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A REMARK ON FOURIER-STIELTJES TRANSFORM

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Let G be a locally compact abelian group, $\Gamma = \mathrm{Hom}(G, \boldsymbol{T})$ and $M(G)$ the Banach algebra of finite complex Baire measures on G with total variation $\|\mu\|$ as norm. $B(\Gamma)$ will denote the algebra consisting of Fourier transforms $\hat{\mu}$ of $\mu \in M(G)$ with the same norm. So $M(G)$ and $B(\Gamma)$ are isometrically isomorphic. $\overline{B(\Gamma)}$ stands for the closure of $B(\Gamma)$ under sup norm. $M_c(G)$, the subalgebra of $M(G)$ consisting of continuous measures, is mapped under Fourier transformation onto a subalgebra of $B(\Gamma)$, denoted by $B_c(\Gamma)$. We write $\overline{B_c(\Gamma)}$ for its uniform closure. The pair of spaces $\big(B_c(\Gamma), M(\Gamma)\big)$ is a dual system which induces a topology in each of them. The following is proved in [1] (Theorem 1):

(*) *If, for a bounded continuous function f on Γ and every bounded net $\lambda_a \in M(\Gamma)$, the convergence*

$$\int_\Gamma \hat{\mu}(x)\, d\lambda_a(x) = \int_G \hat{\lambda}_a d\mu \to 0 \qquad \big(\forall \mu \in M_c(G)\big)$$

implies

$$\int_\Gamma f(x)\, d\lambda_a(x) \to 0,$$

then $f \in \overline{B_c(\Gamma)}$, and conversely.

There was left as an open problem whether in this characterization a general net can be replaced by a simple sequence $\{\lambda_n\}$. Here we give its affirmative solution.

THEOREM 1. *If a sequence $\{\hat{\mu}_n\}$ $(\mu_n \in M(G))$ tends uniformly to 0, and ϱ_n and σ_n denote, respectively, the continuous and the discrete parts of μ_n, then $\{\hat{\varrho}_n\}$ and $\{\hat{\sigma}_n\}$ both tend uniformly to 0* (1).

Proof. Let $\varepsilon > 0$. For n sufficiently large we have $|\hat{\mu}_n(x)| < \varepsilon/2$ $(x \in \Gamma)$. Fix such an n. By a known theorem of Wiener ([3], p. 118), the set

$$A_n = \{x \in \Gamma\colon\ |\hat{\varrho}_n(x)| < \varepsilon/2\}$$

has density 1 which means that its characteristic function has invariant

(1) Added in proof. This was also stated by Wells in [4].

mean value 1. So we have $|\hat{\sigma}_n(x)| < \varepsilon$ for $x \in A_n$. But since the function $\hat{\sigma}_n$ is almost periodic, this holds on the whole of Γ. Hence $\{\hat{\sigma}_n\}$ tends uniformly to 0, and so does $\{\hat{\varrho}_n\}$.

COROLLARY 1. *Let μ_c and μ_d denote, respectively, the continuous and the discrete parts of μ. Then the norms $\|\hat{\mu}\|_\infty$ and $\|\hat{\mu}_c\|_\infty + \|\hat{\mu}_d\|_\infty$ give raise to the same fundamental sequences.*

The proof is obvious.

COROLLARY 2. *If $f \in \overline{B(\Gamma)}$, then f decomposes in a unique way into some $f_1 \in \overline{B_c(\Gamma)}$ and $f_2 \in AP(\Gamma)$ ($AP(\Gamma)$ is the space of almost periodic functions on Γ). The norms $\|f\|_\infty$ and $\|f_1\|_\infty + \|f_2\|_\infty$ are equivalent, i.e. there is a constant C such that $\|f_1\|_\infty + \|f_2\|_\infty \leqslant C\|f\|_\infty$.*

In fact, take a sequence $\{\mu_n\}$ such that $\{\hat{\mu}_n\}$ tends uniformly to f. The "continuous" and the "discrete" parts of $\hat{\mu}_n$ are uniformly convergent by the preceding Corollary. So they furnish f_1 and f_2 as their uniform limits. The uniqueness of the decomposition follows directly from Theorem 1, and the equivalence of norms from the two-norms-theorem.

THEOREM 2. *If f is a bounded continuous function on Γ, then $f \in \overline{B_c(\Gamma)}$ is equivalent to the following implication: if*

$$(\alpha)\quad \lambda_n \in M(\Gamma), \qquad \|\lambda_n\| \leqslant 1 \qquad \text{and} \qquad \lim_n \int \hat{\lambda}_n(t)\,d\mu = 0 \qquad \big(\forall \mu \in M_c(G)\big),$$

then

$$(\beta)\qquad \lim_n \int f(x)\,d\lambda_n(x) = 0.$$

Proof. In view of $(*)$ only the sufficiency must be proved. From a result of Ramirez [2] (see also [1]) it follows that the implication $(\alpha) \Rightarrow (\beta)$ ensures $f \in \overline{B(\Gamma)}$ (and conversely). So, in view of Corollary 2, $f = f_1 + f_2$, where f_1 is in $\overline{B_c(\Gamma)}$ and f_2 in $AP(\Gamma)$. By $(*)$ we have $(\alpha) \Rightarrow (\beta)$ for f_1 instead of f. Consequently, the same is true for f_2. Once we have shown that $f_2 = 0$, the proof will be completed. If $f_2 \neq 0$, then there is at least one character $t_0 \in G$ with $\hat{f}_2(t_0) \neq 0$. So (t_0, x) is represented as $\hat{f}_2(t_0)^{-1}\mathfrak{M}\big(f_2(x-u)\overline{(t_0, u)}\big)$ with $\mathfrak{M}$ denoting the mean value on Γ with respect to u. Thus t_0 can be uniformly approximated by linear combinations of translates of f_2. Since $(\alpha) \Rightarrow (\beta)$ holds for f_2, it also holds for its translates.

Since the set of functions satisfying (β) for a fixed sequence $\{\lambda_n\}$ with $\|\lambda_n\| \leqslant 1$ is closed in sup norm, the implication $(\alpha) \Rightarrow (\beta)$ holds for t_0 (instead of f). But this is impossible, because there exists a sequence $\{\hat{\lambda}_n\}$ in $A(G)$, bounded in A-norm and converging to 0 everywhere except one point which may be chosen in advance ([3], p. 49). If we choose t_0, we have (α) without having

$$\lim_n \int_\Gamma \overline{(t_0, x)}\,d\lambda_n(x) = \lim_n \hat{\lambda}_n(t_0) = 0.$$

An analogous sequential characterization is possible also for the algebra $B_c(\Gamma)$ itself. So we have, as a strengthened form of Theorem 2 of [1] (where nets are used),

THEOREM 3. *For a bounded continuous function, $f \in B_c(\Gamma)$ is equivalent to the following implication: if*

$$(\alpha') \quad \lambda_n \in M(\Gamma), \quad \|\hat{\lambda}_n\|_\infty \leqslant 1 \quad \textit{and} \quad \lim_n \int_G \hat{\lambda}_n(t)\, d\mu = 0 \quad (\forall \mu \in M_c),$$

then

$$(\beta) \qquad \lim_n \int_\Gamma f(x)\, d\lambda_n(x) = 0.$$

The proof is even easier than that of Theorem 2, because it does not require Theorem 1. From a result of Ramirez, analogous to that used in the preceding proof ([2], see also [1], p. 183) it follows that $f \in B(\Gamma)$. The decomposition $f = f_1 + f_2$ with $f_1 \in B_c(\Gamma)$ and $f_2 \in AP(\Gamma)$ is obvious and the rest of the proof can be repeated without any change.

Let now E be a discrete subset of Γ and $B(E)$ ($B_c(E)$) the algebra of restrictions $\hat{\mu}|E$ for $\mu \in M(G)$ ($\mu \in M_c(G)$) with quotient norm. In [1] it is proved (Theorem 8) that $f \in \overline{B_c(E)}$ (the uniform closure of $B_c(E)$) if and only if the implication in (*) holds after replacing $\lambda_a \in M(\Gamma)$ by $\lambda_a \in M(E)$. Whether in this general case nets can be replaced by simple sequences seems to remain an open question (**P 940**). If the answer is in general negative, we can ask what are those sets $E \subset \Gamma$, other than Γ, for which $\overline{B_c(E)}$ admits such a sequential characterization. A similar question for $B(E)$ has an affirmative answer for all E [1].

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