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TRANSFORMATION PROPERTIES OF THE GENERALIZED MUIRHEAD OPERATORS

BY

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For $d, \tau, \gamma \in \mathbf{C}$ the *generalized Muirhead operators* are

$$(1)\qquad \begin{aligned}\mathcal{D}_i^{(\tau,\gamma)} &= x_i(1-x_i)\frac{\partial^2}{\partial x_i^2}\\ &+\left\{\gamma-\frac{d}{2}(r-1)-\left[\tau+1-\frac{d}{2}(r-1)\right]x_i+\frac{d}{2}\sum_{j\neq i}\frac{x_i(1-x_i)}{x_i-x_j}\right\}\frac{\partial}{\partial x_i}\\ &-\frac{d}{2}\sum_{j\neq i}\frac{x_j(1-x_j)}{x_i-x_j}\cdot\frac{\partial}{\partial x_j}\qquad (1\le i\le r).\end{aligned}$$

For $\alpha, \beta \in \mathbf{C}$, the system of partial differential equations

$$(2)\qquad (\mathcal{D}_i^{(\alpha+\beta,\gamma)}-\alpha\beta)y=0 \qquad (1\le i\le r)$$

specializes for $r = 1$ to the classical hypergeometric equation. For $r > 1$ and $d = 1$ it was studied in [5], for general d in [4]. We write X for $(x_1, \ldots, x_r)$. When $d > 0$, and $d/2-\gamma$ is not a natural number, a generalized hypergeometric series $F^{(d)}(\alpha, \beta; \gamma; X)$, convergent for $|x_i| < 1$ $(1 \le i \le r)$, can be constructed; it is the unique solution of the system (2) subject to the conditions (i) y is symmetric in $x_1, \ldots, x_r$, (ii) y is analytic at $(0, \ldots, 0)$, (iii) $y(0, \ldots, 0) = 1$. For $d = 1$ this result is due to Muirhead [5]; for general d it was claimed in [4], but the proof had a gap due to the difficulty of seeing whether the generalized binomial coefficients in the sense of Muirhead are independent of r (as they happen to be when $d = 1$). A complete proof for general d was then given by Z. Yan [6].

The function F is a hypergeometric function in the sense of Heckman and Opdam [2]; it corresponds to the root system BC_r and an eigenvalue parameter on the diagonal edge of the Weyl chamber. This is a non-trivial fact, proved for $r = 2$ by Yan [6], and for general r by R. Beerends and E. Opdam (not yet published).

By general theory the space of solutions of the system (2) is 2^r-dimensional; it is of interest to study this space in detail. As a first step, we want here to find solutions that are symmetric in $x_1, \ldots, x_r$; these form an invariant subspace for a fairly large ("diagonal") subgroup of the monodromy group.

One can find such solutions by generalizing Kummer's twenty-four integrals of the classical hypergeometric equation. The key for finding these is to describe the action of fractional linear transformations on the hypergeometric equation. We are going to do this for the system (2) in Proposition 1. After this, Kummer's table can, in principle, be computed. To make the computation manageable we first find the appropriate generalization of the *normal form* of the hypergeometric equation (cf. e.g. [3], p. 31). It is not immediately obvious what this generalization has to be; a transformation eliminating all the first derivatives as in the one-variable case does not seem possible here. Nonetheless, we find a transformed equation on which the signed permutations of the parameters act in a simple way, just as in the classical case. The generalized Kummer table then follows easily.

If T is a diffeomorphism, for functions f we write $T^*f = f \circ T$, and for linear differential operators $\mathcal{D}$,

$$T_*\mathcal{D} = T^{*-1} \circ \mathcal{D} \circ T^* .$$

(This amounts to "changing the variable" from X to $T(X)$ in the operator.) We note that if $\mathcal{D}$ is a 0th order operator, i.e. multiplication by a function φ, then $T_*\mathcal{D}$ is multiplication by $\varphi \circ T^{-1}$.

Let G be the group of fractional linear transformations T in one variable which permute the points $0, 1, \infty$. We shall consider G as a group acting on $\mathbf{C}^r$ by the transformations $X \mapsto T(X)$, where $T(X) = (T(x_1), \ldots, T(x_r))$. We list the elements of G as $e = T_0, T_1, \ldots, T_5$, and to each of these elements we associate a matrix as follows:

$$\tilde{T}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{T}_1 = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{T}_2 = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{T}_3 = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{T}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{T}_5 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

These matrices form a representation of G (which is, of course, isomorphic with the symmetric group over three elements). The action of $T \in G$ on a single variable ξ is then given as $(a\xi + b)(c\xi + d)^{-1}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the left upper corner of $\tilde{T}$. To each $T \in G$ we also associate the linear function ψ_T given, on $\mathbf{C}$, by

$$\psi_T(\xi) = c\xi + d \tag{3}$$

where c, d are as above.

We define $q = 1 + d(r-1)/2$ and we also define an action of G on $\mathbf{C}^2$ by

$$T_*(\tau,\gamma) = (\tau',\gamma')$$

when $(\gamma',\tau',q) = (\gamma,\tau,q)^t\widetilde{T}$, where ${}^t\widetilde{T}$ denotes the transposed matrix.

Now we can state our first result.

PROPOSITION 1. *The generalized Muirhead operators transform under each $T \in G$ by*

$$T_*\mathcal{D}_i^{(\tau,\gamma)} = \psi_{T^{-1}}(x_i)\mathcal{D}_i^{T_*(\tau,\gamma)}. \tag{4}$$

(*The factor on the right stands for a multiplication operator.*)

It seems easiest to prove this result by first looking at the operators $\mathcal{E}_i^{(\tau,\gamma)}$ obtained by dividing $\mathcal{D}_i^{(\tau,\gamma)}$ by $x_i(1-x_i)$:

$$\mathcal{E}_i^{(\tau,\gamma)} = \frac{\partial^2}{\partial x_i^2} + \left(\frac{\gamma - \frac{d}{2}(r-1)}{x_i} + \frac{\tau+1-\gamma}{x_i-1}\right)\frac{\partial}{\partial x_i} + \frac{d}{2}\sum_{j\neq i}\frac{1}{x_i-x_j}\left(\frac{\partial}{\partial x_i} - \frac{x_j(1-x_j)}{x_i(1-x_i)}\frac{\partial}{\partial x_j}\right). \tag{5}$$

Writing $g(\xi) = \xi(1-\xi)$, a small computation shows that

$$g\circ T = (T')^2\psi_T g \tag{6}$$

where T' denotes the derivative of T. Hence (4) is equivalent to

$$T_*\mathcal{E}_i^{(\tau,\gamma)} = (T^{-1}(x_i)')^{-2}\mathcal{E}_i^{T_*(\tau,\gamma)} \qquad (1\le i\le r). \tag{7}$$

Since both sides describe an action of G, it is enough to prove (7) for a set of generators of G, e.g. for T_1 and T_2. This is a relatively short computation, left to the reader.

Remark. Written out in detail, the Proposition amounts to the five equalities

$$\begin{aligned}
(T_1)_*\mathcal{D}_i^{(\gamma,\tau)} &= -x_i\mathcal{D}_i^{(q-\gamma,q-\tau)},\\
(T_2)_*\mathcal{D}_i^{(\gamma,\tau)} &= \mathcal{D}_i^{(\tau,\tau-\gamma+q)},\\
(T_3)_*\mathcal{D}_i^{(\gamma,\tau)} &= (x_i-1)\mathcal{D}_i^{(q-\gamma,\tau-\gamma+q)},\\
(T_4)_*\mathcal{D}_i^{(\gamma,\tau)} &= (x_i-1)\mathcal{D}_i^{(\gamma-\tau,\gamma)},\\
(T_5)_*\mathcal{D}_i^{(\gamma,\tau)} &= -x_i\mathcal{D}_i^{(\gamma-\tau,q-\tau)}.
\end{aligned}$$

The ambitious sounding formulations (4), (7) serve for making it clear that it is enough to make the computation for T_1 and T_2.

To proceed, we introduce the notations $X^\rho = \prod x_i^\rho$, $(1-X)^\sigma = \prod(1-x_i)^\sigma$. Similarly we shall write $\psi_T(X) = \prod\psi_T(x_i)$. A computation, made easy by using logarithmic differentiation, gives

$$X^{-\rho}(1-X)^{1-\sigma}\mathcal{E}_i^{(\tau,\gamma)}\circ X^\rho(1-X)^\sigma = \mathcal{E}_i^{(\tau+2(\sigma+\rho),\gamma+2\rho)} + \left[\frac{\rho(\rho+\gamma-q)}{x_i^2} + \frac{\sigma(\sigma+\tau-\gamma)}{(x_i-1)^2} + \frac{\rho(\sigma+\tau-\gamma+q)+\sigma(\rho+\gamma)}{x_i(x_i-1)}\right]. \tag{8}$$

(We used a small circle to indicate that $X^\rho(1-X)^\sigma$ is regarded as a multiplication operator, to be followed by $\mathcal{E}_i^{(\tau,\gamma)}$.)

The system (2) can equivalently be written in the form

$$(\mathcal{E}_i^{(\alpha+\beta,\gamma)} - \alpha\beta/g(x_i))y = 0 \qquad (1 \le i \le r).$$

We transform it by choosing

$$\rho = \tfrac{1}{2}(2A-\gamma), \quad \sigma = \tfrac{1}{2}(\gamma-\alpha-\beta-A-1), \tag{9}$$

where, as an abbreviation, we wrote

$$A = \tfrac{d}{6}(r-1).$$

From (8) we get

$$\begin{aligned} X^{-\rho}(1-X)^{-\sigma}(\mathcal{E}_i^{(\alpha+\beta,\gamma)} - \alpha\beta/g(x_i))\circ X^\rho(1-X)^\sigma \\ = \mathcal{E}_i^{(A-1,2A)} + A/g(x_i) - R(\lambda,\mu,\nu;x_i) \qquad (1 \le i \le r) \end{aligned} \tag{10}$$

where we used the abbreviations

$$\begin{aligned} \lambda &= q-\gamma, \\ \nu &= \gamma-\alpha-\beta, \\ \mu &= \beta-\alpha, \end{aligned} \tag{11}$$

and

$$R(\lambda,\mu,\nu;\xi) = -\frac{1}{4}\left[\frac{(A+1)^2-\lambda^2}{\xi^2} + \frac{(A+1)^2-\nu^2}{(\xi-1)^2} - \frac{(A+1)^2-\lambda^2+\mu^2-\nu^2}{\xi(\xi-1)}\right].$$

We denote by $\mathcal{N}_i^{(\lambda,\mu,\nu)}$ the right hand side of (10). Its first two terms are independent of α,β,γ, and its third term transforms under G by

$$R(\lambda,\mu,\nu;T(\xi)) = \frac{1}{T'(\xi)^2}R(\widehat{T}^{-1}(\lambda,\mu,\nu);\xi), \tag{12}$$

where $\widehat{T}$ stands for the same permutation of λ,μ,ν as the one induced by T on $0,\infty,1$ (in this order). To prove (12) it is again enough to verify it for T_1 and T_2.

The system $\mathcal{N}_i^{(\lambda,\mu,\nu)}Y = 0$ $(1 \le i \le r)$ is from our point of view the proper "normal form" of the system (2). Just as in the one-variable case, it has the following crucial property.

PROPOSITION 2. *For each $T \in G$ and $1 \le i \le r$,*

$$T_*(\psi_T(X)^{-1}\mathcal{N}_i^{(\lambda,\mu,\nu)} \circ \psi_T(X)) = \frac{1}{(T^{-1}(x_i)')^2}\mathcal{N}_i^{\widehat{T}(\lambda,\mu,\nu)} .$$

The proof for general r is only a little more troublesome than for $r = 1$. Since $\psi_T(X) = \pm X^{\rho_T}(1-X)^{\sigma_T}$ with ρ_T and σ_T equal to 0 or 1, the formula (8) can be used to find the ψ_T-conjugate of $\mathcal{N}_i^{(\lambda,\mu,\nu)}$. Then one uses (7), (12) and a little computation to finish the proof. ∎

One can now construct the general version of Kummer's twenty-four integrals of the hypergeometric equation (cf. [3], p. 31). In fact, let α, β, γ be fixed. By (9) and (11) we compute the corresponding ρ, σ and λ, μ, ν. For each T in G we take $\widehat{T}^{-1}(\lambda,\mu,\nu)$ and put $\pm$ signs in front of the images of λ and μ in all possible ways. This gives 24 triples (λ', μ', ν'). For each we calculate the corresponding α', β', γ' by (11) and ρ', σ' by (9). Then, by Proposition 2,

$$\psi_T(X)X^{\rho}(1-X)^{\sigma}T(X)^{-\rho'}(1-T(X))^{-\sigma'}F(\alpha', \beta'; \gamma'; T(X))$$

is a solution of the system (2). The computation is easy and gives the following result.

PROPOSITION 3. *Each entry in Kummer's table* ([3], p. 33, or [1], p. 105), *after changing* 1 *and* 2 *to q and $2q$ everywhere in the exponents and parameters, and interpreting powers of X, $1-X$ as described above, gives a solution of the system* (2).

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