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# ON THE CARDINALITY OF SET PRODUCTS IN GROUPS

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**1. Introduction.** Let $G$ be a group and let $F(G)$ denote the collection of all finite non-empty complexes (subsets) of $G$. For $A, B \in F(G)$ the set product of $A$ and $B$ is defined by

$$AB = \{ab \mid a \in A, b \in B\}.$$

If $A \in F(G)$, then $|A|$ denotes the cardinality of the set $A$. In [3], p. 5, Theorem 2.2, and [6], p. 14, Exercise 2.11, it is stated incorrectly that

$$|AB|\,|A \cap B| = |A|\,|B| \tag{1}$$

for any two complexes $A$ and $B$ of $G$. The truth of this identity would imply that

$$|AB| = |BA| \tag{2}$$

for all complexes $A$ and $B$, and this equality does not hold even in the symmetric group $S_3$. Of course, (1) does hold in case where $A$ and $B$ are finite subgroups of $G$. In this note* we prove that if (2) holds for all 3-element subsets $A$ and $B$, then $G$ must be Abelian. We characterize those groups in which (2) holds for all 2-element subsets, and those groups in which (2) holds for all 4-element subsets. The quaternion group $Q$ of order 8 plays a central role in our development. In Section 4 we prove that $\big||AB| - |BA|\big| \leqslant 1$ for all $A, B \in F(G)$ if and only if $G$ is Abelian or the quaternion group $Q$.

**2. Preliminaries.** We say that a group $G$ satisfies *condition* $\mu(k)$ if $|AB| = |BA|$ for all $A, B \in F(G)$ such that $|A| = |B| = k$, and $G$ satisfies *condition* $\mu(0)$ if $|AB| = |BA|$ for all $A, B \in F(G)$. If $A \in F(G)$ and $x \in G$, then the set $xA$ ($Ax$) is called a *left* (*right*) *translate* of $A$. The left

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(right) translation classes form a partition of $F(G)$. The center of $G$ will be denoted by $Z(G)$, the centralizer in $G$ of $x \in G$ by $C(x)$, and the normalizer of $A \in F(G)$ in $G$ by $N(A)$. For $x \in G$, $\langle x \rangle$ denotes the cyclic subgroup generated by $x$. If $A \in F(G)$, then $A^{-1} = \{a^{-1} \mid a \in A\}$. For subsets $X$ and $Y$ of $G$, $X \setminus Y$ denotes the set of elements of $X$ that do not belong to $Y$. The quaternion group of order 8 is given by

$$Q = \langle a, b \mid a^4 = b^4 = 1,\ a^2 = b^2,\ ba = a^3 b \rangle.$$

The following list contains some of the properties of $Q$ that will be needed for our exposition.

(1) If $A \in F(Q)$ and $|A| = 2$, then the left and right translation classes of $A$ are identical.

If this property holds for every 2-element subset of a group $G$, then we say that $G$ satisfies *condition* $\delta(2)$.

(2) The group $Q$ satisfies condition $\mu(2)$.

To see this, let $A, B \in F(Q)$ be such that $|A| = |B| = 2$. Then $A = xA_1$ and $B = yB_1$, where $A_1 = \{1, x_1\}$ and $B_1 = \{1, y_1\}$. It is easy to verify that $|A_1 B_1| = |B_1 A_1|$. Then

$$|AB| = |xA_1 y B_1| = |xzA_1 B_1| = |A_1 B_1|$$

and

$$|BA| = |yB_1 x A_1| = |ywB_1 A_1| = |B_1 A_1|.$$

Thus $|AB| = |BA|$.

Note that this argument is valid for any group that satisfies condition $\delta(2)$.

(3) $Q$ does not satisfy $\mu(3)$.

Let $A = \{1, a, b\}$ and $B = \{a^3, b, ab\}$. Then $|AB| = 7$ and $|BA| = 6$.

(4) $Q$ does not satisfy $\mu(4)$.

Let $A = \{1, a, a^2, b\}$ and $B = \{1, b, ab, a^2 b\}$. Then $|AB| = 7$ and $|BA| = 8$.

(5) Mann proved in [5], Theorem 1, that if $A, B \in F(G)$, then either $AB = G$ or $|A| + |B| \leqslant |G|$. It follows that if $|G| < 2k$, then $G$ satisfies $\mu(k)$. Thus $Q$ satisfies $\mu(r)$ for $r \geqslant 5$.

## 3. Set products and the conditions $\mu(k)$.

LEMMA 1. *If $G = Q \times V$, where $V$ is an elementary Abelian group of exponent 2, then $G$ satisfies condition $\delta(2)$.*

Proof. Let $A = \{a_1 x_1, a_2 x_2\}$, where $a_1, a_2 \in Q$ and $x_1, x_2 \in V$. Let $a_3 x_3 \in G$. If $a_1 = a_2$, let $a_3 a_1 = a_1 a_4$. Then $a_3 x_3 A = A a_4 x_3$. Suppose that $a_1 \neq a_2$ and let $a_3\{a_1, a_2\} = \{a_1, a_2\} a_4$. If $a_3 a_1 = a_1 a_4$ and $a_3 a_2 = a_2 a_4$, then $a_3 x_3 A = A a_4 x_3$. In the other case we have $a_3 a_1 = a_2 a_4$ and $a_3 a_2 = a_1 a_4$. Then

$$a_3 x_3 A = A\, a_4 x_3 x_1 x_2.$$

THEOREM 1. *For a group $G$, the following are equivalent:*

(i) *$G$ satisfies condition $\mu(2)$.*

(ii) *$G$ is Abelian or $G$ is Hamiltonian of the form $Q\times V$, where $V$ is an elementary Abelian group of exponent 2.*

(iii) *$G$ satisfies condition $\delta(2)$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that $G$ is not Abelian and let $x, y\in G$ be such that $xy\neq yx$. Then

$$\{x, y\}\{1, y^{-1}x\} = \{x, y, xy^{-1}x\}$$

and

$$\{1, y^{-1}x\}\{x, y\} = \{x, y, y^{-1}x^2, y^{-1}xy\}.$$

Thus $y = y^{-1}x^2$, and so $y^2 = x^2$. Now $xy^3\neq y^3x$, and thus $y^6 = x^2 = y^2$. Therefore, $y\notin Z(G)$ implies $y^4 = 1$. Clearly, then $y^4 = 1$ for all $y\in G$. Now let $x\in G$. If $y\notin C(x)$, then $xy\notin C(x)$, and so $(xy)^2 = x^2 = y^2$. Thus $xyxy = y^2$, and so $yxy^{-1} = x^{-1} = x^3$. It follows that $\langle x\rangle$ is normal in $G$. Thus $G$ is Hamiltonian with no elements of odd order. By [4], Theorem 12.5.4, p. 190, $G = Q\times V$, where $V$ is a group of exponent 2.

That (ii) implies (iii) is the content of Lemma 1. The fact that (iii) implies (i) is noted after the proof of fact (2) concerning the quaternion group $Q$.

THEOREM 2. *A group $G$ satisfies condition $\mu(0)$ if and only if $G$ is Abelian.*

Proof. $\mu(0)$ implies $\mu(3)$ and $\mu(2)$. By (ii) of Theorem 1, $G$ is Abelian or $G = Q\times V$. Since $Q$ does not satisfy $\mu(3)$, $G$ is Abelian.

LEMMA 2. *If a group $G$ satisfies $\mu(k)$ for some natural number $k\geqslant 3$ and if*

$$|G| > p(k) = 2(k-1)^3+2(k-1)^2+(k-1),$$

*then $G$ satisfies $\mu(r)$ for all $r$ such that $2\leqslant r\leqslant k$.*

Proof. It suffices to show that $G$ satisfies $\mu(k-1)$. Let $A, B\in F(G)$ be such that $|A| = |B| = k-1$. Let

$$C = A\cup ABB^{-1}\cup B^{-1}BA$$

and choose $x\in G\setminus C$. Let

$$D = B\cup A^{-1}AB\cup x^{-1}AB\cup A^{-1}xB\cup BAA^{-1}\cup BAx^{-1}\cup BxA^{-1}$$

and choose $y\in G\setminus D$. This choice of $x$ and $y$ can be made since $|G| > p(k)$ and $p(k)$ is an upper bound for $|C|$ and $|D|$. Then

$$(A\cup x)(B\cup y) = AB\cup xB\cup Ay\cup xy,$$

$$(B\cup y)(A\cup x) = BA\cup Bx\cup yA\cup yx.$$

The choice of $x$ and $y$ guarantees that the unions above are disjoint. Thus $|AB| = |BA|$, and so $G$ satisfies condition $\mu(k-1)$.

THEOREM 3. *If $G$ is an infinite group satisfying $\mu(k)$ for some natural number $k \geqslant 3$, then $G$ is Abelian.*

Proof. By Lemma 2, $G$ satisfies $\mu(3)$ and $\mu(2)$. By Theorem 1 (ii), $G$ is Abelian since $Q$ does not satisfy $\mu(3)$.

LEMMA 3. *The dihedral groups $D_n$ do not satisfy $\mu(k)$ for $n \geqslant k \geqslant 3$.*

Proof. Let

$$D_n = \langle a, b \mid a^n = b^2 = 1,\ ba = a^{n-1}b\rangle.$$

Let

$$A = \{1, a, a^2, \ldots, a^{k-2}, b\}, \qquad B = \{1, b, ab, a^2b, \ldots, a^{k-2}b\}.$$

Then, if $n > 2k-4$,

$$|AB| = 4k-6 \quad \text{and} \quad |BA| = 3k-4 < 4k-6.$$

If $n \leqslant 2k-4$, then

$$|AB| = 2n \quad \text{and} \quad |BA| = n+k-1 \leqslant 2n-1 < 2n.$$

LEMMA 4. *If $G$ is a group satisfying $\mu(k)$ for some natural number $k \geqslant 2$ and if $H$ is a subgroup of $G$ such that $|H| \geqslant k$ and $[G:H] \geqslant k$, then $H$ is normal in $G$.*

Proof. Suppose that $H$ is not normal in $G$. Then there exist $g \in G$ and $h \in H$ such that $ghg^{-1} \notin H$. Thus the cosets $H$, $Hg$, and $Hgh$ are distinct. Let the cosets $Hx_1, \ldots, Hx_{k-3}$ be distinct and different from $H$, $Hg$, and $Hgh$. For $k = 2$ let $A = \{g, gh\}$ and $B = \{1, h\}$. Then $|AB| \leqslant 3$ while $|BA| = 4$. Suppose that $k \geqslant 3$ and let

$$A = \{1, g, gh, x_1, \ldots, x_{k-3}\}.$$

Choose distinct elements $h_1, \ldots, h_{k-2} \in H$ different from 1 and $h$, and let

$$B = \{1, h, h_1, \ldots, h_{k-2}\}.$$

Then $|BA| = k^2$ while $|AB| \leqslant k^2-1$, since we obtain $gh \in AB$ in at least two ways.

THEOREM 4. *A group $G$ satisfies $\mu(3)$ if and only if $G$ is Abelian.*

Proof. Because of Lemma 2, if $|G| > p(3) = 26$, then $G$ satisfies $\mu(2)$, and so $G$ is Abelian. It remains to consider those groups $G$ for which $|G| \leqslant 26$. The non-Abelian groups of orders 6, 10, 14, 22, and 26 are dihedral, and so, by Lemma 3, do not satisfy $\mu(3)$. The non-Abelian groups of order 8 are $Q$ and $D_4$ neither of which satisfies $\mu(3)$. If $|G| = 12$, then, by Lemma 4, the 2-Sylow subgroup and the 3-Sylow subgroup are normal, and so $G$ is Abelian. A similar argument is valid for groups of order 20, 21, and 24. Thus we have left to consider groups of orders 18 and 16.

Suppose that $|G| = 18$. Let $H$ be the normal 3-Sylow subgroup of $G$ and let $b \in G$, where $o(b) = 2$. By Lemma 4, a subgroup of order 3 is normal in $G$. Let $o(a) = 3$. Then $\langle a, b\rangle$ is either $Z_6$ or $D_3$. Since $D_3$ does not satisfy $\mu(3)$, we have $\langle a, b\rangle = Z_6$, and so $ab = ba$. If $H = Z_3 \times Z_3$, then $b \in Z(G)$. Hence all subgroups of $G$ are normal and $G$ is Abelian. Suppose that $H = Z_9$ and let $H = \langle x\rangle$. Then $b^{-1}x^3b = x^3$, and so $b^{-1}xb = x, x^4$, or $x^7$. Also $x = b^{-2}xb^2$. These conditions imply $b^{-1}xb = x$. Thus $b \in Z(G)$ and $G$ is Abelian.

Suppose that $|G| = 16$. By Lemma 4, subgroups of $G$ of order 4 are normal. Subgroups of order 8 are also normal. Suppose, by the way of contradiction, that $G$ is not Abelian. Note that $G$ is not Hamiltonian. Thus there exists a $b \in G$ such that $o(b) = 2$ and $\langle b\rangle$ is not normal in $G$. Let $x \in G$ be such that $x^{-1}bx \neq b$. If $a \in G$ and $o(a) = 4$, then $\langle a\rangle$ is normal in $G$. Also $b \neq a^2$. Thus, if $N = \langle a, b\rangle$, then $|N| = 8$, and so $ab = ba$. Now, if $y \notin N$ implies $o(y) = 2$, then $o(xb) = 2$ and $x^{-1}bx = b$, contrary to our supposition. Thus we may assume that $o(x) = 8$. Then $o(x^2) = 4$, and so $bx^2b^{-1} = x^2$. Thus $bxb^{-1} = x^5$ or $x^7$. If $bxb^{-1} = x^7$, then $G = \langle x, b\rangle$ is isomorphic to $D_8$, a contradiction. Thus $bxb^{-1} = x^5$, and so

$$G = \langle x, b\rangle = \langle x, b \mid x^8 = b^2 = 1,\ bx = x^5b\rangle.$$

Let $A = \{1, b, xb\}$ and $B = \{x, x^2, xb\}$. Then $|AB| = 9$ and $|BA| = 6$, and $G$ does not satisfy $\mu(3)$.

THEOREM 5. *A group $G$ satisfies $\mu(4)$ if and only if $G$ is Abelian or $G = D_3$.*

Proof. By Theorem 1 of Mann [5], $D_3$ does satisfy $\mu(4)$. By Lemma 2, if $|G| > p(4) = 75$, then $G$ satisfies $\mu(3)$ and $\mu(2)$, and so $G$ is Abelian. There remains to consider those groups $G$ for which $|G| \leqslant 75$. Many of the cases follow easily by the use of Lemmas 3 and 4, and the Sylow theorems. The cases for which $|G| = 16$, 32, or 64 are handled in essentially the same way as the case $|G| = 16$ in the proof of Theorem 4. We include here 3 of the cases that are more or less representative of the several cases that must be considered.

Let $|G| = 18$ and suppose (by the way of contradiction) that $G$ is not Abelian. Let $H$ be the normal 3-Sylow subgroup of $G$ and let $b \in G$, where $o(b) = 2$. If $H = \langle x\rangle$, where $o(x) = 9$, then $bxb^{-1} = x^8$ and $G = D_9$; but $D_9$ does not satisfy $\mu(4)$. Thus $H = Z_3 \times Z_3$. If there exist $x, y \in H$ such that $o(x) = o(y) = 3$ and $bxb^{-1} = y \notin \langle x\rangle$, let $A = \{1, x, x^2, b\}$ and $B = \{1, b, xb, x^2b\}$. Then $|AB| = 8$ and $|BA| = 12$, a contradiction. Thus we may suppose that $bxb^{-1} = x^2$ for all $x \in H$. Let $y \in H \setminus \langle x\rangle$. Then, if we let $A = \{x, x^2, xy, xb\}$ and $B = \{1, b, xb, yb\}$, we have $|AB| = 10$ and $|BA| = 12$. Thus a group of order 18 satisfying $\mu(4)$ must be Abelian.

Let $|G| = 24$. We know that groups of order 12 that satisfy $\mu(4)$ are Abelian. Let $H$ be a 2-Sylow subgroup and $J$ a 3-Sylow subgroup. Now $H$ is Abelian as $D_4$ and $Q$ do not satisfy $\mu(4)$. Let $K$ be a subgroup of order 4. Then, by Lemma 4, $K$ is normal in $G$. Thus $KJ$ is an Abelian subgroup of $G$ of order 12. Therefore, $KJ \subseteq N(J)$, and so $J$ is normal in $G$. Now let $L$ be a subgroup of order 2. Then $L$ is contained in a 2-Sylow subgroup $M$ of $G$. Since $M$ is Abelian, $M \subseteq N(L)$. Let $M_1$ be a subgroup of $M$ containing $L$, where $|M_1| = 4$. Then $M_1 J$ is Abelian, and so $J \subseteq N(L)$. Thus $L$ is normal in $G$. If $H$ contains distinct subgroups of order 4, then $H$ is normal and $G$ is Abelian. Otherwise $H$ is cyclic (see [2], Theorem VI, p. 132). Let $H = \langle x \rangle$ and $J = \langle a \rangle$. Then $x^{-1}ax = a^2$. Thus

$$G = \langle a, x \mid x^8 = a^3 = 1,\ ax = xa^2 \rangle.$$

Let $A = \{x, x^2, x^3, xa\}$ and $B = \{a, xa, x^2a, a^2\}$. Then $|AB| = 10$ and $|BA| = 12$. Thus a group of order 24 satisfying $\mu(4)$ is Abelian.

Let $|G| = 27$ and suppose that $G$ is not Abelian. Then we have the following two cases (see [6], Problem 5.42, p. 94).

Case 1. We have

$$G = \langle a, b \mid a^9 = b^3 = 1,\ b^{-1}ab = a^4 \rangle.$$

Let $A = \{a, a^2, a^3, ab\}$ and $B = \{b, ab, a^2b, b^2\}$. Then $|AB| = 11$ and $|BA| = 13$.

Case 2. We have

$$G = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, ca = ac, cb = bc, ab = cba \rangle.$$

Let $A = \{1, a, a^2, b\}$ and $B = \{b, ab, a^2b, b^2\}$. Then $|AB| = 9$ and $|BA| = 15$.

We have not pursued the conditions $\mu(k)$ for $k \geqslant 5$. In order to do so, it would be worth-while to sharpen the inequality in Lemma 2. As the value of $k$ is increased, we will of course find groups of small order that satisfy $\mu(k)$. For example, $D_3$, $D_4$, and $Q$ satisfy $\mu(5)$. In closing this section, we ask the following question:

Is there a non-Abelian group $G$ that satisfies condition $\mu(k)$ $(k \geqslant 5)$, where $|G| \geqslant 2k$? **(P 1165)**

**4. Set products and condition** $\varepsilon(1)$. We say that a group $G$ satisfies *ondition* $\varepsilon(1)$ provided that

$$\big||AB| - |BA|\big| \leqslant 1$$

for all $A, B \in F(G)$. In this section we prove that the only groups that satisfy $\varepsilon(1)$ are the Abelian groups and the quaternion group $Q$ of order 8.

LEMMA 5. *If a group $G$ satisfies condition $\varepsilon(1)$, then every finite subgroup of $G$ is normal.*

Proof. Let $H$ be a non-trivial finite subgroup of $G$, and suppose that $H$ is not normal in $G$. Then there exists an $a \in G$ such that $Ha \neq aH$. Without loss of generality, we may assume that $aH \nsubseteq Ha$. Let $x \in aH \setminus Ha$. Then $x \neq a$. Now

$$\{x, a\}H = xH \cup aH = aH \quad \text{and} \quad H\{x, a\} = Hx \cup Ha.$$

Consequently, $Hx \neq Ha$ since $x \notin Ha$. Thus

$$\big||H\{x, a\}| - |\{x, a\}H|\big| = |H| \geqslant 2,$$

contradicting the supposition that $G$ satisfies condition $\varepsilon(1)$.

LEMMA 6. *If $G$ is a non-Abelian group satisfying condition $\varepsilon(1)$, then $G$ is Hamiltonian.*

Proof. Suppose $y \in G$ is such that $\langle y \rangle$ is not normal in $G$. Then, because of Lemma 5, we have $o(y) = \infty$. Let $x \in G \setminus N(\langle y \rangle)$. Then $xy \neq yx$. Now

$$\{x, y\}\{1, x^{-1}y, y^{-1}x\} = \{x, y, yx^{-1}y, xy^{-1}x\}$$

and

$$\{1, x^{-1}y, y^{-1}x\}\{x, y\} = \{x, y, x^{-1}yx, x^{-1}y^2, y^{-1}x^2, y^{-1}xy\}.$$

Thus at least two of the words of the second product must be equal. Only the following 4 equalities are consistent with the assumption that $xy \neq yx$:

(1) $x = x^{-1}y^2$ which implies $x^2 = y^2$;

(2) $x^{-1}yx = y^{-1}x^2$ which implies $x^{-1}y = y^{-1}x$;

(3) $x^{-1}yx = y^{-1}xy$;

(4) $x^{-1}y^2 = y^{-1}x^2$.

Now $x(xy) \neq (xy)x$ and replacing $y$ in (1)-(4) by $xy$ we infer that at least one of the following equalities must hold:

(1′) $x^2 = (xy)^2$;

(2′) $x^{-1}(xy) = (xy)^{-1}x$;

(3′) $x^{-1}(xy)x = (xy)^{-1}x(xy)$;

(4′) $x^{-1}(xy)^2 = (xy)^{-1}x^2$.

Now (1′) implies $x^{-1}y^{-1}x = y$, and so cannot hold under the assumption that $x \notin N(\langle y \rangle)$. Equality (2′) implies $y^2 = 1$ contrary to the fact that $o(y) = \infty$. Thus one of (1)-(4) holds and (3′) or (4′) holds. We present here only 2 of 8 possible cases.

Case 1. Suppose that (2) and (3′) hold. Then $x^{-1}y = y^{-1}x$ and $yx = y^{-1}xy$. Thus $y^3x^{-1}y = y^2x = xy$, and so $y^3 = x^2$. Also $x^{-1}yx = y^{-1}x^2 = y^2$. Thus

$$y^6 = (x^{-1}yx)^3 = x^{-1}y^3x = x^2 = y^3,$$

contradicting the supposition that $o(y) = \infty$.

Case 2. Suppose that (4) and (4′) hold. Then $x^{-1}y^2 = y^{-1}x^2$ and $yxy = y^{-1}x$. Thus $y^2xy = x$, and so $y^{-1}x^{-1}y^{-2} = x^{-1}$. Therefore, $y^{-1}x^{-1} = x^{-1}y^2 = y^{-1}x^2$, and so $x^3 = 1$. Then $y^{-1}xy = x^2$, since $\langle x\rangle$ is normal in $G$ and $xy \neq yx$. Thus $x^{-1}y^3 = y^{-1}x^2y = x$, and so $y^3 = x^2$, which implies $y^9 = 1$ contrary to the assumption that $o(y) = \infty$. Thus every cyclic subgroup of $G$ is normal, and so $G$ is Hamiltonian.

THEOREM 6. *The only groups that satisfy condition $\varepsilon(1)$ are the Abelian groups and the quaternion group $Q$ of order* 8.

Proof. Suppose that $G$ satisfies $\varepsilon(1)$. Then, by Lemma 6, $G$ is Abelian or Hamiltonian. Suppose that $G$ is Hamiltonian. Then $G = Q \times V$, where $V$ is Abelian. Suppose that $V \neq 1$ and let $x \in V$, $x \neq 1$. Let $A$ and $B$ be the 3-element subsets of $Q$ as given in (3) of Section 2. Then $|AB| = 7$ and $|BA| = 6$. Let $C = A \cup Ax$ and $D = B \cup Bx$. Then

$$CD = (A \cup Ax)(B \cup Bx) = AB \cup ABx \cup ABx^2$$

and

$$DC = (B \cup Bx)(A \cup Ax) = BA \cup BAx \cup BAx^2.$$

If $x^2 = 1$, then $|CD| = 14$ and $|DC| = 12$. If $x^2 \neq 1$, then $|CD| = 21$ and $|DC| = 18$. Thus the only possibility is that $G = Q$. The fact that $Q$ does satisfy condition $\varepsilon(1)$ can easily be verified on a computer. We offer our thanks to Professor Albert Newhouse for his help in writing the program.

We end this note with the observation that one could consider conditions $\varepsilon(k)$ for $k \geqslant 2$. As the value of $k$ is increased, the groups of small order will, of course, satisfy $\varepsilon(k)$. For example, $S_3$ along with $Q$ and the Abelian groups satisfy $\varepsilon(2)$. The dihedral group $D_4$ does not satisfy $\varepsilon(2)$.

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