COLLOQUIUM MATHEMATICUM

VOL. LVI | 1988 | FASC. 1

WEAK DIFFERENCE PROPERTIES OF HIGHER ORDERS FOR THE CLASS $L_p(G)$

BY

ZBIGNIEW GAJDA (KATOWICE)

1. Introduction. In this paper, $(G, +)$ denotes a locally compact Abelian group and $(B, \|\cdot\|)$ stands for a Banach space. Many authors considered the following problem:

Suppose that $f\colon G \to B$ and for each $h \in G$ the function $\Delta_h f$ defined by

$$\Delta_h f(x) := f(x+h) - f(x), \qquad x \in G,$$

belongs to a class $\mathscr{F} \subset B^G$. What can be said about the function f?

In [2]–[4] and [12] it was shown, under various assumptions on G and B, that for a large number of classes $\mathscr{F}$ (among others for the class of all continuous functions) f may be represented in the form

(1) $f = \gamma + g$, where $g \in \mathscr{F}$ and $\gamma\colon G \to B$ is an additive function, i.e., $\gamma(x+y) = \gamma(x) + \gamma(y)$, $x, y \in G$.

It was pointed out (cf. [5] and [12]) that for the class $L_p(G)$ with $1 \leqslant p \leqslant \infty$, instead of (1) the following decomposition is appropriate:

(2) $f = \gamma + g + s$, where $\gamma\colon G \to B$ is additive, g is an L_p-function and $s\colon G \to B$ is such that, for each $h \in G$, $\Delta_h s = 0$ almost everywhere on G.

The class $\mathscr{F} \subset B^G$ is said to have the *difference property* (resp., the *weak difference property*) if any function $f\colon G \to B$ with all differences $\Delta_h f$ belonging to $\mathscr{F}$ admits decomposition (1) (resp., (2)).

Kemperman [11] initiates the study of the following more general question: describe all functions $f\colon G \to B$ satisfying the condition

$$(3) \qquad \Delta_h^n f \in \mathscr{F} \quad \text{for any fixed } h \in G,$$

where $\Delta_h^n := \Delta_{h\ldots h}$ (n times) denotes the n-th order difference operator defined recurrently as follows:

$$\Delta_h^0 f := f, \quad \Delta_h f(x) := f(x+h) - f(x), \quad x, h \in G,$$
$$\Delta_{h_1\ldots h_n} f := \Delta_{h_n}(\Delta_{h_1\ldots h_{n-1}} f), \quad h_1, \ldots, h_n \in G.$$

Kemperman has proved[1] that for various classes $\mathscr{F}$ any function f fulfilling (3) may be written in the form

(4) $f = \gamma + g$ for a certain $g \in \mathscr{F}$ and for an n-th order polynomial function $\gamma\colon G \to B$.

Recall that a function $\gamma\colon G \to B$ is said to be a *polynomial function of n-th order* if and only if

$$\Delta_h^{n+1} \gamma(x) = 0 \quad \text{for all } x, h \in G$$

or, equivalently ([7], Theorem 3),

$$\gamma = \gamma_0 + \gamma_1 + \ldots + \gamma_n, \tag{5}$$

where γ_0 is a constant and there exist k-additive and symmetric functions $\tilde{\gamma}_k\colon G^k \to B$ such that

$$\gamma_k(x) = \tilde{\gamma}_k(x, \ldots, x), \quad x \in G,\ k = 1, \ldots, n.$$

γ_k is sometimes called a *monomial function of k-th order.*

If for every function $f\colon G \to B$ condition (3) implies (4) we say that the class $\mathscr{F}$ has the *difference property of n-th order*. In our previous paper [9] we have obtained difference properties of any orders for the class of all Banach-valued continuous functions defined on an arbitrary locally compact Abelian group and for the class of all Banach-valued Riemann integrable functions on a compact second countable Abelian group.

In the present paper we introduce the concept of weak difference property of higher orders.

DEFINITION. A class $\mathscr{F} \subset B^G$ is said to have the *weak difference property of n-th order* if and only if for any function f satisfying (3) there exist an n-th order polynomial function $\gamma\colon G \to B$, a function $g \in \mathscr{F}$ and a function $s\colon G \to B$ such that, for each $h \in G$, $\Delta_h^n s = 0$ almost everywhere and $f = \gamma + g + s$.

Remark 1. In the above definition we may require γ to be a monomial function of n-th order. Indeed, if γ has representation (5), we put

$$\gamma^* := \gamma_n, \quad s^* := s + \gamma_0 + \gamma_1 + \ldots + \gamma_{n-1}.$$

Then $f = \gamma^* + g + s^*$ and $\Delta_h^n s^* = \Delta_h^n s = 0$ almost everywhere.

The main purpose of this paper is to prove weak difference properties of higher orders for the class of all L_p-functions and, as a simple consequence, for Orlicz spaces, generalizing well-known results of Carroll and Koehl [6] and Woyczyński [16].

2. Preliminary remarks and results. Let $\boldsymbol{R}$ denote the set of all real numbers and let $L_1^{\mathrm{loc}}(\boldsymbol{R})$ be the class of all functions $f\colon \boldsymbol{R} \to \boldsymbol{R}$ which are

[1] Unpublished results; private information.

Lebesgue integrable over each finite interval $[a, b]$. This class has no difference properties of any positive order. We shall show even something more.

It is well known (cf. [15]) that there exists a non-measurable function $\varphi\colon \boldsymbol{R} \to \boldsymbol{R}$ taking only two values (0 and 1) and such that, for every $h \in \boldsymbol{R}$, $\Delta_h \varphi(x) = 0$ holds except for countably many values of $x \in \boldsymbol{R}$.

Given a positive integer n we choose an arbitrary polynomial P of degree $n-1$ and put

$$f(x) := P(x) \cdot \varphi(x), \qquad x \in \boldsymbol{R}.$$

The function f has the following properties:

(i) for each $h \in \boldsymbol{R}$, $\Delta_h^n f = 0$ almost everywhere;

(ii) for any integer k with $0 \leqslant k \leqslant n-1$ and for any $h \in \boldsymbol{R} \setminus \{0\}$, $\Delta_h^k f$ is non-measurable;

(iii) f is bounded on every bounded interval.

Property (iii) is obvious. To prove (ii) and (i) we shall show inductively that for any integer k with $0 \leqslant k \leqslant n$ and each $h \in \boldsymbol{R}$ we have

$$\Delta_h^k f(x) = \varphi(x) \cdot \Delta_h^k P(x) \quad \text{for almost all } x \in \boldsymbol{R}. \tag{6}$$

For $k = 0$, (6) is satisfied by the definition of f. Suppose (6) holds for some k $(0 \leqslant k < n)$. Then, using properties of φ, for almost all $x \in \boldsymbol{R}$ we have

$$\begin{aligned}
\Delta_h^{k+1} f(x) &= \Delta_h \Delta_h^k f(x) = \Delta_h\big(\varphi(x) \cdot \Delta_h^k P(x)\big) \\
&= \varphi(x+h) \cdot \Delta_h^k P(x+h) - \varphi(x) \cdot \Delta_h^k P(x) \\
&= \Delta_h \varphi(x) \cdot \Delta_h^k P(x+h) + \varphi(x) \cdot \Delta_h^{k+1} P(x) = \varphi(x) \cdot \Delta_h^{k+1} P(x).
\end{aligned}$$

The induction completes the proof of (6).

Since $\Delta_h^n P = 0$, from (6) applied for $k = n$ we derive that $\Delta_h^n f = 0$ almost everywhere, which proves (i).

If $k \leqslant n-1$ and $h \in \boldsymbol{R} \setminus \{0\}$, then $\Delta_h^k P$ is a non-zero polynomial of degree $n-1-k$. By virtue of (6), adding the roots of the polynomial $\Delta_h^k P$ to the exceptional set of values of x, we conclude that

$$\varphi(x) = \frac{\Delta_h^k f(x)}{\Delta_h^k P(x)}$$

holds for almost all $x \in \boldsymbol{R}$. If $\Delta_h^k f$ were a measurable function, so would be φ, a contradiction. Consequently, (ii) is proved.

Evidently, for each $h \in \boldsymbol{R}$ the function $\Delta_h^n f$ is Lebesgue integrable over any interval, and putting $\gamma := 0$, $g := 0$, $s := f$, one may represent f in the form $f = \gamma + g + s$ with a monomial function γ, an integrable function g and a function s having all n-th differences equal to zero almost everywhere. We shall show, however, that f does not admit any decomposition of the form

$$f = \gamma^* + g^* + s^*, \tag{7}$$

where γ^* is a polynomial function, g^* is a measurable function and for a certain integer k such that $0 \leqslant k \leqslant n-1$ all k-th differences of s^* vanish almost everywhere. Suppose to the contrary that there exist functions γ^*, g^*, and s^* having the above properties and that (7) holds true. Then, choosing an $h \in \boldsymbol{R} \setminus \{0\}$, we can write

$$\Delta_h^k \gamma^* = \Delta_h^k f - \Delta_h^k g^* - \Delta_h^k s^* = \Delta_h^k f - \Delta_h^k g^* \text{ almost everywhere.}$$

Since $\Delta_h^k g^*$ is measurable, (ii) implies that $\Delta_h^k \gamma^*$ is a non-measurable function. Moreover, (iii) together with the fact that $\Delta_h^k g^*$ being measurable is bounded on a set of positive measure guarantees the boundedness of $\Delta_h^k \gamma^*$ on a set of positive measure. On the other hand, $\Delta_h^k \gamma^*$ as a polynomial function bounded on a set of positive measure must be continuous (see, e.g., [10], Theorem 6): a contradiction.

In particular, the above considerations disprove the difference property of n-th order for the class $L_1^{\mathrm{loc}}(\boldsymbol{R})$ as well as for the class of all Lebesgue measurable functions from $\boldsymbol{R}$ into $\boldsymbol{R}$. However, from our further results the weak difference property of an arbitrary order for the class $L_1^{\mathrm{loc}}(\boldsymbol{R})$ follows easily.

Now, we give some results which will be useful in the sequel. The following lemma may be found in [1] (p. 72 and p. 102, Exercise 1):

LEMMA 1. *Let $(G, +)$ be a compact Abelian group and let μ be the normed Haar measure on G. Every continuous epimorphism $T\colon G \to G$ of the group G preserves the measure μ, i.e. $\mu(T^{-1}(A)) = \mu(A)$ for any Borel set $A \subset G$.*

COROLLARY 1. *If $(G, +)$ is a compact Abelian group in which the division by an integer k is (possibly not uniquely) performed, then the transformation $T\colon G \to G$ determined by $T(x) := k \cdot x$, $x \in G$, preserves the normed Haar measure on G.*

In what follows we always assume that G is compact. For a function $f\colon G \to B$ the symbol $[f]$ will stand for the class of all functions which coincide with f almost everywhere in the sense of the Haar measure. Throughout this paper, $L_p(G, B)$ is regarded as a Banach space of all classes $[f]$ determined by strongly measurable functions $f\colon G \to B$ equipped with the norm

$$\|[f]\|_p := \begin{cases} \left(\int \|f(x)\|^p \, d\mu(x)\right)^{1/p} < \infty & \text{if } 1 \leqslant p < \infty, \\ \operatorname{ess\,sup} \|f\| & \text{if } p = \infty. \end{cases}$$

According to the notation used in [12] we precede the integral sign of a function $f\colon G \to B$ (a function $F\colon G \to L_p(G, B)$) with B (resp., with L_p). Consequently, $B\int f(x)\, d\mu(x)$ denotes an element of the space B, and $L_p \int F(x)\, d\mu(x)$ is an element of the space $L_p(G, B)$.

Given an $x \in G$ we denote by T_x the translation operator defined by

$$T_x f(y) := f(x+y), \qquad y \in G,\ f \in B^G.$$

LEMMA 2. *Let $(G, +)$ be a compact Abelian group divisible (not necessarily uniquely) by an integer $k \geqslant 1$. If $[f] \in L_p(G, B)$, where $1 \leqslant p < \infty$, then*

$$L_p \int [T_{kx} f]\, d\mu(x) = \left[B \int f(x)\, d\mu(x)\right].$$

Proof. Corollary 1 implies

$$L_p \int [T_{kx} f]\, d\mu(x) = L_p \int [T_x f]\, d\mu(x).$$

To complete the proof it remains to use Koehl's Lemma 3.2 from [12].

3. Main results.

THEOREM 1. *Let $1 \leqslant p \leqslant \infty$ and let $(G, +)$ be a compact Abelian group in which the division by 2, 3 up to n is (not necessarily uniquely) performed. Let $f\colon G \to B$ be a function such that for each $h \in G$ we have $[\Delta_h^n f] \in L_p(G, B)$. Then $f = g + \gamma + s$, where $[g] \in L_p(G, B)$, $\gamma\colon G \to B$ is a monomial function of n-th order, and $s\colon G \to B$ is such that, for each $h \in G$, $[\Delta_h^n s] = 0$.*

Proof. We distinguish two cases.

Case 1. $1 \leqslant p < \infty$.

Define a function $F\colon G \to L_p(G, B)$ by

$$F(h) := [\Delta_h^n f], \qquad h \in G.$$

Since we deal with two types of difference operators, the first one defined on functions with values in the space B and the other on functions assuming values in the space $L_p(G, B)$, we distinguish the latter by the use of boldface letters. Moreover, we put

$$\boldsymbol{\Delta}_h^n [f] := [\Delta_h^n f] \quad \text{and} \quad \mathbf{T}_x [f] := [T_x f].$$

It is easy to check that the above operations are correctly defined. Now,

$$\begin{aligned}
\boldsymbol{\Delta}_h^n F(x) &= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} F(x+jh) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} [\Delta_{x+jh}^n f] \\
&= \left[\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} T_{k(x+jh)} f \right] \\
&= \left[\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} T_{kx} \left(\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} T_{jkh} f \right) \right] \\
&= \left[\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} T_{kx} \Delta_{kh}^n f \right] = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \mathbf{T}_{kx} F(kh).
\end{aligned}$$

For each integer k with $0 \leqslant k \leqslant n$, the transformation

$$G \ni x \to T_{kx} F(kh) \in L_p(G, B)$$

is continuous (cf. [14], 1.1.5 and the remark made at the beginning of the proof of Lemma 3.2 from [12]). Thus $\Delta_h^n F$ is continuous for each fixed $h \in G$.

By Theorem 2 from [9] we have

$$F = H + \Gamma_1 + \ldots + \Gamma_n, \tag{8}$$

where H: $G \to L_p(G, B)$ is a continuous function, and Γ_k: $G \to L_p(G, B)$ is a monomial function of k-th order for $k = 1, \ldots, n$.

Using the notation

$$H_n := F, \quad H_k := F - \Gamma_{k+1} - \ldots - \Gamma_n, \ k = 0, 1, \ldots, n-1,$$

we have

$$H_0 = H,$$

(9) for each $h \in G$, $\Delta_h^k H_k$ is a continuous function for $k = 1, \ldots, n$,

$$\Gamma_k(h) = \frac{1}{k!} L_p \int \Delta_h^k H_k(x) \, d\mu(x), \quad h \in G, \ k = 1, \ldots, n. \tag{10}$$

Indeed, from (8) we infer that

$$\Delta_h^n F = \Delta_h^n H + n! \, \Gamma_n(h)$$

and

$$\Delta_h^k F = \Delta_h^k H + k! \, \Gamma_k(h) + \Delta_n^k(\Gamma_{k+1} + \ldots + \Gamma_n), \quad k = 1, \ldots, n-1.$$

Consequently, we get

$$\Delta_h^k H_k = \Delta_h^k H + k! \, \Gamma_k(h), \quad h \in G, \ k = 1, \ldots, n. \tag{11}$$

Now, (9) follows immediately by the continuity of H. Moreover, since H being continuous is integrable on G and since μ is a translation invariant measure, we obtain

$$\begin{aligned} L_p \int \Delta_h^k H(x) \, d\mu(x) & \\ &= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} L_p \int H(x+jh) \, d\mu(x) \\ &= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} L_p \int H(x) \, d\mu(x) = 0, \quad h \in G, \ k = 1, \ldots, n. \end{aligned}$$

This together with (11) yields

$$L_p \int \Delta_h^k H_k(x) \, d\mu(x) = L_p \int k! \, \Gamma_k(h) \, d\mu(x) = k! \, \Gamma_k(h),$$

$h \in G$, $k = 1, \ldots, n$, which completes the proof of (10).

Put

$$\gamma(h) := \frac{1}{n!} B \int \Delta_h^n f(x)\, d\mu(x), \qquad h \in G,$$

$$[g] := (-1)^n L_p \int H(x)\, d\mu(x).$$

Then $[g] \in L_p(G, B)$ and it is easy to check that γ is a monomial function of n-th order. We put $s := f - \gamma - g$. Using the commutativity of the difference operators we compute

$$\begin{aligned}\Delta_h^n(\Delta_x^k F(y)) &= \Delta_h^n\Big(\sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} F(y+jx)\Big)\\ &= \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j}\Delta_h^n[\Delta_{y+jx}^n f] = \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j}\Delta_{y+jx}^n[\Delta_h^n f]\\ &= \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j}\Delta_{y+jx}^n F(h), \qquad h, x, y \in G,\ k = 1, \dots, n.\end{aligned}$$

Since, for any fixed x, $h \in G$ and $j = 0, 1, \dots, k$, the mapping

$$G \ni y \to \Delta_{y+jx}^n F(h) = \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} T_{iy+ijx} F(h) \in L_p(G, B)$$

is continuous, and therefore integrable over G, we obtain

(12) $$\begin{aligned}&L_p \int \Delta_h^n(\Delta_x^k F(y))\, d\mu(y)\\ &= \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} L_p \int \Delta_{y+jx} F(h)\, d\mu(y)\\ &= \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} L_p \int \Delta_y F(h)\, d\mu(y) = 0, \qquad x, h \in G,\ k = 1, \dots, n.\end{aligned}$$

Now we are going to show inductively that

(13) $$\Delta_h^n \Gamma_k(x) = 0 \quad \text{for } h, x \in G,\ k = 1, \dots, n.$$

For, let us first note that Δ_h^n is a linear bounded operator in $L_p(G, B)$. This is readily seen from the formula

$$\Delta_h^n = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} T_{jh}$$

and from the fact that

$$\|T_{jh}[f]\|_p = \|[f]\|_p \quad \text{for any } [f] \in L_p(G, B),$$

which implies the boundedness of T_{jh}, $j = 0, 1, \dots, n$. Therefore, if Φ: $G \to L_p(G, B)$ is an integrable function, then (cf., e.g., [8], Theorem III. 6.20)

$$\Delta_h^n L_p \int \Phi\, d\mu = L_p \int \Delta_h^n \circ \Phi\, d\mu.$$

Thus, by (12) and (10), we have

$$\Delta_h^n \Gamma_n(x) = \frac{1}{n!} \Delta_h^n L_p \int \Delta_x^n F(y)\, d\mu(y)$$

$$= \frac{1}{n!} L_p \int \Delta_h^n (\Delta_x^n F(y))\, d\mu(y) = 0, \qquad x, h \in G,$$

which proves (13) for $k = n$. Suppose that (13) holds true for each of $k+1, \ldots, n$ with some $k \geqslant 1$. Then

$$\Delta_h^n \big(\Delta_x^k (\Gamma_{k+1}(y) + \ldots + \Gamma_n(y))\big)$$

$$= \Delta_h^n \Big(\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (\Gamma_{k+1}(y+jx) + \ldots + \Gamma_n(y+jx)) \Big)$$

$$= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (\Delta_h^n \Gamma_{k+1}(y+jx) + \ldots + \Delta_h^n \Gamma_n(y+jx)) = 0,$$

$$h, x, y \in G.$$

Hence, using again (12) and (10), we derive

$$\Delta_h^n \Gamma_k(x) = \Delta_h^n \Big(\frac{1}{k!} L_p \int \Delta_x^k H_k(y)\, d\mu(y) \Big)$$

$$= \frac{1}{k!} \Delta_h^n \big(L_p \int \Delta_x^k (F(y) - \Gamma_{k+1}(y) - \ldots - \Gamma_n(y))\, d\mu(y) \big)$$

$$= \frac{1}{k!} L_p \int \big(\Delta_h^n (\Delta_x^k F(y)) - \Delta_h^n \big(\Delta_x^k (\Gamma_{k+1}(y) + \ldots + \Gamma_n(y))\big)\big)\, d\mu(y)$$

$$= \frac{1}{k!} L_p \int \Delta_h^n (\Delta_x^k F(y))\, d\mu(y) = 0, \qquad h, x \in G.$$

The induction ensures that (13) holds for every integer k with $1 \leqslant k \leqslant n$.

In the following calculations we apply (13) and Lemma 2:

$$[\Delta_h^n g] = (-1)^n \Delta_h^n L_p \int H(x)\, d\mu(x)$$

$$= (-1)^n L_p \int \Delta_h^n H(x)\, d\mu(x)$$

$$= (-1)^n L_p \int (\Delta_h^n F(x) - \Delta_h^n \Gamma_1(x) - \ldots - \Delta_h^n \Gamma_n(x))\, d\mu(x)$$

$$= (-1)^n L_p \int \Delta_h^n F(x)\, d\mu(x) = (-1)^n L_p \int \Delta_x^n F(h)\, d\mu(x)$$

$$= (-1)^n L_p \int \Big(\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} T_{kx} F(h) \Big) d\mu(x)$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_p \int T_{kx} F(h)\, d\mu(x)$$

$$= F(h) + \sum_{k=1}^{n} (-1)^k \binom{n}{k} [B \int \Delta_h^n f(x) d\mu(x)]$$

$$= F(h) - [B \int \Delta_h^n f(x) d\mu(x)] = F(h) - [n!\, \gamma(h)], \quad h \in G.$$

Finally,

$$[\Delta_h^n s] = [\Delta_h^n f] - [\Delta_h^n \gamma] - [\Delta_h^n g]$$
$$= F(h) - [n!\, \gamma(h)] - [\Delta_h^n g] = 0, \quad h \in G,$$

which completes the proof of our theorem in the case where $1 \leqslant p < \infty$.

Case 2. $p = \infty$.

Let us define the function $F: G \to L_\infty(G, B)$ by the same formula as in Case 1. Fix an arbitrary $p \in [1, \infty)$. It has been shown previously that, for each $h \in G$, $\Delta_h^n F$ is continuous if it is regarded as a function from G into $L_p(G, B)$. Hence

$$F = H + \Gamma_1 + \ldots + \Gamma_n,$$

where, with the aid of the notation

$$H_n := F, \quad H_k := F - \Gamma_{k+1} - \ldots - \Gamma_n, \ k = 0, 1, \ldots, n-1,$$

we have

(14) $H_0 = H: G \to L_p(G, B)$ is continuous;

(15) for any integer k ($1 \leqslant k \leqslant n$) and for each $h \in G$ the function $\Delta_h^k H_k: G \to L_p(G, B)$ is continuous;

(16) $$\Gamma_k(h) = \frac{1}{k!} L_p \int \Delta_h^k H_k(x) d\mu(x), \quad h \in G.$$

All these results are derived in the proof of Case 1.

It is clear that if a function

$$\Phi: G \to \bigcap_{1 \leqslant p < \infty} L_p(G, B)$$

is integrable over G as a function from G into the space $L_p(G, B)$ for each fixed $p \in [1, \infty)$, then the integral $L_p \int \Phi d\mu$ does not depend on p and it is an element of $\bigcap_{1 \leqslant p < \infty} L_p(G, B)$. Consequently, it is not difficult to show inductively that, actually, the functions Γ_k and H_k are defined independently of the choice of a $p \in [1, \infty)$,

$$\Gamma_k: G \to \bigcap_{1 \leqslant p < \infty} L_p(G, B), \quad k = 1, \ldots, n,$$

$$H_k: G \to \bigcap_{1 \leqslant p < \infty} L_p(G, B), \quad k = 0, 1, \ldots, n,$$

and for each fixed $p \in [1, \infty)$ conditions (14)–(16) are satisfied.

Put

$$[g] := (-1)^n L_p \int H(x)\, d\mu(x), \quad 1 \leqslant p < \infty.$$

The last integral is also independent of p and we have

$$[g] \in \bigcap_{1 \leqslant p < \infty} L_p(G, B).$$

Bearing in mind what has been shown in the proof of Case 1, in order to complete our present proof, it remains to check that $[g] \in L_\infty(G, B)$. For this purpose let us define

$$\tilde{\Gamma}_k(h_1, \ldots, h_k) := \frac{1}{k!} L_p \int \Delta_{h_1 \ldots h_k} H_k(x)\, d\mu(x),$$

$$h_1, \ldots, h_k \in G, \; 1 \leqslant p < \infty, \; k = 1, \ldots, n,$$

and

$$\tilde{\Gamma}_0 := L_p \int H_0(x)\, d\mu(x).$$

Then

$$\tilde{\Gamma}_k \colon G^k \to \bigcap_{1 \leqslant p < \infty} L_p(G, B)$$

is a k-additive symmetric function such that $\Gamma_k(h) = \tilde{\Gamma}_k(h, \ldots, h)$ for all $h \in G$, $k = 1, \ldots, n$, and $(-1)^n \tilde{\Gamma}_0 = [g]$.

Now, we shall prove that for each $k = 0, 1, \ldots, n$, the following estimations are valid:

(17) $\forall h_1, \ldots, h_k \in G \; \exists M_k(h_1, \ldots, h_k) > 0 \; \forall x \in G \; \forall 1 \leqslant p < \infty$

$$\|\Delta_{h_1 \ldots h_k} H_k(x)\|_p \leqslant M_k(h_1, \ldots, h_k),$$

(18) $\forall h_1, \ldots, h_k \in G \; \forall 1 \leqslant p < \infty \; \|\tilde{\Gamma}_k(h_1, \ldots, h_k)\|_p \leqslant \frac{1}{k!} M_k(h_1, \ldots, h_k),$

where for $k = 0$ condition (17) is understood as follows:

$$\exists M_0 > 0 \; \forall x \in G \; \forall 1 \leqslant p < \infty \; \|H_0(x)\|_p \leqslant M_0,$$

and similarly for $k = 0$ condition (18) means that

$$\forall 1 \leqslant p < \infty \; \|\tilde{\Gamma}_0\|_p \leqslant M_0.$$

The proofs of (17) and (18) proceed simultaneously by induction on k passing backwards from n to zero. In the first step we are going to obtain (17) and (18) for $k = n$.

Fix arbitrarily elements $h_1, \ldots, h_n \in G$. Applying Djoković's Theorem 2 from [7], by our assumptions we get the expansion

$$\Delta_{h_1 \ldots h_n} H_n(x) = \sum_{i \in I} r_i \Delta_{u_i}^n H_n(x + v_i), \qquad x \in G,$$

where I is a finite set of indices, r_i's are rationals and u_i, $v_i \in G$ are determined by $h_1, \ldots, h_n$ for $i \in I$. Putting

$$M_n(h_1, \ldots, h_n) := \sum_{i \in I} |r_i| \sum_{j=0}^{n} \binom{n}{j} \|F(ju_i)\|_\infty$$

and applying properties of the L_p-norm we obtain

$$\begin{aligned}
\|\Delta_{h_1 \ldots h_n} H_n(x)\|_p &= \Big\|\sum_{i \in I} r_i \Delta_{u_i}^n H_n(x+v_i)\Big\|_p \\
&\leqslant \sum_{i \in I} |r_i| \, \|\Delta_{u_i}^n F(x+v_i)\|_p \\
&= \sum_{i \in I} |r_i| \left\| \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} T_{j(x+v_i)} F(ju_i) \right\|_p \\
&\leqslant \sum_{i \in I} |r_i| \sum_{j=0}^{n} \binom{n}{j} \|T_{j(x+v_i)} F(ju_i)\|_p \\
&= \sum_{i \in I} |r_i| \sum_{j=0}^{n} \binom{n}{j} \|F(ju_i)\|_p \leqslant M_n(h_1, \ldots, h_n)
\end{aligned}$$

for every $x \in G$ and $1 \leqslant p < \infty$, which yields (17) with $k = n$. Now, we have

$$\begin{aligned}
\|\tilde{\Gamma}_n(h_1, \ldots, h_n)\|_p &= \frac{1}{n!} \Big\| L_p \int \Delta_{h_1 \ldots h_n} H_n(x) \, d\mu(x) \Big\|_p \\
&\leqslant \frac{1}{n!} \int \|\Delta_{h_1 \ldots h_n} H_n(x)\|_p \, d\mu(x) \leqslant \frac{1}{n!} M_n(h_1, \ldots, h_n), \qquad 1 \leqslant p < \infty,
\end{aligned}$$

and so (18) holds for $k = n$.

To continue the induction procedure assume (17) and (18) are true for each integer from $k+1$ to n with some k such that $n > k \geqslant 0$. Let us choose elements $h_1, \ldots, h_k \in G$ (if $k = 0$, we choose no elements, whereas the symbols $\Delta_{h_1 \ldots h_k} H_k(x)$ and $\tilde{\Gamma}_k(h_1, \ldots, h_k)$ take the form of $H_0(x)$ and $\tilde{\Gamma}_0$, respectively). Putting

$$A_n := \bigcap_{1 \leqslant p < \infty} \{x \in G \colon \|\Delta_{h_1 \ldots h_k} H_k(x)\|_p \leqslant n\}$$

and taking into account the fact that by (14) or (15) the function

$$\Delta_{h_1 \ldots h_k} H_k \colon G \to L_p(G, B)$$

is continuous for every $p \in [1, \infty)$, we infer that A_n is a closed subset of G for each positive integer n.

Now, we show that

$$\bigcup_{n=1}^{\infty} A_n = G. \tag{19}$$

For, let us choose an arbitrary $x \in G$. Since $H_k = F - \Gamma_{k+1} - \ldots - \Gamma_n$, for any $y \in G$ one can find an integer $n(y)$ such that

$$\begin{aligned}\|H_k(y)\|_p &\leqslant \|F(y)\|_p + \|\Gamma_{k+1}(y)\|_p + \ldots + \|\Gamma_n(y)\|_p \\ &\leqslant \|F(y)\|_\infty + \frac{1}{(k+1)!} M_{k+1}(y, \ldots, y) + \ldots \\ &\quad + \frac{1}{n!} M_n(y, \ldots, y) \leqslant n(y), \qquad 1 \leqslant p < \infty.\end{aligned}$$

Hence, if $k = 0$, then $\|H_0(x)\|_p \leqslant n(x)$, $1 \leqslant p < \infty$, whereas for $k > 0$ we have

$$\begin{aligned}\|\Delta_{h_1 \ldots h_k} H_k(x)\|_p &= \Big\| \sum_{\varepsilon_1 \ldots \varepsilon_k = 0}^{1} (-1)^{k-\varepsilon_1-\ldots-\varepsilon_k} H_k(x+\varepsilon_1 h_1 + \ldots + \varepsilon_k h_k) \Big\|_p \\ &\leqslant \sum_{\varepsilon_1 \ldots \varepsilon_k = 0}^{1} \|H_k(x+\varepsilon_1 h_1 + \ldots + \varepsilon_k h_k)\|_p \\ &\leqslant \sum_{\varepsilon_1 \ldots \varepsilon_k = 0}^{1} n(x+\varepsilon_1 h_1 + \ldots + \varepsilon_k h_k), \qquad 1 \leqslant p < \infty.\end{aligned}$$

Consequently, there exists an integer $N(x)$ such that

$$x \in A_{N(x)} \subset \bigcup_{n=1}^{\infty} A_n,$$

which implies (19).

By the Baire Category Theorem we can find an integer N such that A_N contains a non-void open subset V of G. Thus, G is covered by a finite family of sets $V + l_1, \ldots, V + l_r$ for some $l_1, \ldots, l_r \in G$. We put

$$M_k(h_1, \ldots, h_k) := \max_{1 \leqslant i \leqslant r} \big(2M_{k+1}(l_i, h_1, \ldots, h_k) + N\big)$$

with the usual agreement that for $k = 0$

$$M_0 := \max_{1 \leqslant i \leqslant r} \big(2M_1(l_i) + N\big).$$

Given an arbitrary $x \in G$ we find an index i $(1 \leqslant i \leqslant r)$ such that $x \in V + l_i$. Thus, $x = y + l_i$ for a certain $y \in V$. The observation that $H_k = H_{k+1} - \Gamma_{k+1}$ leads us to the formula

$$\Delta_{l_i h_1 \ldots h_k} H_k(y) = \Delta_{l_i h_1 \ldots h_k} H_{k+1}(y) - (k+1)!\, \tilde{\Gamma}_{k+1}(l_i, h_1, \ldots, h_k),$$

and hence

$$\begin{aligned}\|\Delta_{h_1 \ldots h_k} H_k(x)\|_p &= \|\Delta_{h_1 \ldots h_k} H_k(y + l_i)\|_p \\ &\leqslant \|\Delta_{l_i h_1 \ldots h_k} H_k(y)\|_p + \|\Delta_{h_1 \ldots h_k} H_k(y)\|_p\end{aligned}$$

$$\leqslant \|\Delta_{l_i h_1 \dots h_k} H_{k+1}(y)\|_p + (k+1)!\,\|\tilde{\Gamma}_{k+1}(l_i, h_1, \dots, h_k)\|_p$$
$$+\|\Delta_{h_1 \dots h_k} H_k(y)\|_p$$
$$\leqslant 2M_{k+1}(l_i, h_1, \dots, h_k) + N \leqslant M_k(h_1, \dots, h_k), \qquad 1 \leqslant p < \infty.$$

Since x has been arbitrarily chosen, we conclude that

$$\|\Delta_{h_1 \dots h_k} H_k(x)\|_p \leqslant M_k(h_1, \dots, h_k) \quad \text{for all } x \in G \text{ and } 1 \leqslant p < \infty.$$

Moreover,

$$\|\tilde{\Gamma}_k(h_1, \dots, h_k)\|_p = \left\| \frac{1}{k!} L_p \int \Delta_{h_1 \dots h_k} H_k(x)\, d\mu(x) \right\|_p$$
$$\leqslant \frac{1}{k!} \int \|\Delta_{h_1 \dots h_k} H_k(x)\|_p\, d\mu(x) \leqslant \frac{1}{k!} M_k(h_1, \dots, h_k), \qquad 1 \leqslant p < \infty.$$

This completes the proof of (17) and (18).

From (18) with $k = 0$ it follows that

(20) $$\|[g]\|_p = \|(-1)^n \tilde{\Gamma}_0\|_p \leqslant M_0 \quad \text{for every } p \in [1, \infty).$$

If $[g]$ were not in $L_\infty(G, B)$, there would exist a Borel set $E \subset G$ of positive measure such that $\|g(x)\| \geqslant M_0 + 1$ for all $x \in E$. Consequently, we would have

$$\|[g]\|_p = \left(\int \|g(x)\|^p d\mu(x)\right)^{1/p} \geqslant (M_0 + 1)\left(\mu(E)\right)^{1/p}.$$

The last expression tends to $M_0 + 1$ as $p \to \infty$, contrary to (20). This completes our proof.

In terms of weak difference properties of higher orders the above theorem may be formulated as follows:

COROLLARY 2. *Under the assumptions of Theorem* 1 *on the group G the class $L'_p(G, B)$ $(1 \leqslant p \leqslant \infty)$ consisting of functions whose equivalence classes remain in $L_p(G, B)$ has the weak difference property of n-th order.*

In the particular case of the compact group K of reals mod 1 we have

COROLLARY 3. *The class $L'_p(K, B)$ admits the weak difference property of an arbitrary order.*

Remark. 2. Let us consider a discrete cyclic group G of a finite order r with the uniformly distributed normed measure. The group G is not divisible by any integer k such that r and k have a non-trivial common divisor. Nevertheless, for any p $(1 \leqslant p \leqslant \infty)$, the class $L'_p(G, R)$ has weak difference properties (and, in fact, difference properties) of any orders. This example shows that the divisibility assumptions on G in Theorem 1 and Corollary 2 are not necessary. We do not know yet whether our results remain true without any divisibility assumptions.

4. Conclusions concerning Orlicz spaces. In the sequel, $L_\Phi(G, B)$ denotes the Orlicz space equipped with the Orlicz or Luxemburg norm generated by a Young function Φ (for definitions see, e.g., [6] and [13]).

LEMMA 3. *Let $(G, +)$ be a compact Abelian group. If $g: G \to B$ is such that*

$$B\int g d\mu = 0 \quad \textit{and} \quad [\Delta_h^n g] \in L_\Phi(G, B) \quad \textit{for every } h \in G,$$

then $[g] \in L_\Phi(G, B)$.

The proof of Lemma 3 for $n = 1$ is contained in the crucial part of the proof of Theorem 3.1 from [6]. For $n > 1$ one can obtain Lemma 3 by a simple induction.

The analogue of Corollary 2 for Orlicz spaces reads as follows:

THEOREM 2. *Under the assumptions of Theorem 1 on the group G the class $L'_\Phi(G, B)$ consisting of functions whose equivalence classes belong to the Orlicz space $L_\Phi(G, B)$ has the weak difference property of n-th order.*

Proof. Choose a function $f: G \to B$ such that, for each $h \in G$,

$$\Delta_h^n f \in L'_\Phi(G, B) \subset L'_1(G, B).$$

By Corollary 2 we have $f = \gamma + g + s$, where γ is a monomial function of n-th order, $g \in L'_1(G, B)$ and, for every $h \in G$, $\Delta_h^n s = 0$ almost everywhere. Without loss of generality we may assume that $B\int g d\mu = 0$. Since

$$[\Delta_h^n g] = [\Delta_h^n f] \in L_\Phi(G, B)$$

holds for each $h \in G$, Lemma 3 guarantees that $[g] \in L_\Phi(G, B)$, which was to be shown.

REFERENCES

[1] J. R. Brown, *Ergodic Theory and Topological Dynamics*, New York–San Francisco–London 1976.

[2] N. G. de Bruijn, *Functions whose differences belong to a given class*, Nieuw Arch. Wisk. (2) 23 (1951), pp. 194–218.

[3] — *A difference property for Riemann integrable functions and for some similar classes of functions*, Nederl. Akad. Wetensch. Proc. Ser. A 55 (1952), pp. 145–151.

[4] F. W. Carroll, *Difference properties for continuity and Riemann integrability on locally compact groups*, Trans. Amer. Math. Soc. 102 (1962), pp. 248–252.

[5] — *Functions whose differences belong to $L^p[0, 1]$*, Nederl. Akad. Wetensch. Proc. Ser. A 67 (1964), pp. 250–255.

[6] — and F. S. Koehl, *Difference properties for Banach-valued functions on compact groups*, ibidem 72 (1969), pp. 327–332.

[7] D. Z. Djoković, *A representation theorem for $(X_1-1)(X_2-1)\ldots(X_n-1)$ and its applications*, Ann. Polon. Math. 22 (1969), pp. 189–198.

[8] N. Dunford and J. T. Schwartz, *Linear Operators*, Part I, New York 1958.

[9] Z. Gajda, *Difference properties of higher orders for continuity and Riemann integrability*, Colloq. Math. 53 (1987), pp. 119–132.

[10] R. Ger, *Convex functions of higher orders in Euclidean spaces*, Ann. Polon. Math. 25 (1972), pp. 293–302.

[11] J. H. B. Kemperman, *On a generalized difference property*, Aequationes Math. 4 (1970), p. 231.

[12] F. S. Koehl, *Difference properties for Banach-valued functions*, Math. Ann. 181 (1969), pp. 288–296.

[13] M. A. Krasnosel'skiĭ and Y. B. Rutickiĭ, *Convex Functions and Orlicz Spaces*, Groningen 1961.

[14] W. Rudin, *Fourier Analysis on Groups*, New York 1962.

[15] W. Sierpiński, *Sur les translations des ensembles linéaires*, Fund. Math. 19 (1932), pp. 22–28.

[16] W. A. Woyczyński, *Orlicz spaces have the weak difference property*, Nederl. Akad. Wetensch. Proc. Ser. A 71 (1968), pp. 95–100.

Reçu par la Rédaction le 10.10.1983