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ON POINTS OF ABSOLUTE CONTINUITY. FUNCTIONS OF SEVERAL VARIABLES

BY

WŁADYSŁAW WILCZYŃSKI (ŁÓDŹ)

Let us recall following definitions (cf. Saks [3], p. 169, and Radó [2], p. 540).

Definition 1. Given a function $F(x, y)$, continuous on an interval $I = [a, b; c, d]$, let us denote for every $x \in [a, b]$ by $W_1(F; x; c, d)$ the absolute variation of the function $F(x, y)$ with respect to the variable y on the interval $[c, d]$, and for every $y \in [c, d]$ by $W_2(F; y; a, b)$ that of the function $F(x, y)$ with respect to x on $[a, b]$. If

$$\int_a^b W_1(F; x; c, d)\,dx < +\infty \quad \text{and} \quad \int_c^d W_2(F; y; a, b)\,dy < +\infty,$$

then the function F is said to be of *bounded variation on I in the Tonelli sense* (BVT).

Definition 2. A continuous function $F(x, y)$ will be called *absolutely continuous on an interval $I = [a, b; c, d]$ in the Tonelli sense* (ACT), if it is BVT on I, absolutely continuous with respect to x for almost every value of $y \in [c, d]$ and absolutely continuous with respect to y for almost every value of $x \in [a, b]$.

Definition 3. A continuous function $F(x, y)$ will be called *absolutely continuous on an interval $I = [a, b; c, d]$ in the L. C. Young sense* (ACY), if it is BVT on I and if there exist two Borel sets B_1, B_2 such that $F(x, y)$ is absolutely continuous, as a function of x, on the intersection B_{1y} of B_1 with the horizontal line at altitude y for a.e. $y \in [c, d]$, $F(x, y)$ is absolutely continuous, as a function of y, on the intersection B_{2x} of B_2 with the vertical line corresponding to a given x for a.e. $x \in [a, b]$, and $B_1 \cup B_2 = I$.

Definition 4. A continuous function $F(x, y)$ is called *absolutely continuous on an interval $I = [a, b; c, d]$ in the sense of a rectangle function* (ACR), if the interval function Φ defined for the rectangle $I_0 = [a_0, b_0;$

$c_0, d_0] \subset I$ by the formula

$$\Phi(I_0) = F(b_0, d_0) - F(a_0, d_0) - F(b_0, c_0) + F(a_0, c_0)$$

is absolutely continuous.

After Chakrabarty and Lipiński [1] let us introduce the following definition:

Definition 5. A point (x, y) is said to be an *ACT-point* (*ACY-point*, *ACR-point*, respectively) of F if there exists a rectangular neighbourhood of (x, y) in the closure of which F is ACT (ACY, ACR, respectively).

It is easy to see that for every continuous function F defined on the whole plane the set G_T (G_Y, G_R) of all ACT-points (ACY-points, ACR-points, respectively) of F is open. We shall prove the converse (even in a stronger form).

THEOREM. *For each open set G in the plane there exists a continuous function F defined on the whole plane and such that $G = G_T = G_Y = G_R$.*

Proof. Let G be an open set on the plane and let B denote the complementary set of G, $B = G'$.

Let D_n be the grid consisting of all squares of the form $[k\cdot 2^{-n}, (k+1)\cdot 2^{-n}; m\cdot 2^{-n}, (m+1)\cdot 2^{-n}]$, where k, m are integers; let E_i^1 be squares of D_1 which lie in G; $E_1 = \bigcup_i E_i^1$, E_i^2 — squares of D_2 which lie in G but not in $\operatorname{int} E_1$, and so on if, of course, such squares do exist.

We have $G = \bigcup_{n=1}^{\infty} \bigcup_i E_i^n$, where $E_i^n \in D_n$ and $\operatorname{int} E_i^n \cap \operatorname{int} E_j^m = \emptyset$ for $i \neq j$ or $n \neq m$.

Similarly, $\operatorname{int} B = \bigcup_{n=1}^{\infty} \bigcup_i B_i^n$, where $B_i^n \in D_n$ and $\operatorname{int} B_i^n \cap \operatorname{int} B_j^m = \emptyset$ for $i \neq j$ or $n \neq m$.

One of the sets G and $\operatorname{int} B$, of course, can be empty.

Now we shall define some auxiliary functions.

Function $f_n(x)$ is defined on the interval $[0, 2^{-n}]$ as follows:

$$f_n(x) = \begin{cases} 0 & \text{for } x = 2i\cdot 2^{-4n-1},\ i = 0, 1, \ldots, 2^{3n}; \\ 2^{-n} & \text{for } x = (2i+1)\cdot 2^{-4n-1},\ i = 0, 1, \ldots, 2^{3n}-1; \\ \multicolumn{2}{l}{\text{is linear in the intervals } [i\cdot 2^{-4n-1}, (i+1)\cdot 2^{-4n-1}],} \\ & \qquad\qquad i = 0, 1, \ldots, 2^{3n+1}-1. \end{cases}$$

g_n is a continuous, non-differentiable function on the interval $[0, 2^{-n}]$ such that

$$g_n(0) = g_n(2^{-n}) = 0, \quad 0 < g_n(x) < 2^{-n} \text{ for } x \in (0, 2^{-n}).$$

Let us put

$$F(x,y)=\begin{cases} f_n(x-k\cdot 2^{-n})\cdot f_n(y-m\cdot 2^{-n}) \\ \qquad \text{if } (x,y)\in E_i^n=[k\cdot 2^{-n},(k+1)\cdot 2^{-n};\ m\cdot 2^{-n},(m+1)\cdot 2^{-n}]; \\ g_n(x-k\cdot 2^{-n})\cdot g_n(y-m\cdot 2^{-n}) \\ \qquad \text{if } (x,y)\in B_i^n=[k\cdot 2^{-n},(k+1)\cdot 2^{-n};\ m\cdot 2^{-n},(m+1)\cdot 2^{-n}]; \\ 0 \quad \text{if } (x,y)\in \operatorname{Fr}B. \end{cases}$$

$F(x,y)$ is a continuous function on the whole plane.

I. We shall prove that $G=G_T$ for F.

If $(x,y)\in G$, then (x,y) belongs simultaneously to at most four squares, say $E_{i_1}^{n_1}, E_{i_2}^{n_2}, E_{i_3}^{n_3}, E_{i_4}^{n_4}$. If the rectangular neighbourhood of (x,y) is sufficiently small to be included in the union of these squares, then $F(x,y)$ fulfills the Lipschitz condition in the closure of this neighbourhood, and so F is ACT there.

If $(x,y)\in\operatorname{int}B$, then in the closure of every rectangular neighbourhood of (x,y) the function F is not BVT, so (x,y) is not an ACT-point of F.

To prove that the point $(x,y)\in\operatorname{Fr}B$ is not an ACT-point of F let us consider the square

$$E_i^n=[k\cdot 2^{-n},(k+1)\cdot 2^{-n};\ m\cdot 2^{-n},(m+1)\cdot 2^{-n}].$$

From the definitions of $F(x,y)$ and $f_n(x)$ we have

$$\int_{k\cdot 2^{-n}}^{(k+1)\cdot 2^{-n}} W_1\big(F;x;m\cdot 2^{-n},(m+1)\cdot 2^{-n}\big)\,dx$$
$$=(2^{3n+1})^2\cdot\int_{k\cdot 2^{-n}}^{k\cdot 2^{-n}+2^{-4n-1}} W_1(F;x;m\cdot 2^{-n},m\cdot 2^{-n}+2^{-4n-1})\,dx.$$

For $x\in[k\cdot 2^{-n},k\cdot 2^{-n}+2^{-4n-1}]$ we have

$$F(x,y)=(x-k\cdot 2^{-n})\cdot 2^{3n+1}\cdot f_n(y-m\cdot 2^{-n}).$$

Hence

$$W_1(F;x;m\cdot 2^{-n},m\cdot 2^{-n}+2^{-4n-1})=(x-k\cdot 2^{-n})\cdot 2^{2n+1}$$

and

$$\int_{k\cdot 2^{-n}}^{k\cdot 2^{-n}+2^{-4n-1}} W_1(F;x;m\cdot 2^{-n},m\cdot 2^{-n}+2^{-4n-1})\,dx$$
$$=(2^{-4n-1})^2\cdot 2^{-1}2^{2n+1}=2^{-6n-2}.$$

Hence

$$\int_{k\cdot 2^{-n}}^{(k+1)\cdot 2^{-n}} W_1\big(F;x;m\cdot 2^{-n},(m+1)\cdot 2^{-n}\big)\,dx=1.$$

If $(x, y) \in \operatorname{Fr} B$, then every rectangular neighbourhood of (x, y) contains a sequence of squares $E_{i_j}^{n_j}$ such that $n_j \to \infty$ and $\varrho\big((x, y), E_{i_j}^{n_j}\big) \to 0$. Hence, in every rectangular neighbourhood of (x, y) the function F is not BVT and so (x, y) is not an ACT-point of F.

II. Next we shall prove that $G = G_Y$ for F.

If $(x, y) \in G$, then (x, y) is an ACT-point of F, so (x, y) is also an ACY-point of F. If $(x, y) \in B$, then in the closure of every rectangular neighbourhood of (x, y) the function F is not BVT, so (x, y) is not an ACY-point of F.

III. At last we shall prove that $G = G_R$ for F.

Let $(x, y) \in G$. In this case, similarly to I, we can find a rectangular neighbourhood of (x, y) which has points in common with at most four squares of type E_i^n. For the rectangle $P = [x_1, x_2; y_1, y_2]$ included in the common part of the closure of this neighbourhood and of such a square, say

$$E_{i_1}^{n_1} = [k \cdot 2^{-n_1}, (k+1) \cdot 2^{-n_1}; m \cdot 2^{-n_1}, (m+1) \cdot 2^{-n_1}],$$

we have

$$\Phi(P) = [f_{n_1}(x_2 - k \cdot 2^{-n_1}) - f_{n_1}(x_1 - k \cdot 2^{-n_1})] \cdot [f_{n_1}(y_2 - m \cdot 2^{-n_1}) - f_{n_1}(y_1 - m \cdot 2^{-n_1})].$$

If f_{n_1} fulfills the Lipschitz condition, then Φ is AC on this set. Hence (x, y) is an ACR-point of F.

Let $(x, y) \in \operatorname{int} B$. We shall prove that in every rectangular neighbourhood of (x, y) function Φ treated as a rectangle function is not BV. We know that (x, y) belongs simultaneously to at most four squares of type B_i^n. Let us consider common part of one of these squares, say

$$B_{i_1}^{n_1} = [k \cdot 2^{-n_1}, (k+1) \cdot 2^{-n_1}; m \cdot 2^{-n_1}, (m+1) \cdot 2^{-n_1}],$$

with the closure of an arbitrary rectangular neighbourhood of (x, y). Evidently, this is a rectangle, say $P = [a_0, b_0; c_0, d_0]$. Let us choose $y_1, y_2 \in [c_0, d_0]$ and $a_i, b_i \in [a_0, b_0]$, $i = 1, \ldots, p$, such that $g_{n_1}(y_1 - m \cdot 2^{-n_1}) \neq g_{n_1}(y_2 - m \cdot 2^{-n_1})$ and that rectangles $P_i = [a_i, b_i; y_1, y_2]$, $i = 1, \ldots, p$, have disjoint interiors. Since $P_i \subset B_{i_1}^{n_1}$, we have

$$\varphi(P_i) = F(b_i, y_2) - F(b_i, y_1) - F(a_i, y_2) + F(a_i, y_1)$$
$$= [g_{n_1}(y_2 - m \cdot 2^{-n_1}) - g_{n_1}(y_1 - m \cdot 2^{-n_1})] \cdot [g_{n_1}(b_i - k \cdot 2^{-n_1}) - g_{n_1}(a_i - k \cdot 2^{-n_1})].$$

Hence

$$\sum_{i=1}^{p} |\Phi(P_i)|$$
$$= |g_{n_1}(y_2 - m \cdot 2^{-n_1}) - g_{n_1}(y_1 - m \cdot 2^{-n_1})| \sum_{i=1}^{p} |g_{n_1}(b_i - k \cdot 2^{-n_1}) - g_{n_1}(a_i - k \cdot 2^{-n_1})|$$

and the last sum can be made arbitrarily large, because g_{n_1} is not BV in any interval included in $[0, 2^{-n_1}]$. So Φ is not BV in the considered neighbourhood and (x, y) is not an ACR-point of F.

Let $(x, y) \in \mathrm{Fr} B$. Consider the square

$$E_i^n = [k\cdot 2^{-n}, (k+1)\cdot 2^{-n}; m\cdot 2^{-n}, (m+1)\cdot 2^{-n}]$$

and put

$$P_{rs} = [k\cdot 2^{-n}+r\cdot 2^{-4n-1}, k\cdot 2^{-n}+(r+1)\cdot 2^{-4n-1};$$
$$m\cdot 2^{-n}+s\cdot 2^{-4n-1}, m\cdot 2^{-n}+(s+1)\cdot 2^{-4n-1}], \qquad r, s = 0, \ldots, 2^{3n+1}-1.$$

Then

$$\bigcup_{r=0}^{2^{3n+1}-1} \bigcup_{s=0}^{2^{3n+1}-1} P_{rs} = E_i^n,$$

squares P_{rs} have disjoint interiors and for each r, s there is $|\Phi(P_{rs})| = 2^{-2n}$. Therefore variation of Φ over E is greater than or equal to

$$\sum_{r=0}^{2^{3n+1}-1} \sum_{s=0}^{2^{3n+1}-1} |\Phi(P_{rs})| = 2^{4n+2}.$$

If $(x, y) \in \mathrm{Fr} B$, then every rectangular neighbourhood of (x, y) contains a sequence of squares $E_{i_j}^{n_j}$ such that $n \to \infty$ and $\varrho\big((x, y), E_{i_j}^{n_j}\big) \to 0$. Hence Φ is not BV in the closure of this neighbourhood and (x, y) is not an ACR-point of F.

Generalization of this theorem to a space of a greater number of dimensions offers no difficulty.

REFERENCES

[1] M. C. Chakrabarty and J. S. Lipiński, *On points of absolute continuity III*, this fascicle, p. 281–284.

[2] T. Radó, *Length and area*, New York 1948.

[3] S. Saks, *Theory of the integral*, Warszawa–Lwów 1937.

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