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DIRECT SUMS IN GENERAL ALGEBRA *

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§ 1. Introduction. To study the decompositions of algebras into direct sums of their subalgebras, different frameworks have been proposed. The most general seems to be the one used by B. Jónsson and A. Tarski in [1]. They deal with algebras of arbitrary type with the only restriction that among primitive operations there is a constant 0 which is a one-element subalgebra and a binary operation $+$ with the property that for all x's there is $x+0 = 0+x = x$. The authors show that under this assumption the relation "A is a direct sum of its subalgebras B and C" may be defined in such a way that projections ϱ_1, ϱ_2 of A onto B and C are unique and $a = \varrho_1 a + \varrho_2 a$ holds for every a in A.

Let us remark that:

(I) If we restrict ourselves to algebras of a fixed similarity type, then the algebras studied by Jónsson and Tarski form an equational class. Hence this class is closed with respect to direct (outer) products of any power, and every direct (outer) product $B\times C$ of algebras belonging to this class is decomposable into a direct (inner) sum of its subalgebras B_1 and C_1, isomorphic respectively to B and C.

(II) To built the theory of direct sums in the way proposed by Jónsson and Tarski we do not need to assume that among primitive operations of the algebra are 0 and $+$. It is enough to assume that there exist "terms", i. e. operations ζ^A and a^A, defined by compositions of the primitive operations, such that

$$\zeta^A(x) = \zeta^A(y), \qquad a^A\big(x, \zeta^A(x)\big) = a^A\big(\zeta^A(x), x\big) = x,$$

and that the value $\zeta^A(x)$ is a one-element subalgebra. If it is so, we may define $0 = \zeta^A(x)$ and $x+y = a(x, y)$, and the whole theory of direct sum holds.

Both remarks are rather trivial, but they seem to have some explanatory meaning for what follows.

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The most important freature of direct sums are projections. In many special cases (groups, rings and so on) we know how to define the direct sum by means of projections. Usually, in order to prove that the definition by means of projections defines really what we want to call direct sum, we use special properties of algebraic operations (in fact these assumed by Jónsson and Tarski). We do not know whether such properties are really essential. To answer this question we should give a definition of the direct sum by means of projections only and with no assumption on operations, and then check what we can infer on operations from assumption of the existence of direct sums in this sense.

In the following sections we present a partial answer to this question. First we give a definition of the direct sum and then we show, that if a class of algebras has the properties mentioned in the remark (I), then there are two terms, ζ and a, having the properties mentioned in the remark (II) in every algebra of that class.

§ 2. Definition and lemmas. Let A be an algebra and let B_1, B_2 be subalgebras of A.

Definition. A is a *direct sum* of B_1 and B_2, $A = B_1 \oplus B_2$, if there exist two endomorphisms (projections) $\varrho_i : A \to B_i$, $i = 1, 2$, such that

(1) $\varrho_i b_i = b_i$ for b_i in B_i, $i = 1, 2$;

(2) if b_i is in B_i, $i = 1, 2$, then there exists a unique element a in A such that $\varrho_1 a = b_1$ and $\varrho_2 a = b_2$;

(3) if B_i^0 is a subalgebra of B_i, $i = 1, 2$, then $[B_1^0, B_2^0] = \overset{-1}{\varrho_1} B_1^0 \cap \overset{-1}{\varrho_2} B_2^0$ ($[B_1^0, B_2^0]$ denotes the subalgebra of A generated by B_1^0 and B_2^0).

In the following lemmas we shall always assume that $A = B_1 \oplus B_2$ and that ϱ_1, ϱ_2 are endomorphisms with properties (1)-(3). By C_1 and C_2 we shall denote the subalgebras $\varrho_1 B_2$ and $\varrho_2 B_1$, by $B_1 \times B_2$ the (outer) direct product of B_1 and B_2, and by π_i the homomorphism $B_1 \times B_2 \to B_i$ defined by $\pi_i : \langle b_1, b_2 \rangle \to b_i$, $i = 1, 2$.

LEMMA 1. *The mapping $\varphi : a \to \langle \varrho_1 a, \varrho_2 a \rangle$ is an isomorphism of A onto $B_1 \times B_2$ such that for each $i = 1, 2$ both diagrams*

$$B_i \overset{\varrho_i}{\leftarrow} A \overset{\varphi}{\to} B_1 \times B_2 \overset{\pi_i}{\to} B_i$$

$$B_i \overset{\pi_i}{\leftarrow} B_1 \times B_2 \overset{\varphi^{-1}}{\to} A \overset{\varrho_i}{\to} B_i$$

commute.

Proof follows from the usual properties of homomorphisms and the property (2).

LEMMA 2. *C_i is for $i = 1, 2$ the minimal subalgebra of B_i.*

Proof. Let B_1^0 be a subalgebra of B_1. Following (3) we have

$$[B_1^0, B_2] = \overset{-1}{\varrho_1} B_1^0 \cap \overset{-1}{\varrho_2} B_2 = \overset{-1}{\varrho_1} B_1^0 \cap A = \overset{-1}{\varrho_1} B_1^0 .$$

Hence $B_2 \subset \overset{-1}{\varrho_1} B_1^0$ and therefore $\varrho_1 B_2 = C_1 \subset B_1^0$.

LEMMA 3. *There is either* $B_1 \cap B_2 = C_1 = C_2$ *and then it is a one-element subalgebra minimal in* A, *or* $B_1 \cap B_2 = 0$.

Proof. Obviously, $B_1 \cap B_2 \subset C_i$ for $i = 1, 2$ and therefore if $B_1 \cap B_2 \neq 0$, then by Lemma 2 the equality holds.

Let us suppose that c_1 and c_2 are two elements in $C_1 = C_2$. Following (2) there exists an element a in A with $\varrho_1 a = c_1$, $\varrho_2 a = c_2$. By virtue of (3) a belongs to $[C_1, C_2] = C_i$. Hence and from (1), $\varrho_1 a = \varrho_2 a = a = c_1 = c_2$.

Now let c be the only element in $C_1 = C_2$ and let A^0 be an arbitrary subalgebra of A. Since $\varrho_1 A^0$ being a subalgebra of B_1 contains c, $\overset{-1}{\varrho_1} c \cap A^0 = B_2 \cap A^0$ is not empty and as a subalgebra of B_2 contains c.

Let us call a direct sum $A = B_1 \oplus B_2$ with $C_1 = C_2$ a *normal* one.

LEMMA 4. *If the direct sum* $A = B_1 \oplus B_2$ *is normal, then the projections* ϱ_1, ϱ_2 *are unique.*

Proof. Let us suppose ϱ_1', ϱ_2' to be a second pair of endomorphisms with properties (1)-(3). It follows from lemma 3 that $\varrho_1' B_2 = \varrho_2' B_1 = \varrho_1 B_2 = \varrho_2 B_1 = \{c\}$. Let us construct isomorphisms

$$\varphi : a \to \langle \varrho_1 a, \varrho_2 a \rangle \quad \text{and} \quad \varphi' : a \to \langle \varrho_1' a, \varrho_2' a \rangle .$$

If b_1 is in B_1, then $\varphi(b_1) = \langle b_1, c \rangle = \varphi'(b_1)$. Similarly, if b_2 is in B_2, then $\varphi(b_2) = \varphi'(b_2)$.

As B_1 and B_2 generate A, then $\varphi = \varphi'$. From Lemma 1 it follows that $\varrho_1 = \varrho_1'$ and $\varrho_2 = \varrho_2'$.

LEMMA 5. *If* $B_{11} \oplus B_{12} = B_1$ *and* $A = B_1 \oplus B_2$, *then the direct sum* $B_{11} \oplus B_{12}$ *is normal.*

Proof. It follows from Lemma 2 that B_1 has a minimal subalgebra, and therefore $B_{11} \cap B_{12} \neq 0$.

Example. Let $B = \{b_0, b_1, \ldots, b_n = b_0\}$ be an algebra with one unary operation $F(b_i) = b_{i+1}$ for $i = 0, 1, \ldots, n-1$. In the direct product $A = B \times B$, the sets

$$B_1 = \{\langle b_0, b_0 \rangle, \ldots, \langle b_{n-1}, b_{n-1} \rangle\} \quad \text{and}$$
$$B_2 = \{\langle b_1, b_0 \rangle, \langle b_2, b_1 \rangle, \ldots, \langle b_0, b_{n-1} \rangle\}$$

are disjoint subalgebras of A. The mappings $\varrho_1 : \langle b_i, b_j \rangle \to \langle b_i, b_i \rangle$ and $\varrho_2 : \langle b_i, b_j \rangle \to \langle b_{j+1}, b_j \rangle$, where $i, j = 0, 1, \ldots, n-1$, are endomorphisms of A. For $n > 2$, ϱ_1 and ϱ_2 have properties (1), (2) but not (3). For $n = 2$ all three properties hold and $A = B_1 \oplus B_2$.

§ 3. Terms in equational classes. Let $\mathfrak{A}$ be an equational class of algebras, i.e. a class of similar algebras closed with respect to taking subalgebras, homomorphic images and direct products of any power. Without loss of generality we shall assume that the operations in algebras of $\mathfrak{A}$ are finite.

Let to every algebra A in $\mathfrak{A}$ be assigned an n-ary operation $\tau^A : A^n \to A$ in such a way that if $h : A \to B$ is a homomorphism and both A and B are in $\mathfrak{A}$, then $h\tau^A(x_1, \ldots, x_n) = \tau^B(hx_1, \ldots, hx_n)$ for $x_1, \ldots, x_n$ in A. Then τ is called an *n-term* in $\mathfrak{A}$.

LEMMA 1. *If F is a free algebra in $\mathfrak{A}$ freely generated by $f_1, \ldots, f_n$, then to every element v in F there exists one and only one n-term τ_v such that $\tau_v^F(f_1, \ldots, f_n) = v$.*

Proof. Let A be an arbitrary algebra in $\mathfrak{A}$ and let $x_1, \ldots, x_n$ be arbitrary elements in A. Since F is free, then there exists a unique homomorphism $h : F \to A$ such that $hf_i = x_i$, $i = 1, \ldots, n$. Let us define: $\tau_v^A(x_1, \ldots, x_n) = hv$. Since A and x_i's have been arbitrary, τ_v is well defined. We shall prove that it is a term. Let futher $h : A \to B$ be a homomorphism and let $x_1, \ldots, x_n$ be elements in A. Let $h_1 : F \to A$ and $h_2 : F \to B$ be homomorphisms with $h_1 f_i = x_i$ and $h_2 f_i = hx_i$. The diagram

$$B \xleftarrow{h_2} F \xrightarrow{h_1} A \xrightarrow{h} B$$

commutes. It follows that $hh_1 v = h_2 v$. By the definition of τ_v,

$$\tau_v^A(x_1, \ldots, x_n) = h_1 v, \qquad \tau_v^B(hx_1, \ldots, hx_n) = h_2 v,$$

hence $h\tau_v^A(x_1, \ldots, x_n) = \tau_v^B(hx_1, \ldots, hx_n)$ as required. To prove the unicity of τ_v, let τ be a term with $\tau(f_1, \ldots, f_n) = v$. Then for every $h : F \to A$ we have

$$\tau_v^A(hf_1, \ldots, hf_n) = hv = \tau^A(hf_1, \ldots, hf_n).$$

Since $hf_1, \ldots, hf_n$ may be arbitrary in an arbitrary algebra in $\mathfrak{A}$, we get $\tau_v = \tau$.

An easy consequence of Lemma 1 is the following

LEMMA 2. *If an algebra A in $\mathfrak{A}$ is generated by a set of its elements X, then to every a in A there exist a number n, an n-term τ in $\mathfrak{A}$, and n elements $x_1, \ldots, x_n$ in X such that $a = \tau^A(x_1, \ldots, x_n)$. In particular, if A is generated by two elements b and c, and a belongs to A, then $\tau(b, c) = a$ for a suitable term τ.*

Let us remark that the Lemma 1 states that we can identify n-terms with elements of a free algebra with n generators and therefore that terms defined as above are identical with operations defined by compositions of primitive operations of algebras in the considered class.

§ 4. Direct sums in classes of algebras. Let $\mathfrak{A}$ be an equational class of algebras and let F be a free algebra in $\mathfrak{A}$ with two free generators.

THEOREM 1. *If the direct product $F\times F$ is decomposable into a normal direct sum $F\times F = F_1 \oplus F_2$ of algebras F_1, F_2 both isomorphic to F, then there exist two terms in $\mathfrak{A}$: one 1-term ζ such that $\zeta^A(x) = \zeta^A(y)$ for every algebra A in $\mathfrak{A}$, and another one, a 2-term a such that*

$$a^A\big(x, \zeta^A(x)\big) = a^A\big(\zeta^A(x), x\big) = x$$

for every algebra A in $\mathfrak{A}$. Moreover, the value of ζ^A (which is unique) is a one-element subalgebra of every algebra A in $\mathfrak{A}$.

Proof. Let f_1, f_1' and f_2, f_2' be free generators in F_1 and F_2, respectively. Let c be the unique element in the common part of F_1 and F_2, and let a be an element of A with $\varrho_1 a = f_1$ and $\varrho_2 a = f_2$.

It follows from Lemma 2 of § 3 that there exists a term ζ such that $\zeta^{F_1}(f_1) = c$. In view of symmetry, $\zeta^{F_1}(f_1') = c$ and therefore $\zeta^{F_1}(f_1) = \zeta^{F_1}(f_1')$. Now let us remark that, as it is well known, an equation holding for free generators of a free algebra in a class holds also for arbitrary algebras in that class. Hence $\zeta^A(x) = \zeta^A(y)$ for x, y in arbitrary algebra A in $\mathfrak{A}$.

From the assumption (3) on the direct sum and Lemma 2 of § 3 we draw the conclusion that there exists a term a such that $a^{F_1\oplus F_2}(f_1, f_2) = a$. Applying homomorphisms ϱ_1 and ϱ_2 to this equation we obtain

$$a^{F_1}(f_1, c) = a^{F_1}\big(f_1, \zeta^{F_1}(f_1)\big) = f_1 \quad \text{and} \quad a^{F_2}(c, f_2) = a^{F_2}\big(\zeta^{F_2}(f_2), f_2\big) = f_2.$$

By the same argument as before we have

$$a^A\big(x, \zeta^A(x)\big) = a^A\big(\zeta^A(x), x\big) = x$$

in every algebra A in $\mathfrak{A}$.

Let x be an element in an algebra A in $\mathfrak{A}$, and let $h\colon F_1 \to A$ be a homomorphism with $hf_1 = x$. Since $hc = h\zeta^{F_1}(f_1) = \zeta^A(hf_1) = \zeta^A(x)$, we infer that $\zeta^A(x)$ is a one-element subalgebra, because it is the homomorphic image of c which is such a one. Moreover, since $\zeta^A(x)$ does not depend on x and belongs to every subalgebra of A generated by one element, it is a minimal subalgebra.

Remark. The assumption that F is a free algebra with more than one generator is essential. Let $\mathfrak{A}$ be a class defined by the equations $(x+x)+y = y$ and $x+(y+z) = (x+y)+z$. The free algebra with one generator in this class is simply a group of order 2, and therefore its direct product is decomposable. But since every algebra with $x+y = y$ belongs to $\mathfrak{A}$, the free algebra with two generators contains no minimal subalgebra.

THEOREM 2. *If $\mathfrak{A}_0$ is a class of algebras closed with respect to direct product of arbitrary power and such that the direct product of any two its algebras B and C is decomposable into a direct sum $B\times C = B_1 \oplus C_1$ of algebras B_1 and C_1 isomorphic to B and C respectively, then in the least equational class $\mathfrak{A}$ containing $\mathfrak{A}_0$ there exist two terms ζ, a such that*

$$\zeta^A(x) = \zeta^A(y) \quad \text{and} \quad a^A\big(x, \zeta^A(x)\big) = a^A\big(\zeta^A(x), x\big) = x$$

in every algebra A in $\mathfrak{A}$.

Proof. Let us observe that the assumption and Lemma 5 of § 2 implies that every direct decomposition of an algebra in $\mathfrak{A}_0$ is normal.

Following a known result of Tarski [2] that the algebra F with two generators free in $\mathfrak{A}$ may be embedded in a direct product of algebras from $\mathfrak{A}_0$, and therefore in an algebra B from $\mathfrak{A}_0$, let us decompose the product $B\times B$ into a direct sum of isomorphic copies of $B: B\times B = B_1 \oplus B_2$. B contains F, and B_1 and B_2 contain isomorphic copies of F. Denoting them by F_1 and F_2 we conclude that the algebra generated by both F_1 and F_2 is a direct sum of F_1 and F_2. Hence the assumptions of Theorem 1 are fulfilled for $\mathfrak{A}$ and therefore the thesis of Theorem 2 follows from Theorem 1.

REFERENCES

[1] B. Jónsson and A. Tarski, *Direct decompositions of finite algebraic systems*, Notre Dame Mathematical Lectures 5 (1947).

[2] A. Tarski, *A remark on functionally free algebras*, Annals of Mathematics 47 (1946), p. 163-165.

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