COLLOQUIUM MATHEMATICUM

VOL. XXII 1971 FASC. 2

# ON POINTS OF ABSOLUTE CONTINUITY, III

BY

M. C. CHAKRABARTY (KALYANI, WEST BENGAL)
AND J. S. LIPIŃSKI (GDAŃSK)

Let the function $f$ be defined and continuous in the closed interval $[a, b]$. In [1] the following definition has been introduced:

Definition. A point $x \in [a, b]$ is said to be a *point of absolute continuity* of $f$ if there is a closed neighbourhood of $x$ in which $f$ is absolutely continuous.

If there is no neighbourhood of $x$ in which $f$ is absolutely continuous, then $x$ is said to be a *point of non-absolute continuity* of $f$.

A point of absolute continuity and a point of non-absolute continuity will be shortly written as ac-point and nac-point, respectively. Let $G$ and $N$ denote, respectively, the set of ac-points and nac-points of $f$ in $[a, b]$. It is easy to see that the set $G$ is open and, consequently, $N$ is closed. Of course, $a \in N$ and $b \in N$. In [1] and [2] we have proved some results on ac-points and nac-points.

Now let us take $G$ to be any open set in $[a, b]$ and $N$ the complement of $G$ with respect to $[a, b]$. In this paper we ask, whether it is possible to construct a continuous function $f$ on $[a, b]$ such that its sets of ac-points and nac-points are precisely the sets $G$ and $N$, respectively. Here we answer the query in the affirmative sense.

THEOREM 1. *For each closed set $N \subset [a, b]$ there exists a function $f$ such that each point of $N \cup \{a\} \cup \{b\}$ is a nac-point of $f$ and each point of $(a, b) \setminus N$ is an ac-point of $f$.*

Proof. Let $(a_i, b_i)$ $(i = 1, 2, \ldots)$ be the components of the set $(a, b) \setminus N = G$. Then $G = \bigcup_i (a_i, b_i)$. Let $(\alpha_i, \beta_i)$ $(i = 1, 2, \ldots)$ be the components of interior of $N$. Put $\operatorname{Int}(N) = N_0$. Let $\mu$ be any positive number with $0 < \mu < 1$ and let $F$ be any continuous non-differentiable function on $[a, b]$. We define two sequences of functions $\{\varphi_n\}$ and $\{\psi_n\}$ as follows:

From the definition of $f$ we infer that $f(\beta)-f(\alpha)>0$. For each $\delta>0$ there exists an open set

$$G=\bigcup_{i=1}^{\infty}(\alpha_i,\beta_i)\subset(\alpha,\beta)$$

such that $(\alpha,\beta)\cap\bigcup_n E_n\subset G$ and $\sum_i(\beta_i-\alpha_i)<\delta$. Of course,

$$\begin{aligned}\sum_{i=1}^{\infty}[f(\beta_i)-f(\alpha_i)] &= \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}[\varphi_n(\beta_i)-\varphi_n(\alpha_i)]\\ &= \sum_{n=1}^{\infty}[\varphi_n(\beta)-\varphi_n(\alpha)]\\ &= f(\beta)-f(\alpha).\end{aligned}$$

Then $f$ cannot be absolutely continuous on an interval containing points of the set $N$. Hence each point of $N$ is a nac-point of $f$. The function $f$ is constant on each component of $(a,b)\setminus N$. Hence each point of $(a,b)\setminus N$ is an ac-point of $f$.

In [1] it is proved that if $f$ is differentiable, then the set of nac-points of $f$ is non-dense. Here we prove the following theorem:

THEOREM 3. *For each non-dense closed set $N\subset[a,b]$ there exists a differentiable function $f$ such that the set of nac-points of $f$ is equal to $N\cup\{a\}\cup\{b\}$.*

Proof. Let $(a_n,b_n)$, $n=1,2,\ldots$, be components of $(a,b)\setminus N$. Let $g(x)=0$ for $x\in N\cup\{a\}\cup\{b\}$ and let

$$g(x)=(b_n-a_n)\sin[2\pi(b_n-a_n)^{-1}(x-a_n)]\quad \text{for } x\in(a_n,b_n).$$

Then $g$ is a continuous function.

Let

$$F(x)=\int_a^x g(t)\,dt.$$

Then $F$ is differentiable and $F(x)=F'(x)=0$ for $x\in N\cup\{a\}\cup\{b\}$. Let $\varphi_n(x)$ be a differentiable function defined on $(a_n,b_n)$ and such that, for each $\varepsilon>0$, $\varphi_n$ is of unbounded variation in $(a_n,a_n+\varepsilon)$ and in $(b_n-\varepsilon,b_n)$, and $\varphi_n'$ is bounded in $(a_n+\varepsilon,b_n-\varepsilon)$, $|\varphi_n(x)|\leqslant F(x)$ in the whole interval $(a_n,b_n)$.

Let

$$f(x)=\begin{cases}F(x) & \text{for } x\in N\cup\{a\}\cup\{b\},\\ \varphi_n(x) & \text{for } x\in(a_n,b_n).\end{cases}$$

It is easy to see that $f$ is differentiable.

$$\varphi_n(x) = \begin{cases} \mu^n(x-a_n)\sin\dfrac{\pi}{x-a_n} & \text{in } a_n < x \leqslant \tfrac{1}{2}(a_n+b_n), \\ \mu^n(b_n-x)\sin\dfrac{\pi}{b_n-x} & \text{in } \tfrac{1}{2}(a_n+b_n) < x < b_n, \\ 0 & \text{elsewhere;} \end{cases}$$

$$\psi_n(x) = \begin{cases} \mu^n\left[F(\beta_n)-F(x)-\dfrac{\beta_n-x}{\beta_n-\alpha_n}\{F(\beta_n)-F(\alpha_n)\}\right] & \text{in } (\alpha_n, \beta_n), \\ 0 & \text{elsewhere.} \end{cases}$$

Let $\varphi = \sum\varphi_n$ and $\psi = \sum\psi_n$. Since $\varphi_n$ and $\psi_n$ are continuous and each series converges uniformly in any interval, the functions $\varphi$ and $\psi$ are continuous and so is the function $f = \varphi+\psi$.

We show that every point of $G$ is an ac-point and every point of $N \cup \{a\} \cup \{b\}$ is a nac-point of $f$. Functions $\varphi_n(x)$ and $f(x)$ are differentiable on the set $G\setminus \bigcup\limits_n \{2^{-1}(a_n+b_n)\}$. Each Dini derivative of $f$ is bounded on each closed interval contained in $(a_n, b_n)$. Then the function $f$ is absolutely continuous on each closed interval contained in $G$. Hence each point of the open set $G$ is an ac-point of $f$. The function $\varphi_n(x)$ is of unbounded variation on each interval containing the point $a_n$ or $b_n$. The function $f$ has also the same property. Hence each point belonging to $\mathrm{Fr}(N \cup \{a\} \cup \{b\})$ is a nac-point of $f$.

If $x$ is an ac-point of $f$, then there exists a neighbourhood of $x$ in which the function $f$ is differentiable a.e. From the definition of $\psi_n(x)$ and $f(x)$ we infer that in each interval containing interior points of $N$ the function $f$ is not differentiable a.e. Hence interior points of $N$ are nac-points of $f$.

In [1] it is proved that if $f$ is a continuous function of bounded variation, then the set of nac-points of $f$ is perfect. We shall prove that this property is a characterization of the set of nac-points of $f$, where $f$ is a continuous function of bounded variation.

THEOREM 2. *For each perfect set $N \subset [a, b]$ there exists non-decreasing continuous function $f$ (so $f$ is of bounded variation) such that the set of nac-points of $f$ is equal to $N \cup \{a\} \cup \{b\}$.*

Proof. Let $\{E_n\}$ be a sequence of perfect sets of measure zero contained in $N$ and such that $\bigcup\limits_n E_n$ is dense in $N$. Let $\varphi_n(x)$ be a singular non-decreasing function transforming the set $E_n$ into the interval $[0, 2^{-n}]$, non-constant on each interval containing points of $E_n$. We shall prove that $f = \sum\varphi_n$ is a function possessing the required property.

If $(\alpha, \beta)$ is an interval such that $(\alpha, \beta)\cap N \neq \emptyset$, then $(\alpha, \beta)\cap\bigcup\limits_n E_n \neq \emptyset$.

If $(\alpha, \beta)$ is an interval such that

$$N \cup \{a\} \cup \{b\} \cap (\alpha, \beta) \neq \emptyset,$$

then there exists $a_n$ or $b_n$ belonging to $(\alpha, \beta)$. Hence $f$ is of unbounded variation on $(\alpha, \beta)$. Then each point of $N \cup \{a\} \cup \{b\}$ is a nac-point of $f$. The remaining points are ac-points of $f$.

Finally, the authors are grateful to Dr. P. C. Bhakta for suggesting the original problem from which the problem of this paper emerges.

*REFERENCES*

[1] M. C. Chakrabarty and P. C. Bhakta, *On points of absolute continuity*, Bulletin of the Calcutta Mathematical Society 59 (1967), p. 115–118.

[2] M. C. Chakrabarty, *On points of absolute continuity, II*, Revue Roumaine des Mathématiques Pures et Appliquées 13. 6 (1968), p. 771–777.

*Reçu par la Rédaction le 20. 2. 1969*