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INTERPOLATION BY THE FOURIER–STIELTJES TRANSFORM OF A POSITIVE COMPACTLY SUPPORTED MEASURE

BY

J. FLOREK (WROCŁAW)

Let G be a locally compact abelian group with additive operation, and $\hat{G}$ its dual group with multiplicative operation.

Definition 1. $\Lambda \subset \hat{G}$ is called a *topological Sidon set* (is said to have the *Fatou–Zygmund property*) if every bounded (hermitian) function Φ on Λ is the restriction to Λ of the Fourier–Stieltjes transform of a bounded Radon measure (positive measure) μ_Φ. In both cases a set $K \subset G$ is said to be *associated with* Λ. if μ_Φ can always be chosen so that $\operatorname{supp}\mu_\Phi \subset K$, where $\operatorname{supp}\mu$ denotes the support of μ.

By the open mapping theorem, there exists a constant C, called a *Sidon constant*, not depending on the function Φ such that $\|\mu_\Phi\| \leqslant C|\Phi|_\infty$. Evidently, every Sidon or Fatou–Zygmund set is uniformly discrete. Thus there exists a neighbourhood U of the neutral element e in $\hat{G}$ such that

(*) $\lambda_1 \cdot U \cap \lambda_2 \cdot U = \emptyset$ for every $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \in \Lambda \cup \{e\}$.

In the whole paper the letters U and V will denote some neighbourhoods of e.

It was proved by Drury [2] that if $\hat{G}$ is discrete and Λ is a Sidon set not containing e, then Λ has the Fatou–Zygmund property. Moreover, for every $\varepsilon > 0$ the measure μ_Φ can be chosen so as to have $|\hat{\mu}_\Phi(\gamma)| \leqslant \varepsilon$ for $\gamma \notin \Lambda \cup \{e\}$. Déchamps-Gondim proved in [1] that if Λ is a topological Sidon set having an associated compact, then, for every U and $\varepsilon > 0$, μ_Φ can be chosen to satisfy $|\hat{\mu}_\Phi(\gamma)| \leqslant \varepsilon$ for $\gamma \notin \Lambda \cdot U$. A closed set Λ is called a *Helson set* iff $c_0(\Lambda) = A(\Lambda)$. It was proved by Smith [6] that every Helson set has the corresponding Fatou–Zygmund property (see Remark 2).

It is well known [5] that every topological Sidon set in a metrizable and separable group has an associated compact set. We shall show that the same is true for every set with the Fatou–Zygmund property. Namely, we shall prove that a symmetric topological Sidon set not containing e and having an associated compact set is a Fatou–Zygmund set also having an associated compact set (may be a larger one). Our main result is as follows:

THEOREM. *Let Λ be a symmetric topological Sidon set not containing e, C its Sidon constant, and $K \subset G$ a compact associated with Λ. For every $\varepsilon > 0$, every U satisfying* (*), *and any hermitian function Φ, $|\Phi|_\infty \leqslant 1$, there exists a positive measure μ_Φ such that for a suitable compact set $K_1 = K_1(C, U, \varepsilon)$ we have*

(i) $\operatorname{supp}\mu_\Phi \subset 2K+K_1$, $\|\mu_\Phi\| \leqslant C_1/\varepsilon$, $C_1 = C_1(C)$;

(ii) $\hat{\mu}_\Phi(\lambda) = \Phi(\lambda)$ *for* $\lambda \in \Lambda$;

(iii) $|\hat{\mu}_\Phi(\gamma)| \leqslant \varepsilon$ *for* $\gamma \notin U \cup \Lambda \cdot U$.

The following notion will be useful in the proof.

DEFINITION 2. A subset F of $\hat{G}$ is called a *dissociate set* with respect to V if F is asymmetric and, for every pair of different finite asymmetric sets $A, B \subset F \cup F^{-1}$, the sets $\big(\prod_{\lambda \in A} \lambda\big) \cdot V$ and $\big(\prod_{\lambda \in B} \lambda\big) \cdot V$ are disjoint.

LEMMA 1. *Let $F \subset \hat{G}$ be finite and dissociate with respect to V. If $f \in L^1_+(G)$ is such that $\operatorname{supp}\hat{f} \subset V$ and $|f|_1 = 1$ then, for every hermitian function Φ on $F \cup F^{-1}$, $|\Phi|_\infty \leqslant \frac{1}{2}$, there exists a measure $P_\Phi \in L^1_+(G)$ satisfying the following conditions*:

(1) $\operatorname{supp} P_\Phi \subset \operatorname{supp} f$, $|P_\Phi|_1 = 1$;

(2) $\hat{P}_\Phi(\lambda \cdot \gamma) = \Phi(\lambda)\hat{f}(\gamma)$ *for* $\lambda \in F \cup F^{-1}$ *and* $\gamma \in V$;

(3) $\hat{P}_\Phi(\gamma) = \hat{f}(\gamma)$ *for* $\gamma \in V$;

(4) $|\hat{P}_\Phi(\gamma)| \leqslant |\Phi|^2_\infty$ *for* $\gamma \notin V \cup F \cdot V \cup F^{-1} \cdot V$.

Proof. We shall use the Riesz product method, so, for any $x \in G$, we set

$$R_\Phi(x) = \prod_{\lambda \in F} g_\lambda(x),$$

where

$$g_\lambda(x) = \begin{cases} 1+\Phi(\lambda)\lambda(x)+\overline{\Phi(\lambda)\lambda(x)} & \text{for } \lambda^2 \neq e, \\ 1+\Phi(\lambda)\lambda(x) & \text{for } \lambda^2 = e. \end{cases}$$

Since $|\Phi|_\infty \leqslant \frac{1}{2}$, we get $R_\Phi \geqslant 0$. For $P_\Phi = R_\Phi \cdot f$ we have

$$\hat{P}_\Phi(\gamma) = \int_G f(x) R_\Phi(x) \overline{\gamma(x)}\, dx = \sum_{S \subset F \cup F^{-1}}^{\mathrm{as}} \prod_{\lambda \in S} \Phi(\lambda) \cdot \hat{f}\Big(\gamma\big(\prod_{\lambda \in S} \lambda\big)^{-1}\Big),$$

where $\sum^{\mathrm{as}}$ indicates that the union is taken over asymmetric sets only. Let $\gamma \in \big(\prod_{\lambda \in S} \lambda\big) V$ for some S. For $S = \emptyset$ this means that $\gamma \in V$. Since F is dissociate with respect to V and $\operatorname{supp}\hat{f} \subset V$, we have

$$\hat{P}_\Phi(\gamma) = \hat{f}\Big(\gamma\big(\prod_{\lambda \in S} \lambda\big)^{-1}\Big) \prod_{\lambda \in S} \Phi(\lambda).$$

For $S = \emptyset$ this means that $\hat{P}_\Phi(\gamma) = \hat{f}(\gamma)$. If $\gamma = \lambda \cdot v$ ($\lambda \in F \cup F^{-1}$, $v \in V$), then

$$\hat{P}_\Phi(\gamma) = \hat{P}_\Phi(\lambda \cdot v) = \hat{f}(v)\Phi(\lambda).$$

If $\gamma \notin V \cup F\cdot V \cup F^{-1}\cdot V$, then since $\gamma \in (\prod_{\lambda\in S} \lambda) V$, we have $|S| \geqslant 2$ and $|\hat{P}_\Phi(\gamma)| \leqslant |\Phi|_\infty^2$.

LEMMA 2. *Let E be a finite dissociate set in an abelian discrete group $\hat{\Omega}$ and let φ: $E \cup E^{-1} \to \hat{G}$ be such that $\varphi(\xi)^{-1} = \varphi(\xi^{-1})$. For $0 \leqslant \varepsilon \leqslant \frac{1}{2}$ and any function $g \in L^1_+(G)$, $|g|_1 = 1$, there exist functions P_1, $P_2 \in L^1(\Omega \times G)$ satisfying*

(1) $P_1 + P_2$, $P_1 - P_2 \in L^1_+(\Omega \times G)$;

(2) $\operatorname{supp} P_1$, $\operatorname{supp} P_2 \subset \Omega \times \operatorname{supp} g$, $|P_1|_1$, $|P_2|_1 \leqslant 1$;

(3) $\hat{P}_1(\xi, \gamma) = 0$, $\hat{P}_2(\xi, \gamma) = \varepsilon\cdot\hat{g}(\varphi(\xi)^{-1}\gamma)$ *for* $\xi \in E \cup E^{-1}$, $\gamma \in \hat{G}$;

(4) $\hat{P}_1(e, \gamma) = \hat{g}(\gamma)$, $\hat{P}_2(e, \gamma) = 0$ *for* $\gamma \in \hat{G}$;

(5) $|\hat{P}_1(\xi, \gamma)|$, $|\hat{P}_2(\xi, \gamma)| \leqslant \varepsilon^2$ *for* $\xi \notin E \cup E^{-1} \cup \{e\}$, $\gamma \in \hat{G}$.

Proof. First we show that the set $F = \{(\xi, \varphi(\xi))\colon \xi \in E\}$ is dissociate with respect to the neighbourhood $e \times \hat{G}$ in the group $\hat{\Omega} \times \hat{G}$. Let A, $B \subset F \cup F^{-1}$ be asymmetric and different. Then the sets

$$A_\varphi = \{\xi \in E \cup E^{-1}\colon (\xi, \varphi(\xi)) \in A\} \quad \text{and} \quad B_\varphi = \{\xi \in E \cup E^{-1}\colon (\xi, \varphi(\xi)) \in B\}$$

are different and asymmetric because if ξ, $\xi^{-1} \in A_\varphi$, then $(\xi, \varphi(\xi)) \in A$ and $(\xi^{-1}, \varphi(\xi^{-1})) = (\xi, \varphi(\xi))^{-1} \in A$, so $\xi = \xi^{-1}$. Since E is dissociate,

$$\prod_{\xi\in A_\varphi} \xi \neq \prod_{\xi\in B_\varphi} \xi.$$

Hence the sets

$$(e \times \hat{G}) \prod_{\lambda\in A} \lambda = (\prod_{\xi\in A_\varphi} \xi) \times \hat{G} \quad \text{and} \quad (e \times \hat{G}) \prod_{\lambda\in B} \lambda = (\prod_{\xi\in B_\varphi} \xi) \times \hat{G}$$

are disjoint.

We define $f \in L^1_+(\Omega \times G)$ by $f(\omega, x) = g(x)$. Then $\operatorname{supp}\hat{f} \subset e \times \hat{G}$ and $\hat{f}(e, \gamma) = \hat{g}(\gamma)$ for $\gamma \in \hat{G}$ because

$$\hat{f}(\xi, \gamma) = \int_G g(x)\overline{\gamma(x)}\big(\int_\Omega \overline{\xi(\omega)}\, d\omega\big) dx.$$

We now use Lemma 1 for $F = \{(\xi, \varphi(\xi))\colon \xi \in E\}$ and $V = e \times \hat{G}$, thus getting two functions P_ε and $P_{-\varepsilon}$ which correspond to the constant functions $\Phi = \varepsilon$ and $\Phi = -\varepsilon$ on $F \cup F^{-1}$. For $P_1 = \frac{1}{2}(P_\varepsilon + P_{-\varepsilon})$ and $P_2 = \frac{1}{2}(P_\varepsilon - P_{-\varepsilon})$ the conditions of Lemma 2 are satisfied. Conditions (2) and (5) follow from (1) and (4) of Lemma 1. Since $\hat{f}(e, \gamma) = \hat{g}(\gamma)$ for $\gamma \in \hat{G}$ and $(\xi, \gamma) = (\xi, \varphi(\xi))(e, \varphi(\xi)^{-1}\gamma)$, we get (3) and (4) from (2) and (3) of Lemma 1.

The following Lemma 3 is the well-known Drury Lemma adapted for topological Sidon sets ([4], Lemma 3.2). The proof will be omitted since it is analogous to that of Drury.

LEMMA 3. *Let Λ be a symmetric topological Sidon set with Sidon constant C associated with a compact $K \subset G$, and Ω a finite group of hermitian* *functions on Λ with values in the torus group $\boldsymbol{T} = \boldsymbol{R}/2\pi\boldsymbol{Z}$. Then, for $\omega \in \Omega$, there exist measures μ_ω, $\mu_\omega^* \in M(G)$ such that*

(α) $\mu_\omega^* + \mu_\omega$, $\mu_\omega^* - \mu_\omega \in M_+(G)$;

(β) $\operatorname{supp} \mu_\omega$, $\operatorname{supp} \mu_\omega^* \subset 2K$, $\|\mu_\omega\|$, $\|\mu_\omega^*\| \leqslant C^2$;

(γ) $\hat{\mu}_\omega(\lambda) = \omega(\lambda)$ *for* $\lambda \in \Lambda$;

(δ) $\|g_\gamma\|_{A(\Omega)}$, $\|g_\gamma^*\|_{A(\Omega)} \leqslant C^2$, *where* $g_\gamma(\omega) = \hat{\mu}_\omega(\gamma)$, $g_\gamma^*(\omega) = \hat{\mu}_\omega^*(\gamma)$ *for* $\omega \in \Omega$, $\gamma \in \hat{G}$.

In the proof of our Theorem we shall use the Drury method ([4], Theorem 3.3).

Proof of the Theorem. It is enough to suppose that Λ is finite. In fact, if a set Ξ is such that for every finite symmetric subset $F \subset \Xi$ there exists a positive measure μ_F satisfying (i)–(iii) (for $\Lambda = F$), then any *-weak accumulation point of the net $\{\mu_F$: F finite, symmetric, $F \subset \Xi\}$ satisfies (i)–(iii) for $\Lambda = \Xi$. Now let Ω consist of all hermitian functions

$$\omega\colon \Lambda \to Z(4) = \{1, i, -1, -i\} \quad \text{and} \quad \pi_\lambda(\omega) = \omega(\lambda).$$

Let Λ_0 be an asymmetric subset of Λ such that $\Lambda_0 \cup \Lambda_0^{-1} = \Lambda$. It is easy to see that $E = \{\pi_\lambda\}_{\lambda \in \Lambda}$ is a dissociate set in $\hat{\Omega}$. Let $\varphi\colon E \to \hat{G}$ be defined by $\varphi(\pi_\lambda) = \lambda$. Since Λ_0 is asymmetric, we may extend this definition to $\varphi\colon E \cup E^{-1} \to \hat{G}$ by putting $\varphi(\pi_\lambda^{-1}) = \lambda^{-1}$. Let $0 < \varepsilon \leqslant \frac{1}{2}$ and $g \in L_+^1(G)$ be such that $|g|_1 = 1$, $\operatorname{supp} g$ is compact, and $|\hat{g}(\gamma)| \leqslant \varepsilon^2$ for $\gamma \notin U$. By Lemma 2 we obtain two functions P_1, $P_2 \in L^1(\Omega \times G)$ satisfying:

(α′) $P_1 + P_2$, $P_1 - P_2 \in L_+^1(\Omega \times G)$;

(β′) $\operatorname{supp} P_1$, $\operatorname{supp} P_2 \subset \Omega \times \operatorname{supp} g$, $|P_1|_1$, $|P_2|_1 \leqslant 1$;

(γ′) $\hat{P}_2(\pi_\lambda, \lambda) = \varepsilon$ for $\lambda \in \Lambda$;

(δ′) $|\hat{P}_1(\xi, \gamma)| \leqslant \varepsilon^2$ for $\xi \in \hat{\Omega}$, $\gamma \notin U$; and $|\hat{P}_2(\xi, \gamma)| \leqslant \varepsilon^2$ for $\xi \in \hat{\Omega}$, $\gamma \notin \Lambda \cdot U$.

For $\omega \in \Omega$ let us set $P_\omega^1(x) = P_1(\omega, x)$, $P_\omega^2(x) = P_2(\omega, x)$ and let μ_ω, μ_ω^* have the same meaning as in Lemma 3. We define

$$\sigma_\omega^* = \int_\Omega P^1_{\omega\alpha^{-1}} * \mu_\alpha^* \, d\alpha \quad \text{and} \quad \sigma_\omega^0 = \int_\Omega P^2_{\omega\alpha^{-1}} * \mu_\alpha \, d\alpha.$$

By (α′) and (α) we have

$$(P^1_{\omega\alpha^{-1}} - P^2_{\omega\alpha^{-1}}) * (\mu_\alpha^* - \mu_\alpha) + (P^1_{\omega\alpha^{-1}} + P^2_{\omega\alpha^{-1}}) * (\mu_\alpha^* + \mu_\alpha) \in L_+^1(G).$$

Hence $\sigma_\omega^* + \sigma_\omega^0 \in L_+^1(G)$. Moreover, by (β′) and (β),

$$\operatorname{supp} \sigma_\omega^*, \ \operatorname{supp} \sigma_\omega^0 \subset 2K + \operatorname{supp} g \quad \text{and} \quad \|\sigma_\omega^*\|, \ \|\sigma_\omega^0\| \leqslant C^2.$$

Using computations from [4], Theorem 3.2, we also have

$$(\sigma_\omega^0)\hat{}(\lambda) = \int_\Omega (P^2_{\omega\alpha^{-1}})\hat{}(\lambda)\,\hat{\mu}_\alpha(\lambda)\,d\alpha = \int_\Omega\int_G P^2_{\omega\alpha^{-1}}(x)\,\overline{\lambda(x)}\,dx\alpha(\lambda)\,d\alpha$$

$$= \int_\Omega\int_G P_2(\omega\alpha^{-1}, x)\,\overline{\lambda(x)}\,\pi_\lambda(\alpha)\,dxd\alpha$$

$$= \int_\Omega\int_G P_2(\omega\alpha^{-1}, x)\,\overline{\lambda(x)}\,\overline{\pi_\lambda(\omega\alpha^{-1})}\,dxd\alpha\omega(\lambda)$$

$$= \hat{P}_2(\pi_\lambda, \lambda)\,\omega(\lambda).$$

Hence, by (γ) and (γ'), $(\sigma_\omega^0)\hat{}(\lambda) = \varepsilon\omega(\lambda)$ for $\lambda \in \Lambda$. Further

$$(\sigma_\omega^0)\hat{}(\gamma) = \int_\Omega (P^2_{\omega\alpha^{-1}})\hat{}(\gamma)\,g_\gamma(\alpha)\,d\alpha = \int_\Omega\int_{\hat\Omega} (P^2_{\omega\alpha^{-1}})\hat{}(\gamma)\,\hat{g}_\gamma(\xi)\,\xi(\alpha)\,d\xi d\alpha$$

$$= \int_{\hat\Omega}\int_\Omega\int_G P_2(\omega\alpha^{-1}, x)\,\overline{\gamma(x)}\,\hat{g}_\gamma(\xi)\,\xi(\alpha)\,dxd\alpha d\xi$$

$$= \int_{\hat\Omega}\int_\Omega\int_G P_2(\omega\alpha^{-1}, x)\,\overline{\gamma(x)}\,\overline{\xi(\omega\alpha^{-1})}\,dxd\alpha\hat{g}_\gamma(\xi)\,\xi(\omega)\,d\xi$$

$$= \int_{\hat\Omega} \hat{P}_2(\xi, \gamma)\,\hat{g}_\gamma(\xi)\,\xi(\omega)\,d\xi,$$

and so, by (δ) and (δ'), $|(\sigma_\omega^0)\hat{}(\gamma)| \leqslant \varepsilon^2 C^2$ for $\gamma \notin \Lambda\cdot U$. Analogously,

$$(\sigma_\omega^*)\hat{}(\gamma) = \int_{\hat\Omega} \hat{P}_1(\xi, \gamma)\,\hat{g}_\gamma^*(\xi)\,\xi(\omega)\,d\xi$$

and arguing as above we obtain $|(\sigma_\omega^*)\hat{}(\gamma)| \leqslant \varepsilon^2 C^2$ for $\gamma \notin U$. Thus we infer that the measure

$$\nu_\omega = \frac{\sigma_\omega^0 + \sigma_\omega^*}{\varepsilon} \in L^1_+(G)$$

satisfies the following conditions:

$$\operatorname{supp}\nu_\omega \subset 2K + \operatorname{supp} g, \qquad \|\nu_\omega\| \leqslant 2C^2/\varepsilon,$$

$$|\hat{\nu}_\omega(\lambda) - \omega(\lambda)| \leqslant \varepsilon C^2 \qquad \text{for } \lambda \in \Lambda$$

and

$$|\hat{\nu}_\omega(\gamma)| \leqslant 2\varepsilon C^2 \qquad \text{for } \gamma \notin U \cup \Lambda\cdot U.$$

Let H denote the set of all hermitian functions from Λ into the closed unit disc. Since the convex hull $\operatorname{co}(Z(4))$ of the set $Z(4)$ contains all complex numbers z such that $|z| \leqslant \cos(\pi/4)$, it follows that $H \subset \sqrt{2}\operatorname{co}(\Omega)$. Replacing ε and g by $\varepsilon/4\sqrt{2}C^2$ and a suitable g' with a compact support K_1

$= K_1(C, U, \varepsilon)$, for every function $\Phi \in H$ we obtain a measure $\nu_\Phi \in L^1_+(G)$ satisfying:

$$\operatorname{supp} \nu_\Phi \subset 2K + K_1, \quad \|\nu_\Phi\| \leqslant |\Phi|_\infty \frac{16C^4}{\varepsilon},$$

$$|\hat{\nu}_\Phi(\lambda) - \Phi(\lambda)| \leqslant |\Phi|_\infty \frac{\varepsilon}{4} \quad \text{for } \lambda \in \Lambda,$$

$$|\hat{\nu}_\Phi(\gamma)| \leqslant |\Phi|_\infty \frac{\varepsilon}{2} \quad \text{for } \gamma \notin \Lambda \cdot U \cup U.$$

Finally, we apply the foregoing inductively to Φ, $\Phi - \hat{\nu}_1$, etc. to obtain ν_1, ν_2, etc. in $L^1_+(G)$ such that

$$\Big|\sum_{k=1}^{n} \nu_k(\lambda) - \Phi(\lambda)\Big| \leqslant \frac{\varepsilon}{4} 2^{-n+1} \quad \text{for } \lambda \in \Lambda,$$

$$|\hat{\nu}_k(\gamma)| \leqslant \frac{\varepsilon}{2} 2^{-k+1} \quad \text{for } \gamma \notin \Lambda \cdot U \cup U,$$

$$\|\nu_k\| \leqslant \frac{16C^4}{\varepsilon} 2^{-k+1}.$$

The measure $\mu_\Phi = \sum_{k=1}^{\infty} \nu_k$ has the desired properties.

Remark 1. If Λ is finite, then, for any $\Phi \in H$, $\mu_\Phi \in L^1_+(G)$.

The next result is similar to the result of Smith [6].

COROLLARY 1. *If* $e \notin \Lambda = \Lambda^{-1} \subset \hat{G}$ *and* Λ *is a topological Sidon set associated with some compact set, then every hermitian function belonging to* $C_0(\Lambda)$ *is the restriction of the Fourier transform of a positive* L_1*-function on* G *with a compact support.*

Proof. Let Φ be a hermitian function, $|\Phi|_\infty \leqslant 1$, $\Phi \in C_0(\Lambda)$, and

$$P_n = \{\lambda \in \Lambda \colon 2^{-n} < |\Phi(\lambda)| \leqslant 2^{-n+1}\}.$$

Since P_n are symmetric, the functions $\Phi_n = \Phi \cdot \chi_{P_n}$ are hermitian on Λ and $|\Phi_n|_\infty \leqslant 2^{-n+1}$. The Theorem yields a positive measure μ_n such that $\hat{\mu}_n|_\Lambda = \Phi_n$, $\|\mu_n\| \leqslant D \cdot 2^{-n+1}$, $\operatorname{supp} \mu_n \subset L$, where D is some constant and L is some compact set. Since P_n are finite, by Remark 1 there exist measures $f_n \in L^1_+(G)$, $|f_n|_1 \leqslant D$, $\operatorname{supp} f_n \subset L$, $\hat{f}_n|_{P_n} = 1$. Hence, for

$$\mu = \sum_{n=1}^{\infty} \mu_n * f_n \in L^1_+(G),$$

we get $\|\mu\| \leqslant 2D^2$, $\operatorname{supp}\mu \subset L^2$ and

$$\hat{\mu}|_\Lambda = \sum_{n=1}^{\infty} \hat{\mu}_n|_{P_n} \cdot \hat{f}_n|_{P_n} = \sum_{n=1}^{\infty} \Phi_n = \Phi.$$

Remark 2. Whereas in [6] it is proved that, for any Helson set $\Lambda \subset \hat{G}$, every hermitian function $\Phi \in C_0(\Lambda)$ can be interpolated by some $\hat{f}$ with $f \in L^1_+(G)$, in Corollary 1 more is supposed, namely the existence of an associated compact set and the assertion obtained is respectively stronger.

It was proved by Hartman [3] that if Λ is a Sidon set in an abelian infinite discrete group, then every function bounded on Λ is the restriction to Λ of the Fourier–Stieltjes transform of a continuous measure. It was shown in [4] that if $e \notin \Lambda$ and if Λ is a Sidon set in an abelian infinite discrete group, then every hermitian function bounded on Λ is the restriction to Λ of the Fourier–Stieltjes transform of a continuous positive Radon measure.

Corollary 2. *If G is a nondiscrete group, $e \notin \Lambda = \Lambda^{-1} \subset \hat{G}$, and Λ is a topological Sidon set associated with some compact set, then every hermitian function bounded on Λ is the restriction of the Fourier–Stieltjes transform of a positive, compactly supported continuous measure.*

Proof. Let μ_d and μ_c denote the discrete and continuous parts of μ, respectively. Analogously as in [4], Theorems 1.4 and 4.3, we are able to prove the following two propositions, respectively:

(A) *If Λ is a topological Sidon set and U satisfies* (*), *then*

$$\sup\{\min(|M|, |N|)\colon M\cdot N \subset \Lambda\cdot U \cup U \text{ and } \operatorname{card}\{M\cdot N \subset \lambda U\} \leqslant 1,\ \lambda \in \Lambda \cup \{e\}\} < \infty.$$

(B) *If G is a nondiscrete group and Λ satisfies the assertion of* (A), *then*

$$\hat{\mu}_d(\hat{G}) \subset \hat{\mu}(\hat{G} \setminus \Lambda \cdot U \cup U)^-.$$

Finally, by the Theorem, there is a positive compactly supported measure μ such that $\hat{\mu}(\lambda) = 1$ for $\lambda \in \Lambda$ and $|\hat{\mu}(\gamma)| \leqslant \frac{1}{2}$ for $\gamma \notin \Lambda \cdot U \cup U$. By (A) and (B) we have $|\hat{\mu}_d|_\infty \leqslant \frac{1}{2}$, and so $|\hat{\mu}_c(\lambda)| \geqslant \frac{1}{2}$ for $\lambda \in \Lambda$. We now use our Theorem again to show that, for a hermitian function $\Phi \in L^\infty(\Lambda)$, there is a positive compactly supported measure ν such that

$$\hat{\nu}|_\Lambda = \frac{\Phi}{\hat{\mu}_c|_\Lambda}.$$

Now $\nu * \mu_c$ is positive, continuous, compactly supported and $(\nu * \mu_c)\hat{}\,|_\Lambda = \Phi$.

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REFERENCES

[1] M. Déchamps-Gondim, *Ensembles de Sidon topologiques*, Ann. Inst. Fourier (Grenoble) 22 (3) (1972), pp. 51–79.

[2] S. W. Drury, *The Fatou–Zygmund property for Sidon sets*, Bull. Amer. Math. Soc. 80 (1974), pp. 532–538.

[3] S. Hartman, *Interpolation par les mesures diffuses*, Colloq. Math. 26 (1972), pp. 339–343.

[4] J. M. Lopez and K. A. Ross, *Sidon sets*, Lecture Notes in Pure and Appl. Math. 13 (1975).

[5] Y. Meyer, *Algebraic Numbers and Harmonic Analysis*, Amsterdam–London 1972.

[6] B. P. Smith, *Helson sets not containing the identity are uniform Fatou–Zygmund sets*, Indiana Univ. Math. J. 27 (1978), pp. 331–347.

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