $A_1, A_2 \in \Pi$; there S can be explicitly written through a cyclic element which we know always exists, while the above operator S_1 is not given explicitly. In some applications the version of [CS3] may be more suitable, and in others the one given here is better. Moreover, the version given here, based on the existence of cyclic elements, applies to all representations $[H, W(z)]$ and not only in the case when $H = \mathcal{L}^2(L^2(\mathbf{R}))$, $W(z) = \Phi(z)$, and to obtain this was the basic aim of this paper.

Appendix. Our results depend on the fact that every unitary representation of the symplectic plane has a cyclic element. Lacking an explicit reference for this fact, we reproduce here a proof by Nolan Wallach [W] of a more general proposition.

Let G be a group and let (π, H) be an infinite-dimensional, irreducible, unitary representation of G. Let H_∞ be a countably infinite direct sum of copies of H and let π_∞ be the corresponding diagonal representation of G.

PROPOSITION. *Let $\{v_n\}$ be an orthonormal basis of H and let $\lambda \in \mathbf{R}$ be such that $0 < \lambda < 1$. Set $\omega = v_1 \oplus \lambda v_2 \oplus \lambda^2 v_3 \oplus \ldots \oplus \lambda^{n-1} v_n \oplus \ldots$ Then ω is a cyclic vector for π_∞.*

Let us first recall a finite-dimensional result that contains all but one of the ideas for the general case. Let G be a group and let (π, V) be a representation of G with $\dim V = n < \infty$. Let V^n be a direct sum of n copies of V and let π^n be the corresponding diagonal representation of G:

$$\pi^n(g)(v_1 \oplus \ldots \oplus v_n) = \pi(g)v_1 \oplus \ldots \oplus \pi(g)v_n.$$

LEMMA. *If $\{v_1, \ldots, v_n\}$ is a basis of V then $\omega = v_1 \oplus \ldots \oplus v_n$ is a cyclic vector for π^n.*

Proof. Suppose that $\lambda \in (V^n)^*$ and that

$$\lambda(\pi^n(g)w) = 0 \quad \text{for all } g \in G. \tag{A.1}$$

Let $\{v_j^*\}$ be the dual basis to $\{v_j\}$. Let $\lambda = \lambda_1 \oplus \ldots \oplus \lambda_n$. Then $\lambda_i = \sum_j \lambda_i(v_j) v_j^*$. Thus (A.1) implies that

$$\sum_{i,j} \lambda_i(v_j) v_j^*(\pi(g)v_i) = 0 \quad \text{for all } g \in G. \tag{A.2}$$

Let Λ be the linear operator on V with matrix $[\lambda_i(v_j)]$ relative to the basis $\{v_j\}$. Then (A.2) says that $\operatorname{Tr}(\Lambda \pi(g)) = 0$ for all $g \in G$. Since π is irreducible, the span of all $\pi(g)$, $g \in G$ is $\operatorname{End}(V)$. Thus $\Lambda = 0$ so $\lambda = 0$. ∎

We now modify this argument so that it applies to the infinite-dimensional case.

Proof of the Proposition. Let $u \in H_\infty$ be such that $\langle \pi_\infty(g)\omega, u\rangle = 0$ for all $g \in G$. Now $u = u_1 \oplus u_2 \oplus \ldots$ Write $u_i = \sum_j \langle u_i, v_j\rangle v_j$. Then our assumption says that

$$\sum_{i,j} \langle v_j, u_i\rangle \langle \pi(g)\lambda^{i-1}v_i, v_j\rangle = 0 \quad \text{for all } g \in G. \tag{A.3}$$

Let Λ be the operator on V with matrix $\langle v_i, u_j\rangle$ relative to the basis $\{v_n\}$. Then Λ is of Hilbert–Schmidt class with HS-norm $\sum_{i,j} |\langle u_i, v_j\rangle|^2 = \sum_i \|u_i\|^2 = \|\Lambda\|^2_{\mathrm{HS}}$. Let D be the operator such that $Dv_n = \lambda^{n-1}v_n$. Then (A.3) says

$$\operatorname{Tr}(\pi(g)D) = 0 \quad \text{for all } g \in G.$$

Notice that this makes sense since both Λ and D are HS.

We now recall the von Neumann density theorem. Let $\mathcal{A}$ be the algebra of operators on H generated by the $\pi(g)$, $g \in G$. Let T be a bounded operator on H, let $\varepsilon > 0$ be given and let $x_j \in H$ be such that $\sum \|x_j\|^2 < \infty$. Then there exists $A \in \mathcal{A}$ such that

$$\sum_i \|(A - T)x_i\|^2 < \varepsilon.$$

Apply this result to see that $u = 0$. Let $x_i = \lambda^{i-1}v_i$. Let $\varepsilon > 0$ be given and let $A_\varepsilon \in \mathcal{A}$ be such that $\sum_i \|(A_\varepsilon - \Lambda^*)x_i\|^2 < \varepsilon$. Now (A.3) implies

$$\operatorname{Tr}(\Lambda(\Lambda^* - A_\varepsilon)D) = \operatorname{Tr}(\Lambda\Lambda^*D). \tag{A.4}$$

On the other hand, $\operatorname{Tr}(\Lambda(\Lambda^* - A_\varepsilon)D) = \sum_i \langle \Lambda(\Lambda^* - A_\varepsilon)x_i, v_i\rangle$. So

$$\begin{aligned} |\operatorname{Tr}(\Lambda\Lambda^*D)| &\le \sum_i |\langle \Lambda(\Lambda^* - A_\varepsilon)x_i, v_i\rangle| = \sum_i |\langle (\Lambda^* - A_\varepsilon)x_i, \Lambda^*v_i\rangle| \\ &\le \sum_i \|(\Lambda^* - A_\varepsilon)x_i\| \cdot \|\Lambda^*v_i\|. \end{aligned}$$

Observe that if $x, y \ge 0$ then $xy \le \varepsilon^{1/2}x^2 + \varepsilon^{-1/2}y^2$. Thus

$$\begin{aligned} |\operatorname{Tr}(\Lambda\Lambda^*D)| &\le \varepsilon^{1/2}\sum_j \|\Lambda^*v_j\|^2 + \varepsilon^{-1/2}\sum_j \|(\Lambda^* - A_\varepsilon)x_j\|^2 \\ &< \varepsilon^{1/2}\|\Lambda\|^2_{\mathrm{HS}} + \varepsilon^{1/2}. \end{aligned}$$

Hence $\operatorname{Tr}(\Lambda\Lambda^*D) = 0$. Now $D = EE^*$, with $Ev_n = \lambda^{(n-1)/2}v_n$. Thus $\operatorname{Tr}((E^*\Lambda)(E^*\Lambda)^*) = 0$. So $E^*\Lambda = 0$. This implies that $\lambda^{(i-1)/2}\langle u_j, v_i\rangle = 0$ for all i, j. So $u = 0$. ∎

Remark. The preceding argument actually shows that $\omega = a_1v_1 \oplus a_2v_2 \oplus \ldots \oplus a_nv_n \oplus \ldots$ is cyclic whenever $\{a_n\} \in l^2$ and $a_n \ne 0$ for all n.

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FACULTAD DE CIENCIAS
UNIVERSIDAD CENTRAL DE VENEZUELA
CARACAS 1040, VENEZUELA

DEPARTMENT OF MATHEMATICS
HOWARD UNIVERSITY
WASHINGTON, D.C. 20059, U.S.A.

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