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WEYL ALMOST PERIODIC POINTS IN TOPOLOGICAL DYNAMICS

BY

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Let X be a compact metric space and T be a homeomorphism of X onto itself. The *trajectory* $\bar{x}(j) = T^j x$ of a single point x can be viewed as an X-valued function on the group of integers. In this paper we study the trajectories that are almost periodic in the sense of Weyl (W-a.p.). A well-known subclass of W-a.p. trajectories consists of the uniformly (or Bohr) almost periodic ones. These are just the equicontinuous (or L-stable) trajectories, which always have a unique invariant measure μ (unique ergodicity) as well as the property that the unitary operator induced by T on $L^2(\mu)$ has pure point spectrum (discrete spectrum). In Section 2 we prove that both properties are preserved in the W-a.p. case (Theorems 1 and 2).

The unique ergodicity of W-a.p. trajectories is obtained by means of an approximation argument for vector-valued W-a.p. functions while the discrete spectrum is a consequence of the Wiener and Wintner theory of sequences having a correlation [17].

In symbolic dynamics, an important class of W-a.p. trajectories are the regular Toeplitz sequences of Jacobs and Keane [8]. Our Theorems 1 and 2 extend Theorems 5 and 6 in [8], respectively. Another class of examples is related to the uniform sequences of Brunel and Keane [4] derived from the unit circle and is obtained via R-a.p. functions considered by Hartman and Ryll-Nardzewski [6]. The latter class contains, in particular, the Sturmian dynamical systems ([7], [10]). The unique ergodicity of Sturmian systems was shown by Klein in [10]. We prove (Section 3) that their discrete spectrum is always irrational.

Finally, we note that nonperiodic Morse sequences [9] are never W-a.p. (Proposition 3). This observation provides us with a vast class of symbolic dynamical systems that are uniquely ergodic and have discrete spectrum without being W-a.p.

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1. Almost periodic functions. A subset S of the integers $\boldsymbol{Z}$ is called *syndetic* (or *relatively dense*) if it has bounded gaps in $\boldsymbol{Z}$. The class of all syndetic sets will be denoted by $\mathcal{S}$. A function g from $\boldsymbol{Z}$ into a complete metric space (X, d) is called *uniformly almost periodic* (u.a.p.) if

$$\forall \varepsilon > 0\ \exists S \in \mathcal{S}\ \forall s \in S\ \sup_j d(g_s(j), g(j)) < \varepsilon,$$

where $g_s(j) = g(j+s)$. The range of a u.a.p. function is easily seen to be relatively compact. If (X, d) is the complex plane, the u.a.p. functions are the classical almost periodic functions of Bohr (in classical theory, however, u.a.p. functions are usually defined on $\boldsymbol{R}$ rather than $\boldsymbol{Z}$). Abstract a.p. functions were first considered by Bochner [3].

For every $L \in \boldsymbol{N}$ the Stepanoff distance D_{S_L} between two functions $\boldsymbol{Z} \to X$ is defined by the formula

$$D_{S_L}(g, h) = \sup_J |J|^{-1} \sum_{j \in J} d(g(j), h(j)),$$

where the supremum is taken over all intervals J of length L in $\boldsymbol{Z}$. Now a function $g\colon \boldsymbol{Z} \to X$ is called *Weyl almost periodic* (W-a.p.) if

$$\forall \varepsilon > 0\ \exists L \in \boldsymbol{N}\ \exists S \in \mathcal{S}\ \forall s \in S\ D_{S_L}(g_s, g) < \varepsilon.$$

Clearly, every purely periodic function is u.a.p. and u.a.p. functions are W-a.p. The classical W-a.p. functions were introduced by Weyl in [16] (see also [2] and [15] for properties of classical W-a.p. functions).

It is easy to verify that for $k \leqslant l$

$$D_{S_l}(g, h) \leqslant (1 + k/l) D_{S_k}(g, h),$$

whence, in the definition of W-a.p. functions, $D_{S_L} < \varepsilon$ can be replaced by the condition $\forall k \geqslant L\ D_{S_k} < \varepsilon$. Moreover, it suffices to consider the intervals J of the form $\{0, 1, \ldots, k\}$ (see [15], Section 3). For bounded functions another equivalent definition reads

$$\forall \varepsilon > 0\ \exists L \in \boldsymbol{N}\ \exists S \in \mathcal{S}\ \forall s \in S\ \left|\{j \in J\colon d(g_s(j), g(j)) > \varepsilon\}\right|/|J| < \varepsilon$$

for every interval J of length $\geqslant L$ in $\boldsymbol{Z}$.

It should be noted that the notion of almost periodicity is independent of the choice of a uniformly equivalent metric on X. Recall that every compact metric space embeds homeomorphically into a Hilbert space.

The following proof is based on an approximation argument ([2], Chapter 2, Section 4).

LEMMA 1. *Let g be a W-a.p. function taking values in a compact convex subset K of a Banach space. Then for every $\varepsilon > 0$ there exists a K-valued u.a.p. function h such that $D_{S_L}(g, h) < \varepsilon$ for some L.*

Proof. We let $d(x, y) = \|x-y\|$ in K. Now choose $S \in \mathcal{S}$ with gaps $< m$ and $L \in N$ such that $D_{S_L}(g_s, g) < \varepsilon$ for $s \in S$. There exists a bisequence $t_j \in S$ $(j \in Z)$ with

$$t_j \in \{jm, \ldots, (j+1)m-1\}.$$

The set $J = \{t_j\colon j \in Z\}$ is syndetic with gaps $< 2m$. By compactness and using diagonal procedure we can find a sequence $T_n \to \infty$ such that

$$\lim_n |J_n|^{-1} \sum_{t \in J_n} g(j+t) = h(j) \in K$$

exists for every $j \in Z$, where $J_n = J \cap \{-T_n, \ldots, T_n\}$. We have

$$L^{-1} \sum_{j=k}^{k+L-1} \|g(j)-h(j)\| \leqslant \underline{\lim}_n |J_n|^{-1} \sum_{t \in J_n} L^{-1} \sum_{j=k}^{k+L-1} \|g(j)-g(j+t)\|$$

$$\leqslant \underline{\lim}_n |J_n|^{-1} \sum_{t \in J_n} D_{S_L}(g, g_t) \leqslant \varepsilon,$$

so $D_{S_L}(g, h) \leqslant \varepsilon$. It remains to show that h is a u.a.p. function. Given $\delta > 0$, there exist $P \in \mathcal{S}$ and $l \in N$ such that

$$D_{S_l}(g_s, g) < \delta/2m$$

for all s in P. By the definition of h,

$$\|h(j+s)-h(j)\| \leqslant \underline{\lim}_n |J_n|^{-1} \sum_{t \in J_n} \|g(j+s+t)-g(j+t)\|$$

$$\leqslant \overline{\lim_n} \frac{2T_n+1}{|J_n|} (2T_n+1)^{-1} \sum_{t=-T_n}^{T_n} \|g(j+s+t)-g(j+t)\|.$$

Since $(2T_n+1)/|J_n| \to m$, and $2T_n+1 \geqslant l$ implies

$$(2T_n+1)^{-1} \sum_{t=-T_n}^{T_n} \|g(j+s+t)-g(j+t)\| \leqslant 2D_{S_l}(g_s, g),$$

we obtain

$$\sup_j \|h(j+s)-h(j)\| \leqslant 2mD_{S_l}(g_s, g) < \delta$$

for all $s \in P$.

The Weyl distance D_W between two functions from Z into X is defined by the formula

$$D_W(g, h) = \overline{\lim_L} D_{S_L}(g, h)$$

(cf. [2], II, Section 2). It is clear that a D_W-limit of W-a.p. functions is itself W-a.p. Consequently, Lemma 1 characterizes vector W-a.p. functions with relatively compact range as D_W-limits of u.a.p. functions (hence of trigonometric polynomials, see [12], II, Section 2).

By Bochner's criterion, a Banach space valued function $g\colon \mathbf{Z} \to E$ is u.a.p. iff its orbit $\{g_s\colon s \in \mathbf{Z}\}$ is relatively compact in the space of E-valued bounded sequences with supremum norm. An easy consequence of this fact is that the functions $g_i\colon \mathbf{Z} \to X_i$ $(i = 1, 2, \ldots)$, where the X_i are compact metric, are u.a.p. iff the diagonal function

$$g = (g_1, g_2, \ldots)\colon \mathbf{Z} \to \prod X_i$$

is u.a.p. (cf. [12], I, Section 2). We generalize this to the case of W-a.p. functions.

LEMMA 2. *Let $g_i\colon \mathbf{Z} \to X_i$ with (X_i, d_i) compact metric, $i = 1, 2, \ldots$ Then the function $g = (g_1, g_2, \ldots)$ is W-a.p. (u.a.p.) iff each g_i is W-a.p. (u.a.p.).*

Proof. It suffices to prove the sufficiency in the W-a.p. case. Assume $d_i \leqslant 1$ and endow $X = \prod X_i$ with the metric $d = \sum 2^{-i} d_i$. Without loss of generality we may assume that each X_i is a convex compact subset of a Banach space. By Lemma 1, for every $\varepsilon > 0$ there exist u.a.p. functions $h_i\colon \mathbf{Z} \to X_i$ such that $D_W(g_i, h_i) < \varepsilon/2$. Now the function $h = (h_1, h_2, \ldots)$ is u.a.p. and for any $L \geqslant 1$, $k \in \mathbf{Z}$ we have

$$L^{-1} \sum_{j=k}^{k+L-1} d(g(j), h(j)) = \sum_{i=1}^{\infty} 2^{-i} L^{-1} \sum_{j=k}^{k+L-1} d_i(g_i(j), h_i(j))$$
$$< \sum_{i=1}^{m} 2^{-i} L^{-1} \sum_{j=k}^{k+L-1} d_i(g_i(j), h_i(j)) + \varepsilon/2$$

provided $2^{1-m} < \varepsilon$. Since $D_{S_L}(g_i, h_i) < \varepsilon/2$ for L large enough and $i = 1, 2, \ldots, m$, we obtain

$$D_W(g, h) = \overline{\lim_L}\, D_{S_L}(g, h) \leqslant \varepsilon.$$

This proves that g can be D_W-approximated by u.a.p. functions, so it is W-a.p.

Our next lemma gives another equivalent definition of compact valued almost periodic functions.

LEMMA 3. *Let (X, d) be a compact metric space and F be a family of continuous functions on X taking values in a metric space. Assume in addition that the functions in F separate the points of X. Then $g\colon \mathbf{Z} \to X$ is W-a.p. (u.a.p.) iff the functions $f \circ g$ are W-a.p. (u.a.p.) for all $f \in F$.*

Proof. Let $f_1, f_2, \ldots$ be a separating sequence in F. The mapping $\varphi(x) = (f_1(x), f_2(x), \ldots)$ establishes a homeomorphic embedding of X into the product space $Y = \prod f_i(X)$. Put $h_i(j) = f_i(g(j))$ and let $h = (h_1, h_2, \ldots)$. Since $h(j) = \varphi(g(j))$, g is W-a.p. iff h has the same property. Now apply Lemma 2.

We recall that a complex-valued function g is almost periodic in the sense of Besicovitch (B-a.p.) if for every $\varepsilon > 0$ there exists a trigonometric polynomial p such that

$$\overline{\lim_{L}}(2L+1)^{-1} \sum_{j=-L}^{L} |g(j)-p(j)| < \varepsilon$$

(see [2]). Every complex-valued W-a.p. function is known to be B-a.p.

For a compact metric space (X, d) we say that g: $\boldsymbol{Z} \to X$ is *B-a.p.* if $f \circ g$ is a (bounded) B-a.p. function for each continuous complex-valued function f on X. It follows from Lemma 3 that X-valued W-a.p. functions are B-a.p.

Remark 1. Note that complex-valued bounded B-a.p. functions form a function algebra closed under the uniform norm and conjugation, which by the Stone-Weierstrass theorem makes Lemma 3 valid for B-a.p. functions as well.

2. Almost periodic points. We shall consider a dynamical system (X, T), where (X, d) is a compact metric space and T a homeomorphism of X onto itself. A point x in X will be called *uniformly almost periodic* (u.a.p.) if the X-valued function

$$\bar{x}(j) = T^j x$$

is u.a.p. Analogously, x will be called a *W-a.p. (B-a.p.) point* if $\bar{x}$ is a W-a.p. (B-a.p.) function. We say that x is *strictly transitive* if there is only one invariant Borel probability measure on its orbit closure. The orbit closure of x is then said to be *uniquely ergodic*.

The dynamical properties of u.a.p. points are well understood. In fact, x is u.a.p. iff the orbit closure $\bar{O}(x)$ of x is an equicontinuous subsystem of X. It follows that every $y \in \bar{O}(x)$ is also u.a.p. Moreover, by the Halmos–von Neumann theorem, x is u.a.p. iff $(\bar{O}(x), T)$ is isomorphic (i.e., topologically conjugate) to a minimal rotation of a compact abelian metric group. This, in particular, implies that $(\bar{O}(x), T)$ is uniquely ergodic and has discrete spectrum (see [13] for a thorough discussion of u.a.p. points in continuous time dynamical systems). By Lemma 3 and Remark 1, x is u.a.p. (W-a.p., B-a.p.) iff for every $f \in C(X)$ the function

$$f^x(j) = f(T^j x)$$

is u.a.p. (W-a.p., B-a.p.).

Remark 2. Since the uniform norm, the Stepanoff distance, and the upper limit in the definition of B-a.p. functions are translation invariant, the following useful criterion of almost periodicity is also valid: x is u.a.p. (W-a.p., B-a.p.) if the functions f^x $(f \in F)$ are u.a.p. (W-a.p., B-a.p.), where F is any family of continuous functions such that the functions $f \circ T^j$ $(f \in F,\ j \in \boldsymbol{Z})$

separate the points of X. In particular, if Y is compact metric and $(Y^{\mathbf{Z}}, S)$ is the (left bilateral) shift system, then $x \in Y^{\mathbf{Z}}$ is a u.a.p. (W-a.p., B-a.p.) point iff $x\colon \mathbf{Z} \to Y$ is a u.a.p. (W-a.p., B-a.p.) function (to see this project onto the 0-th coordinate). It is now obvious that in symbolic dynamics, i.e., for Y finite, all u.a.p. points are periodic.

Now we prove that, as in the u.a.p. case, almost periodicity in the sense of Weyl is a property of the whole orbit closure.

LEMMA 4. *If x is a W-a.p. point, then every y in $\bar{O}(x)$ is also W-a.p.*

Proof. Let $\varepsilon > 0$. There exist $L \geqslant 1$ and $S \in \mathscr{S}$ such that $D_{S_L}(\bar{x}_s, \bar{x}) < \varepsilon$ for $s \in S$. This implies

$$L^{-1} \sum_{j=k}^{k+L-1} d(T^{j+s}y, T^j y) < \varepsilon$$

for all $k \in \mathbf{Z}$, $s \in S$, and $y \in \{T^j x\colon j \in \mathbf{Z}\}$. Since the function

$$y \to L^{-1} \sum_{j=k}^{k+L-1} d(T^{j+s}y, T^j y)$$

is continuous, we obtain $D_{S_L}(\bar{y}_s, \bar{y}) \leqslant \varepsilon$ for all $y \in \bar{O}(x)$.

As the following example shows, the B-a.p. points do not enjoy the same property.

EXAMPLE 1. Let Y be a compact metric space. Consider the shift system $(Y^{\mathbf{Z}}, S)$. There exists a B-a.p. point x such that $\bar{O}(x) = Y^{\mathbf{Z}}$. In fact, let A be a countable dense subset of Y. The family of all finite blocks of letters from A is countable, so we may arrange it into a sequence $A_1, A_2, \ldots$ Let a_i be the length of A_i. Now choose a rapidly increasing sequence of natural numbers b_i so that

$$\lim_n \Big(\sum_{i=1}^{n} a_i \Big/ \sum_{i=1}^{n} (a_i + b_i)\Big) = 0$$

and let $B_i = \beta \ldots \beta$ be a block of length b_i, where β is a fixed element of Y. We define x_j to be β for $j < 0$. For $j \geqslant 0$ we define (x_j) as the concatenation $B_1 A_1 B_2 A_2 \ldots$ It is now clear that for every continuous function on Y we have

$$\lim_L (2L+1)^{-1} \sum_{j=-L}^{L} |f(x_j) - f(b)| = 0,$$

so $j \to f(x_j)$ is a B-a.p. function and, consequently, x is a B-a.p. point in $Y^{\mathbf{Z}}$ (see Remark 2). On the other hand, each finite block of A occurs infinitely many times in x, so $\bar{O}(x)$ contains $A^{\mathbf{Z}}$, hence its closure $Y^{\mathbf{Z}}$.

Let (X, T) be a dynamical system. We say that $x_n \to x$ *quasi-uniformly* in X if $D_W(\bar{x}_n, \bar{x}) \to 0$. Quasi-uniform convergence has been studied in [8]. (The quasi-uniform convergence in [8] is slightly stronger since the authors require additionally that $d(x_n, x) \to 0$.)

It follows from Lemma 1 that for any W-a.p. point x in (X, T) the function $\bar{x}$ can be D_W-approximated by the u.a.p. function h taking values in a possibly larger compact convex subset $K \supset X$ of a Banach space. We can consider both h and x (or, more precisely, $\bar{x}$) as points in the shift system $K^{\mathbf{Z}}$. Clearly, X embeds into $K^{\mathbf{Z}}$ via $\varphi(y) = (T^j y)$. In view of Remark 2, h approximates x quasi-uniformly in $K^{\mathbf{Z}}$ and we obtain the following result:

PROPOSITION 1. *There exists a topological embedding of (X, T) into a dynamical system $(\tilde{X}, \tilde{T})$ in which all W-a.p. points of X are quasi-uniform limits of u.a.p. points in $\tilde{X}$.*

Theorem 3 in [8] states that a quasi-uniform limit of strictly transitive points is strictly transitive. Since u.a.p. points are strictly transitive, we have

THEOREM 1. *Every W-a.p. point is strictly transitive.*

It should be noted that an alternative proof of Theorem 1 is possible by combining Theorem 5.5 in [14], Lemma 3, and a property of scalar W-a.p. functions asserting that the limit

$$\lim_{L} L^{-1} \sum_{j=k}^{k+L-1} f(j)$$

exists uniformly in k ([15], Sections 3 and 4).

We recall that a sequence $x \in \{0, 1\}^{\mathbf{Z}}$ is called *Toeplitz* if every symbol (hence every finite block) occurs in x periodically. Regular Toeplitz sequences (see [8] for the definition) are quasi-uniform limits of periodic points in $\{0, 1\}^{\mathbf{Z}}$, so they are W-a.p. Strict transitivity of regular Toeplitz sequences is the contents of Theorem 5 in [8] (see also [18] for a more general approach).

We remark that in general a W-a.p. point cannot be quasi-uniformly approximated by u.a.p. points within the same dynamical system. In fact, if $x \in \{0, 1\}^{\mathbf{Z}}$ is a nonperiodic regular Toeplitz sequence, then the system $X = \bar{O}(x)$ is minimal and x is W-a.p., but there are no u.a.p. points in X whatsoever, as all u.a.p. points in $\{0, 1\}^{\mathbf{Z}}$ are periodic.

The following example shows that it may not be possible to quasi-uniformly approximate a W-a.p. point (or even a u.a.p. point) by periodic ones even at the expense of enlarging the dynamical system. Therefore, the phrase "u.a.p." in Proposition 1 cannot be replaced by "periodic".

EXAMPLE 2. Let X be the unit circle in the complex plane and $T_\alpha x = e^{2\pi i\alpha} x$, α irrational. Then, clearly, all the points are u.a.p. Suppose (X, T_α) is isometrically embedded into a dynamical system $(\tilde{X}, \tilde{T})$. We show that if y is k-periodic in $\tilde{X}$ and $z \in X$, then $D_W(\bar{z}, \bar{y}) \geq 1/2$ (in fact, more is shown as the same inequality is proved for a weaker pseudometric). First note that the sets

$$A_j = \{x \in X\colon\ d(x, \tilde{T}^j y) \geq 1\} \quad (j = 0, 1, \ldots, k-1)$$

are closed and each of them has Lebesgue measure $\lambda(A_j)$ at least $1/2$ (as $X = A_j \cup (-A_j)$). For every $\varepsilon > 0$ there exists a B_j which is a finite union of arcs and satisfies

$$\text{(a) } \lambda(B_j) > (2+\varepsilon)^{-1} \quad \text{and} \quad \text{(b) } d(B_j, \tilde{T}^j y) > 1-\varepsilon.$$

Since $T_\alpha^k = T_{k\alpha}$ is still an irrational rotation, for m large enough we have

$$m^{-1} \sum_{i=0}^{m-1} \chi_{B_j}(T_\alpha^{ki+j} z) > (2+\varepsilon)^{-1} \qquad (j = 0, 1, \ldots, k-1),$$

which means $T_\alpha^{ki+j} z \in B_j$ for at least $m/(2+\varepsilon)$ i's in $\{0, \ldots, m-1\}$. Consequently,

$$m^{-1} \sum_{i=0}^{m-1} d(T_\alpha^{ki+j} z, T_\alpha^{ki+j} y) > (1-\varepsilon)/(2+\varepsilon)$$

for $j = 0, 1, \ldots, k-1$. This clearly implies

$$\overline{\lim_n}\, n^{-1} \sum_{i=0}^{n-1} d(\tilde{T}^i z, \tilde{T}^i y) \geqslant 1/2$$

and $D_W(\bar{z}, \bar{y}) \geqslant 1/2$.

Apart from the regular Toeplitz sequences we single out a vast class of W-a.p. points in shift dynamical systems. Let X, Y be compact metric spaces and let μ be a Borel probability measure on X. By $R(X, \mu, Y)$ we denote the family of all functions $h\colon X \to Y$ such that h has the set of discontinuity points of measure μ zero. The following proposition is similar to Corollary 3.21 in [1]; the idea goes back to Hartman and Ryll-Nardzewski (see [6]).

PROPOSITION 2. *Let x be a W-a.p. point in (X, T) and $h \in R(X, \mu, Y)$ where μ is the unique invariant measure of $(\bar{O}(x), T)$. Then the function $j \to h(T^j x)$ is W-a.p.*

Proof. By Lemma 3 it suffices to consider a real-valued h. For every $\varepsilon > 0$ there exist continuous functions f and g such that

$$f \leqslant h \leqslant g \quad \text{and} \quad \int (g-f)\, d\mu < \varepsilon.$$

By unique ergodicity (Theorem 1), the averages

$$L^{-1} \sum_{j=k}^{k+L-1} (g-f)(T^j x)$$

converge to $\int (g-f)\, d\mu$ uniformly in k (see [14]). This implies

$$\overline{\lim_L}\, D_{S_L}(g^x, f^x) \leqslant \varepsilon$$

and, consequently,

$$D_W(g^x, h^x) \leqslant D_W(g^x, f^x) \leqslant \varepsilon.$$

Since g^x is W-a.p. and ε was arbitrary, h^x is W-a.p.

We remark that if, in Proposition 2, x is u.a.p. and h is a characteristic function with $\int h d\mu > 0$, then $\{j \geqslant 1: h(T^j x) = 1\}$ is a typical uniform sequence in the sense of Brunel and Keane [4]. It is observed in [1] that such sequences $h(T^j x)$ are B-a.p.

The following example shows, in particular, that there exist W-a.p. 0-1 sequences which are not quasi-uniform limits of periodic points in $\{0, 1\}^{\mathbf{Z}}$. More specifically, there is a 0-1 sequence of the above type which is distant in the sense of D_W from any Toeplitz sequence.

EXAMPLE 3. Let (X, T_α) be an irrational rotation of the circle, A be the arc $[0, \pi)$, and $z \in X$ be fixed. By Proposition 2, the 0-1 sequence $x_j = \chi_A(T_\alpha^j z)$ is W-a.p. We shall prove $D_W(x, y) \geqslant 1/2$ for every 0-1 Toeplitz sequence y. Suppose the contrary for some y. Then there exists $L \geqslant 1$ such that the corresponding L-blocks of x and y always agree at more than a half of the coordinates. If we consider only sufficiently large j's, then $T_\alpha^j z$ never hits any of the end points of A. Let $B = y_i \dots y_{i+L-1}$ be an L-block in y positioned sufficiently far to the left. Since y is Toeplitz, B occurs in y with a period m. Among the numbers $T_\alpha^{mk} z$ $(k \geqslant 0)$ there exists one arbitrarily close to $-z$, so, for some k,

$$x_{mk+i+j} = \chi_A(T_\alpha^{mk+i+j} z) = 1 - \chi_A(T_\alpha^{i+j} z) = 1 - x_{i+j},$$

$j = 0, 1, \dots, L-1$. This shows that the blocks

$$C = x_i \dots x_{i+L-1} \quad \text{and} \quad S^{mk} C = x_{mk+1} \dots x_{mk+i+L-1}$$

disagree at each coordinate. On the other hand, B and $S^{mk} B$ coincide (in y). Consequently, the L-blocks $S^{mk} C$ and $S^{mk} B$ disagree at more than a half of the coordinates, a contradiction.

For the rest of the paper we consider spectral properties of W-a.p. points.

Let μ be an invariant measure for (X, T). We say that T has *discrete spectrum* on $L^2(\mu)$ if the eigenfunctions of the unitary operator $f \to f \circ T$ span a dense subspace of $L^2(\mu)$. If (X, T) is uniquely ergodic and $X = \bar{O}(x)$, then x is said to have *discrete spectrum* if T has discrete spectrum on $L^2(\bar{O}(x))$.

Our aim is to prove that W-a.p. points have discrete spectrum. For regular Toeplitz sequences this is Theorem 6 in [8]. Our result follows from the theory of sequences having a correlation and its applications to ergodic theory developed by Wiener and Wintner. In [17], they obtained a connection between ergodic systems with discrete spectrum and complex-valued B-a.p. functions. A convenient version is the following result due to Bellow and Losert:

($*$) *Let T be an ergodic invertible measure preserving transformation of a probability space. Then T has discrete spectrum on L^2 iff for every bounded measurable function g and almost every x the function $j \to g(T^j x)$ is B-a.p.* ([1], Theorem 3.22).

Now we have

LEMMA 5. *Let μ be an ergodic measure on a dynamical system (X, T). Then T has discrete spectrum on $L^2(\mu)$ iff almost every point in X is B-a.p.*

Proof. It suffices to consider a suitable countable set F of continuous (scalar) functions on X and apply $(*)$ along with Lemma 3 for B-a.p. functions (see Remark 1).

In view of Theorem 1, Lemmas 4 and 5 imply

THEOREM 2. *Every W-a.p. point has discrete spectrum.*

3. Morse shifts and Sturmian dynamical systems. We have shown that if x is W-a.p., then $\bar{O}(x)$ is uniquely ergodic and has discrete spectrum on L^2. The converse does not hold: a counterexample exists among the (generalized) Morse sequences (see [9] for definitions and notation). In fact, it has been shown by Lemańczyk [11] that a Morse sequence either is continuous (in the sense of [9], p. 351) or has discrete spectrum. Consequently, by [9], Theorem 9, there exist nonperiodic (strictly transitive) Morse sequences with discrete spectrum. On the other hand, no nonperiodic Morse sequence is W-a.p. This follows from Proposition 3 below.

Let (Y, ϱ) be a compact metric space. For any two L-blocks over Y we define

$$D(a_1 \ldots a_L, b_1 \ldots b_L) = L^{-1} \sum_{i=1}^{L} \varrho(a_i, b_i).$$

Consider the following property of x in $Y^{\mathbf{Z}}$:

(P) There exist $\varepsilon > 0$ and arbitrarily large natural numbers L such that x is a concatenation of certain L-blocks B_γ satisfying $D(B_\alpha, B_\beta) \geqslant \varepsilon$ for $B_\alpha \neq B_\beta$.

(The *concatenation* means $x = \ldots B_{\gamma_{-1}} B_{\gamma_0} B_{\gamma_1} \ldots$, where $B_{\gamma_k} = x_{kL} \ldots x_{(k+1)L-1}$.) If x is a Morse sequence, then $Y = \{0, 1\}$, $\varepsilon = 1$, $L = n_t$, $\gamma = 1, 2$, $B_1 = c^t$, $B_2 = \tilde{c}^t$.

PROPOSITION 3. *Let x satisfy* (P) *in the shift system $Y^{\mathbf{Z}}$. If x is W-a.p., then x is periodic.*

Proof. Choose L in (P) and $S \in \mathscr{S}$ such that $D_{S_L}(\xi_s, \xi) < \varepsilon/2$ for $s \in S$, where $\xi(j) = x_j$. By reducing S (mod L) we can find $s_1 \neq s_2$ in S with $s_1 = s_2$ (mod L), or

$$s_i = k_i L + t \quad (0 \leqslant t < L,\ i = 1, 2).$$

Since $D_{S_L}(\xi_{s_1}, \xi_{s_2}) < \varepsilon$, we have in particular

$$D(B_{\gamma_{k+k_1}}, B_{\gamma_{k+k_2}}) < \varepsilon \quad (k \in \mathbf{Z}),$$

where $B_{\gamma_k} = x_{kL} \ldots x_{(k+1)L-1}$. By (P), this implies $B_{\gamma_{k+k_1}} = B_{\gamma_{k+k_2}}$ $(k \in \mathbf{Z})$, whence $\xi_{s_1} = \xi_{s_2}$. Since $s_1 - s_2 \neq 0$, x is periodic.

Now we recall the definition of the Sturmian dynamical systems ([7], [5], 12.57–12.63, and [10]). Let A be a subset of the unit circle, T_α be an irrational rotation, $0 < \alpha < 1$, and $|z| = 1$. If the boundary of A has Lebesgue measure zero, then by Proposition 2 the sequence

$$x_j = \chi_A(T_\alpha^j z)$$

is W-a.p. (in fact, R-a.p. in the sense of [6]). In particular, the orbit closure $\bar{O}(x)$ in the shift space $(\{0, 1\}^{\mathbf{Z}}, S)$ is uniquely ergodic and has discrete spectrum. (For Sturmian systems the unique ergodicity has been proved in [10].)

If A is an arc $[u, v)$ of length $2\pi\alpha$ (i.e. of Lebesgue measure α), then $\bar{O}(x)$ is called a *Sturmian dynamical system*. It is known that $(\bar{O}(x), S^k)$ is then minimal for $k = 1, 2, \ldots$ ([7], Theorem 6.4). This implies that the S^k-orbit closures of x all coincide with $\bar{O}(x)$. Note that if π_r denotes the projection on the r-th coordinate, then the function

$$j \to \pi_r(S^{kj} x) = \chi_A(T_\alpha^{kj+r} z) = \chi_A\left(T_{k\alpha}^j (e^{2\pi i r\alpha} z)\right)$$

is also W-a.p., which by Lemma 3 implies that x is a W-a.p. point in $(\bar{O}(x), S^k)$. By Theorem 1 each of the systems $(\bar{O}(x), S^k)$, $k \geqslant 1$, is uniquely ergodic with the same invariant measure μ. The measure theoretic dynamical system $(\bar{O}(x), \mu, S)$ is therefore totally ergodic. Consequently, there are no roots of unity (different from 1) in the spectrum of the unitary operator defined by S on $L^2(\mu)$, that is to say we have proved

COROLLARY. *Each Sturmian dynamical system has irrational discrete spectrum.*

Now we shall prove a more general proposition which, in particular, applies to the 0-1 sequences generated by arbitrary Jordan measurable subsets of the unit circle. The proof is independent of [7].

Let z be a W-a.p. point in (X, T). Denote by λ the unique invariant measure on $\bar{O}(z)$. Now fix $h \in R(X, \lambda, Y)$, where Y is a metric compact space. We consider the sequence

$$x_j = h(T^j z) \quad (j \in \mathbf{Z})$$

as a point in the shift dynamical system $(Y^{\mathbf{Z}}, S)$. By Proposition 2 and Lemma 3, x is a W-a.p. point in $Y^{\mathbf{Z}}$.

PROPOSITION 4. *Let (X, d) be a compact metric space and $T, z, \lambda, Y, h, S, x$ be as above. Assume in addition that for every $k \geqslant 1$ the T^k-orbit $\{T^{kj} z \colon j \in \mathbf{Z}\}$ is dense in $\bar{O}(z)$. Then x has irrational discrete spectrum.*

Proof. As before, it suffices to prove that the measure theoretic system $(\bar{O}(x), \mu, S)$ is totally ergodic, where μ denotes the unique invariant measure on $\bar{O}(x)$ (Theorem 1).

First we prove that $\operatorname{supp}\mu$, the topological support of μ, is contained in all the S^k-orbit closures of x. Fix $w \in \operatorname{supp} \mu$ $(= \bar{O}(w))$ and pick a cylindrical neighborhood U_ε of w of the form

$$U_\varepsilon = \{u \in Y^{\mathbf{Z}}: |\varphi(u_i) - \varphi(w_i)| < \varepsilon, \ |i| < p\},$$

where φ: $Y \to \mathbf{R}$ is a continuous function, $t \in \mathbf{Z}$, and $p \geqslant 1$. Note that all such sets form a neighborhood base at w. We want to show that $S^{kj_0} x \in U_\varepsilon$ for some $j_0 \in \mathbf{Z}$. Since $\bar{O}(w)$ is a unique minimal orbit closure in $\bar{O}(x)$, it is not hard to see that

$$M = \{m: \ S^m x \in U_{\varepsilon/4}\} \in \mathscr{S},$$

so the gaps of M are bounded by, say, $r \geqslant 1$. Since the set of discontinuity points of $\varphi \circ h$ has λ measure zero, we can find two continuous real-valued functions f and g on X such that

$$f \leqslant \varphi \circ h \leqslant g \quad \text{and} \quad \int (g-f)\, d\lambda < \varepsilon/(4 \max(r, 2p)).$$

There must exist an $m_0 \in M$ such that

$$g(T^{m_0+i} z) - f(T^{m_0+i} z) < \varepsilon/4 \quad (|i| < p),$$

for otherwise

$$\int (g-f)\, d\lambda = \lim n^{-1} \sum_{j=0}^{n-1} (g-f)(T^j z) \geqslant \varepsilon/(4 \max(r, 2p))$$

(cf. the proof of Proposition 2). Now choose $\delta > 0$ such that the condition $d(T^{m_0} z, v) < \delta$ implies

$$|f(T^{m_0+i} z) - f(T^1 v)| < \varepsilon/4 \quad \text{and} \quad g(T^i v) - f(T^i v) < \varepsilon/4$$

for $|i| < p$. Given $k \geqslant 1$, we can find $j_0 \in \mathbf{Z}$ such that

$$d(T^{m_0} z, T^{kj_0} z) < \delta.$$

This gives

$$\begin{aligned} |\varphi(h(T^{kj_0+i} z)) - \varphi(h(T^{m_0+i} z))| &\leqslant g(T^i(T^{kj_0} z)) - f(T^i(T^{kj_0} z)) \\ &\quad + g(T^{m_0+i} z) - f(T^{m_0+i} z) + |f(T^{m_0+i} z) - f(T^i(T^{kj_0} z))| < 3\varepsilon/4 \end{aligned}$$

$(|i| < p)$. Since $m_0 \in M$, $S^{kj_0} x \in U_\varepsilon$.

By using the projections π_i as before, we infer that x is W-a.p. in each $(\bar{O}(x), S^k)$. Also, for $k \geqslant 1$ the S^k-orbit closure of x contains $\operatorname{supp} \mu$, so we deduce from Theorem 1 that μ is the unique invariant measure for all $(\bar{O}(x), S^k)$. This implies that the measure theoretic dynamical system $(\bar{O}(x), \mu, S)$ is totally ergodic.

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