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GENERALIZED CONVOLUTIONS AND DELPHIC SEMIGROUPS

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I. Introduction. Let $\mathcal{N}$ be the set of all non-negative integers and let $\mathfrak{B}$ be the family of all non-negative measures on $\mathcal{N}$ whose total mass is less than or equal to 1. We say that a sequence of measures $\{P_n\}$ *converges to a measure* P if and only if the relation

$$\lim_n P_n(k) = P(k)$$

holds for all $k \in \mathcal{N}$. It is very easy to verify that under this convergence $\mathfrak{B}$ is a compact metrizable space.

In the sequel, we denote by E_k the measure concentrated at a point $k \in \mathcal{N}$, and by 0 the measure vanishing on $\mathcal{N}$. Further, by $\mathfrak{P}$ we denote the subset of $\mathfrak{B}$ consisting of all probability measures. Moreover, let $\mathfrak{B}_0 = \mathfrak{B} \setminus \{0\}$.

A *generalized convolution* $*$ is a commutative and associative $\mathfrak{B}$-valued binary operation defined on $\mathfrak{B}$ and satisfying the following conditions:

(i) $E_0 * P = P$ for all $P \in \mathfrak{B}$;

(ii) $(aP+bQ)*R = a(P*R)+b(Q*R)$ for all $P, Q, R \in \mathfrak{B}$ and for $a, b \geqslant 0$, $a+b \leqslant 1$;

(iii) if $P_n \to P$, then $P_n * Q \to P*Q$ for all $Q \in \mathfrak{B}$;

(iv) $P*Q \in \mathfrak{P}$ provided $P \in \mathfrak{P}$ and $Q \in \mathfrak{P}$.

Let D be an open unit disk in the complex plane. Generalized convolution $*$ is said *to admit a generating function* if there exists an injective map Φ: $P \mapsto \Phi_P$ from $\mathfrak{B}$ to the space of analytic functions on D such that:

(a) $\Phi_{aP+bQ} = a\Phi_P + b\Phi_Q$ for all $P, Q \in \mathfrak{B}$ and $a, b \geqslant 0$, $a+b \leqslant 1$;

(b) $\Phi_{P*Q} = \Phi_P \Phi_Q$ for all $P, Q \in \mathfrak{B}$;

(c) $\lim_n \Phi_{P_n} = \Phi_P$ uniformly on every compact subset of D if and only if $P_n \to P$.

The function Φ_P is called a *generating function* of P. Generalized convolutions admitting generating function were considered in [3].

Let us quote some simple examples of generalized convolutions.

It is clear that a generalized convolution is determined uniquely by its values on the measures $E_n * E_m$ $(n, m = 0, 1, 2, \ldots)$.

Further on,

$$E_n * E_m = E_{n+m} \quad \text{(ordinary convolution)},$$

$$E_n * E_m = E_{nm+n+m},$$

$$E_n * E_m = E_{\max(n,m)},$$

$$E_n * E_m = \frac{\cosh a(n+m)}{2\cosh(an)\cosh(am)} E_{n+m} + \frac{\cosh a(n-m)}{2\cosh(an)\cosh(am)} E_{|n-m|},$$

where a is a non-negative constant.

Another example of a generalized convolution can be obtained in the following way:

Let $S = (\mathcal{N}, \circ)$ be a commutative semigroup of non-negative integers with 0 as a neutral element. Moreover, we assume that for every pair $i, k \in \mathcal{N}$ the set $\{j: i \circ j = k\}$ is finite. Then the formula $E_n * E_m = E_{n \circ m}$ defines a generalized convolution which is called an *S-convolution*. It was proved in [3] (theorem 3.1) that an S-convolution admits a generating function if and only if the semigroup S is isomorphic with a denumerable, discrete subsemigroup containing 0 of the additive semigroup R^+.

In [4] Kendall introduced the concept of *delphic semigroups*, i.e., commutative, topological semigroups for which, roughly speaking, the central limit theorem for triangular arrays holds.

The aim of this note is to prove that each generalized convolution admitting generating functions defines a delphic semigroup.

Now we give a precise definition of a delphic semigroup.

Let G be a commutative topological (with Hausdorff topology) semigroup having a unique neutral element e. The semigroup operation will be denoted by uv $(u, v \in G)$.

By a *triangular array* we understand a system $\{u(i, j): j = 1, 2, \ldots, i;$ $i = 1, 2, \ldots\}$ of elements of G and we call $u(i) = u(i, 1)u(i, 2) \ldots u(i, i)$ the *i-th marginal product* of the array.

We say that the array is *convergent* if its marginal products $u(1)$, $u(2), \ldots$ converge to some element of G. An element u of G is called *infinitely divisible* if it has a k-th rooth in G for every $k = 2, 3, \ldots$

We say that such a G is a *delphic semigroup* if:

(A) There exists a continuous homomorphism Δ from G into the additive semigroup of non-negative real numbers such that $\Delta(u) = 0$ if and only if u is the neutral element e of G.

(B) For each u in G the factors of u form a compact set.

(C) If a convergent triangular array is null, i.e.,

$$\limsup_{i}\sup_{1\leqslant j\leqslant i} \Delta[u(i,j)] = 0,$$

then its limit is infinitely divisible.

It was proved in [4] that delphic semigroups have all general properties discovered by A. Ya. Khintchine in his study of the arithmetic of probability measures on R. In particular, the elements u of G can be classified as follows:

(I) u is indecomposable,

(II) u is decomposable and has an indecomposable factor,

(III) u is infinitely divisible and has no indecomposable factors.

The elements of the class (III) form a subset of G which is usually denoted by I_0.

In the next section we shall prove the main result of this note. The last section contains an analogue of the famous Raikov's theorem for S-convolutions.

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II. Generalized convolutions and delphic semigroups. We have the following

THEOREM 1. *Suppose that a generalized convolution $*$ admits a generating function. Then $(\mathfrak{P}_0, *)$ with the topology induced from $\mathfrak{P}$ is a delphic semigroup.*

Before proving the theorem we shall show some simple lemmas.

LEMMA 1. *Let*

$$\Delta(P) = -\int_0^{2\pi} \operatorname{Log}|\Phi_P(re^{i\vartheta})|\,d\vartheta,$$

where $P\in\mathfrak{P}_0$ and r is a given constant, $0<r<1$. Then Δ is a sequentially continuous homomorphism of $(\mathfrak{P}_0, *)$ to the additive semigroup of non-negative real numbers.

Proof. First we have $\Phi_0 \equiv 0$. Let $P\in\mathfrak{P}_0$, so $P\neq 0$. In view of the analicity of generating functions and of the injectivity of the map Φ we can write Φ_P in the form $\Phi_P(z) = z^k\varphi(z)$, where φ is an analytic function in D and $\varphi(0)\neq 0$. Moreover, we have

$$\operatorname{Log}|\Phi_P(re^{i\vartheta})| = \log(r^k)+\operatorname{Log}|\varphi(re^{i\vartheta})|.$$

Both parts of the last sum are summable on the circle $K(0,r) = \{z: |z| = r\}$, the second by virtue of the Jensen's formula (see [7]), whence we infer that $\Delta(P)<\infty$ for all $P\in\mathfrak{P}_0$. Further, $\Delta(P)\geqslant 0$, because $|\Phi_P(z)|\leqslant 1$ for $z\in D$ (see [3], formula (2.1)). Additivity and sequential continuity of Δ are simple consequences of conditions (b) and (c) from the introduction

LEMMA 2. *$\Delta(P) = 0$ if and only if $P = E_0$.*

Proof. Suppose that $\Delta(P) = 0$. From the fact that $|\Phi_P(z)| \leqslant 1$ for all $z \in D$ and the definition of the homomorphism Δ we infer that $|\Phi_P(z)| = 1$ for all $z \in K(0, r)$, because the integrand is continuous.

By virtue of the maximum principle, it follows that $|\Phi_P(z)| \equiv 1$ for $z \in D$, because $|\Phi_P(z)|$ attains its maximum inside an analicity area of Φ_P. On the other hand, $\Phi_{E_0} \equiv 1$ and thus theorem 2.1 (see [3]) implies $P = E_0$.

The converse implication is obvious.

Lemmas 1 and 2 assure that the Kendall's axiom (A) holds for the semigroup $(\mathfrak{B}_0, *)$.

Now we show that $(\mathfrak{B}_0, *)$ is a sequentially delphic semigroup in the sense of Davidson [1]. It is rather simple (see also [1]) that a sequentially delphic semigroup is a delphic semigroup if its topology is metrizable. This will give us the assertion, since the topology of a weak convergence of a measures is metrizable (for instance, by Levy's metric).

In order to prove that $(\mathfrak{B}_0, *)$ is a sequentially delphic semigroup we shall make the use of theorem 3 (see [2]), namely we shall show that $(\mathfrak{B}_0, *)$ is an L-semigroup satisfying some additional conditions. In fact, we shall prove that almost all axioms of a sequentially delphic semigroup are satisfied by $(\mathfrak{B}_0, *)$ and the additional conditions (lemmas 4 and 6) assure that the central limit theorem holds.

Let $\mathfrak{M}$ be the family of all convergent sequences $\{P_n\}$ of measures from $\mathfrak{B}_0$ whose limits P also belong to $\mathfrak{B}_0$; so $P \neq 0$. For $\{P_n\} \in \mathfrak{M}$, $P_n \to P$, let us define the mapping $\mathfrak{L}\colon \mathfrak{M} \to \mathfrak{B}_0$ by the formula $\mathfrak{L}(\{P_n\}) = P$.

LEMMA 3. *Semigroup $(\mathfrak{B}_0, *, \mathfrak{M}, \mathfrak{L})$ is an L-semigroup* (see [2]).

Proof. If we put $M = \mathfrak{M}$ and $L = \mathfrak{L}$ in the Davidson's definition of L-semigroup, his axioms read:

(i) If $\{P_n\} \in \mathfrak{M}$ and $\{n'\} \subset \{n\}$, then $\{P_{n'}\} \in \mathfrak{M}$ and $\mathfrak{L}(\{P_{n'}\}) = \mathfrak{L}(\{P_n\})$.

(ii) If $\{P_n\}$ is such that there exists $P \in \mathfrak{B}_0$ with the property that for each $\{n'\} \subset \{n\}$ there is $\{n''\} \subset \{n'\}$ with $\{P_{n''}\} \in \mathfrak{M}$ and $\mathfrak{L}(\{P_{n''}\}) = P$, then $\{P_n\} \in \mathfrak{M}$ and $\mathfrak{L}(\{P_n\}) = P$.

(iii) If $\{P_n\} \in \mathfrak{M}$ and $\{Q_n\} \in \mathfrak{M}$, then $\{P_n * Q_n\} \in \mathfrak{M}$ and $\mathfrak{L}(\{P_n * Q_n\}) = \mathfrak{L}(\{P_n\}) * \mathfrak{L}(\{Q_n\})$.

(iv) If $P_n = P$ for each $n = 1, 2, \ldots$, then $\{P_n\} \in \mathfrak{M}$ and $\mathfrak{L}(\{P_n\}) = P$.

These axioms are satisfied in the obvious way. In particular, condition (ii) is a simple consequence of a metrizability of a weak convergence of measures.

For $P \in \mathfrak{B}$ let $F(P)$ be the set of factors of P in $\mathfrak{B}$, i.e., the set $F(P) = \{Q \in \mathfrak{B}\colon$ there exists $R \in \mathfrak{B}$ such that $Q * R = P\}$.

LEMMA 4. *If $\{P_n\} \in \mathfrak{M}$ and $Q_n \in F(P_n)$ for $n = 1, 2, \ldots$, then there is a subsequence $\{n'\} \subset \{n\}$ such that $\{Q_{n'}\} \in \mathfrak{M}$ and $\mathfrak{L}(\{Q_{n'}\}) \in F(\mathfrak{L}\{P_n\})$.*

Proof. Let $P_n \to P$, $P \in \mathfrak{B}_0$ and $P_n = Q_n * R_n$. By the compactness of the semigroup $\mathfrak{B}$, there exists a subsequence $\{n'\} \subset \{n\}$ such that a sequence $\{Q_{n'}\}$ is convergent with $Q_{n'} \to Q$. Likewise we can choose a convergent subsequence $\{R_{n''}\}$ of the sequence $\{R_{n'}\}$ with $R_{n''} \to R$. Then $P_{n''} = Q_{n''} * R_{n''} \to Q * R$, whence $P = Q * R$. Because $P \in \mathfrak{B}_0$, we have $P \neq 0$, whence $Q \neq 0$, $R \neq 0$. Therefore we have got a subsequence $\{Q_{n'}\} \subset \{Q_n\}$ convergent to $Q \neq 0$, which implies that

$$\{Q_{n'}\} \in \mathfrak{M} \quad \text{and} \quad \mathfrak{L}(\{Q_{n'}\}) = Q \in F(P) = F[\mathfrak{L}(\{P_n\})].$$

LEMMA 5. *If $\Delta(P_n) \to 0$, then $P_n \to E_0$.*

Proof. Suppose P_n does not converge to E_0. By the compactness of $\mathfrak{B}$, there exists a convergent subsequence $\{P_{n_k}\}$ of $\{P_n\}$ with $P_{n_k} \underset{k}{\to} P \neq E_0$. Then

$$\Delta(P_{n_k}) \underset{k}{\to} \Delta(P) = 0,$$

by assumption, and hence lemma 2 implies $P = E_0$, which yields a contradiction.

Let $K(z_0, r) = \{z\colon |z - z_0| = r\}$. Let us consider the family of circles $\{K(z, r) \subset D\colon z = a + bi,\ a, b, r$ — rationals$\}$ and arrange it in a single sequence $\{K_j(z_j, r_j)\}$.

Then put

$$D_j(P) = -\int_{K_j(z_j, r_j)} \operatorname{Log}|\Phi_P(z)|\,dz = -\int_0^{2\pi} \operatorname{Log}|\Phi_P(r_j e^{i\vartheta} + z_j)|\,d\vartheta \qquad (j = 1, 2, \ldots),$$

where $P \in \mathfrak{B}_0$.

LEMMA 6. *For each $j = 1, 2, \ldots$ the mapping $D_j\colon (\mathfrak{B}_0, *) \to (R^+, +)$ is a sequentially continuous homomorphism. Moreover, the sequence $\{\Delta, D_1, D_2, \ldots\}$ separates points of the semigroup $\mathfrak{B}_0$ and, for each $\varepsilon > 0$ and $j \geqslant 1$, there is $\delta_j > 0$ such that, for each $P \in \mathfrak{B}_0$, if $\Delta(P) \leqslant \delta_j$, then $D_j(P) \leqslant \varepsilon$.*

Proof. The proof of the fact that each D_j is a well defined, sequentially continuous and additive homomorphism is analogous to that of lemma 1.

Suppose now that the last assertion does not hold. Then there are $\varepsilon_0 > 0$ and j_0 such that for each n there exists $P_n \in \mathfrak{B}_0$ with $\Delta(P_n) \leqslant 1/n$ and $D_{j_0}(P_n) \geqslant \varepsilon_0$. Since $\Delta(P_n) \to 0$, we have, by lemma 5, $P_n \to E_0$, and it follows from the sequential continuity of D_{j_0} that $D_{j_0}(P_n) \underset{n}{\to} 0$ which yields a contradiction with $D_{j_0}(P_n) \geqslant \varepsilon > 0$.

Let us now suppose that $D_j(P) = D_j(Q)$ for $j = 1, 2, \ldots$ and $P, Q \in \mathfrak{B}_0$.

For $0 < r < 1$ let us denote by $D^{(r)}$ the circle $D^{(r)} = \{z\colon |z| \leqslant r\}$. Given a measure R from $\mathfrak{B}_0$, it is clear that its generating function $\Phi_R(z)$ vanishes only in a finite number of points $z \in D^{(r)}$. Therefore we can choose from $\{K_j\}$ infinitely many circles, whose centers are dense in $D^{(r)}$ and in whose interiors the function $\mathrm{Log}|\Phi_R(z)|$ is harmonic.

Then

$$D_j(R) = -\int\limits_{K_j(z_j,r_j)} \mathrm{Log}|\Phi_R(z)|\,dz = -\mathrm{Log}|\Phi_R(z_j)|.$$

Since $P, Q \neq 0$ $(P, Q \in \mathfrak{B}_0)$, we have $|\Phi_P(z)| = |\Phi_Q(z)|$ for a dense subset of $D^{(r)}$, i.e., for infinitely many centers of the circles $\{K_j\}$ contained in $D^{(r)}$, and this implies, by the continuity of generating functions, that $|\Phi_P(z)| = |\Phi_Q(z)|$ for all $z \in D^{(r)}$. Taking $r \to 1$, we obtain the last equality for all $z \in D$. Hence, from theorem 2.1 (see [3]), we infer that $P = Q$, which completes the proof of the lemma.

By virtue of theorem 3 (see [2]), it follows that in the semigroup $\mathfrak{B}_0$ the central limit theorem holds, so $\mathfrak{B}_0$ satisfies the Kendall's condition (C). Axiom (B) is a simple consequence of lemma 4 if we take $P_n = P$ for each n. The theorem is thus proved.

Let us note that the assumption of admitting a generating function by a generalized convolution $*$ is essential. In fact, consider the example of a convolution given by the formula $E_n * E_m = E_{\max(n,m)}$. This convolution does not admit a generating function, because $E_1 * E_1 = E_1$ and, consequently, either $\Phi_{E_1} \equiv 1$ or $\Phi_{E_1} \equiv 0$, a contradiction with the injectivity of the map Φ. Moreover, if Δ would be a homomorphism of a delphic semigroup, then $\Delta(E_1 * E_1) = \Delta(E_1)$ and hence $\Delta(E_1) = 0$, which is impossible by axiom (A).

III. Some examples of the elements from the class I_0 for S-convolution. Let $*$ be an S-convolution admitting generating function and defined by the formula

$$E_n * E_m = E_{n \circ m} \quad \text{for } n, m \in \mathcal{N},$$

and let h be the corresponding to this S-convolution isomorphism of the semigroup $S = (\mathcal{N}, \circ)$ into a subsemigroup containing 0 of the additive semigroup R^+.

For the given convolution $*$ let us denote by $P(\lambda, E)$ a $*$-Poisson measure, i.e., the measure

$$P(\lambda, E) = e^{-\lambda} E_0 + \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} E^{*n},$$

where $E \in \mathfrak{P}$ and $\lambda > 0$. The power E^{*n} is taken in the sense of the convolution $*$.

We have the following

THEOREM 2. *Let $*$ be an S-convolution admitting generating function and let $k \in \mathcal{N}$. If $P(\lambda, E_k) = Q*R$ with $Q, R \in \mathfrak{B}$, then Q and R are $*$-Poisson measures and $Q = P(\alpha, E_k)$, $R = P(\beta, E_k)$, where $\alpha, \beta \geqslant 0$ and $\alpha+\beta = \lambda$.*

The proof of the theorem needs some simple lemmas.

By $\operatorname{supp} S$ we denote the support of a measure $S \in \mathfrak{B}$, i.e., the set

$$\operatorname{supp} S = \{n \in \mathcal{N}\colon S(n) > 0\}.$$

LEMMA 7. *If $P(\lambda, E_k) = Q*R$, then* $\operatorname{supp} Q \subset \operatorname{supp} P(\lambda, E_k)$ *and, analogously,* $\operatorname{supp} R \subset \operatorname{supp} P(\lambda, E_k)$.

Proof. $\operatorname{supp} P(\lambda, E_k) = \{0, k, k^{\circ 2}, k^{\circ 3}, \ldots\}$, because

$$(E_k)^{*n} = E_k * E_k * \ldots * E_k = E_{k\circ k\circ \ldots \circ k} = E_{k^{\circ n}}.$$

If $k = 0$, then we have $\operatorname{supp} P(\lambda, E_0) = \{0\}$, because $P(\lambda, E_0) = E_0$. Suppose that $E_0 = Q*R$, where

$$Q = \sum_{m=0}^{\infty} Q(m) E_m \quad \text{and} \quad R = \sum_{n=0}^{\infty} R(n) E_n.$$

Obviously, Q and R are not measures equal to 0. Therefore there exist non-negative integers m' and n' for which $Q(m') > 0$ and $R(n') > 0$.

By the formula

$$Q*R = \sum_m \sum_n Q(m) R(n) E_{m\circ n},$$

we have $(Q*R)(m'\circ n') \geqslant Q(m')R(n') > 0$ and, consequently, $m'\circ n' \in \operatorname{supp}(Q*R) = \{0\}$.

If, for example, $m' \neq 0$, then $h(m'\circ n') = h(m')+h(n') > 0$ which implies $m'\circ n' \neq 0$, since $h(r) = 0$ if and only if $r = 0$. This yields a contradiction. Therefore $Q(m) = 0$ for $m > 0$ and $R(n) = 0$ for $n > 0$, so we get $Q = R = E_0$. Note that the just proved part of lemma 7 (for the case $k = 0$) completes the proof of theorem 2 for $k = 0$. So it remains only to show the validity of theorem 2 for $k \neq 0$.

Now we return to the proof of lemma 7 in the case $k \neq 0$. If $k \neq 0$, then $h(k) > 0$. By the formula $h(k^{\circ r}) = rh(k) > 0$, we infer that $(E_k)^{*m} \neq (E_k)^{*n}$ for $m \neq n$, $m, n \in \mathcal{N}$ ($E_k^0 = E_0$ by definition).

Thus we have $P(\lambda, E_k)(0) = e^{-\lambda} = (Q*R)(0) = Q(0)R(0) > 0$, and hence $0 \in (\operatorname{supp} Q) \cap (\operatorname{supp} R)$.

If it would be $m \notin \operatorname{supp} P(\lambda, E_k)$ and, for example, $m \in \operatorname{supp} Q$, i.e., $Q(m) > 0$, then we would have

$$(Q*R)(m) \geqslant Q(m)R(0)E_{m\circ 0}(m) = Q(m)R(0) > 0,$$

since 0 is the neutral element of the semigroup S. But this means that $m \in \operatorname{supp}(Q*R) = \operatorname{supp} P(\lambda, E_k)$; a contradiction to the assumption. This completes the proof of the lemma.

LEMMA 8. *If for a measure $P \in \mathfrak{P}$ there is a constant c, $0 \leqslant c \leqslant 1$, such that $|\Phi_P(z)| \equiv c$ for $z \in D$, then $P = E_0$.*

Proof. It follows from the properties of the generating functions that

$$|\Phi_{cE_0}(z)| = c|\Phi_{E_0}(z)| = c = |\Phi_P(z)| \quad \text{for } z \in D,$$

which gives us, by theorem 2.1 (see [3]), $P = cE_0$. Since both measures P and E_0 are probability measures on $\mathscr{N}$, we have $1 = P(\mathscr{N}) = cE_0(\mathscr{N}) = c$, hence $P = E_0$.

COROLLARY. *If $P \in \mathfrak{B}_0$ and $cP \neq E_0$, where $c = 1/P(\mathscr{N})$, then $\Phi_P(D)$ and $|\Phi_P(D)| = \{|\Phi_P(z)|: z \in D\}$ are sets of the power of continuum.*

In fact, both sets are connected and each contains at least two points.

Proof of theorem 2 for the case $k \neq 0$. Suppose that $P(\lambda, E_k) = Q * R$. Then $\Phi_{P(\lambda,E_k)}(z) = \Phi_Q(z)\Phi_R(z)$ for $z \in D$. But

$$\Phi_{P(\lambda,E_k)}(z) = \exp\{\lambda[\Phi_{E_k}(z)-1]\}$$

and, by lemma 7,

$$\Phi_Q(z) = \Phi_{\sum\limits_{m=0}^{\infty} Q(m)E_k^{\circ m}}(z) = \sum_{m=0}^{\infty} Q(m)\Phi_{E_k^{*m}}(z) = \sum_{m=0}^{\infty} Q(m)[\Phi_{E_k}(z)]^m$$

and, likewise,

$$\Phi_R(z) = \sum_{n=0}^{\infty} R(n)[\Phi_{E_k}(z)]^n.$$

Let us introduce the notation

$$Q(m) = q_m, \quad R(n) = r_n, \quad q(u) = \sum_{m=0}^{\infty} q_m u^m,$$

$$r(u) = \sum_{n=0}^{\infty} r_n u^n, \quad p(u) = e^{\lambda[u-1]}.$$

Then

$$\Phi_{P(\lambda,E_k)}(z) = \exp\{\lambda[\Phi_{E_k}(z)-1]\} = q[\Phi_{E_k}(z)]\, r[\Phi_{E_k}(z)] \quad \text{for } z \in D,$$

i.e., $p(u) = q(u)r(u)$ for $u \in \Phi_{E_k}(D) \subset \bar{D}$.

Both functions $q(u)$ and $r(u)$ are analytic for $|u| < 1$, because

$$\sum_{m=0}^{\infty} q_m \leqslant 1 \quad \text{and} \quad \sum_{n=0}^{\infty} r_n \leqslant 1$$

with $q_m \geqslant 0$ and $r_n \geqslant 0$. So the functions $p(u)$ and $q(u)r(u)$ are both analytic on D and equal on the set $D \cap \Phi_{E_k}(D)$. Since $E_k \neq E_0$ ($k \neq 0$), we infer, from the corollary of lemma 8, that this set is of the power of continuum, and hence, by analicity, the functions $p(u)$ and $q(u)r(u)$ are equal on D.

Further, analogously as in the well-known Raikov's theorem about the Poisson-law (see [5]), it may be concluded that both functions $q(u)$ and $r(u)$ are entire functions without zeros. Their order cannot exceed the order of $p(u)$, i.e., the order one. By virtue of Hadamard's factorization theorem (see [6]), we infer that $q(u)$, as well as $r(u)$, has exactly order one. Since $q(1) = r(1) = 1$, we see that $q(u) = e^{\alpha(u-1)}$ and $r(u) = e^{\beta(u-1)}$.

The coefficients of $q(u)$ are equal to q_m, i.e., $q_m = e^{-\alpha}\cdot \alpha^m/m!$, and hence

$$0 \leqslant \sum_{m=0}^{\infty} m\cdot q_m = \alpha.$$

A similar argument applies to $r(u)$ and it can be seen that $\alpha \geqslant 0$, $\beta \geqslant 0$ and $\alpha+\beta = \lambda$.

Therefore

$$\Phi_Q(z) = q[\Phi_{E_k}(z)] = \exp\{\alpha[\Phi_{E_k}(z)-1]\}$$

and, likewise,

$$\Phi_R(z) = \exp\{\beta[\Phi_{E_k}(z)-1]\}.$$

Hence, finally, $Q = P(\alpha, E_k)$, $R = P(\beta, E_k)$, $\alpha, \beta \geqslant 0$ and $\alpha+\beta = \lambda$, which completes the proof of theorem 2.

PROBLEM. Give a description of all probability measures Q for which $P(\lambda, Q)\in I_0$, λ is a positive constant. (**P 799**)

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