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ON ESTIMATING A FIFTH ORDER FUNCTIONAL FOR BOUNDED UNIVALENT FUNCTIONS

BY

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1. Introduction and statement of results. The authors are concerned with the class $S(b_1)$ of bounded univalent functions of the form

$$f(z) = b_1 z + b_2 z^2 + \ldots, \quad 0 < b_1 \leqslant 1, \ |z| < 1,$$

which map the unit disc into itself. This class has often been studied in the equivalent form of functions $F = (1/b_1)f$, whose coefficients will be denoted by a_n. Both classes have been developed quite extensively since 1950 (cf., e.g., [5], [1], [6] and [4]). In this paper the authors investigate the problem of finding the sharp estimate for the functional of the fifth order

$$B = |a_5 - pa_2a_4 - qa_3^2 + ra_2^2a_3 - sa_2^4|, \quad p, q, r, s \text{ real},$$

in dependence on p, q, r, s and b_1, where f ranges over $S(b_1)$. In particular, the authors obtain the sharp estimates of the fourth coefficients

$$B_4 = 4a_5 - 4a_2a_4 - 2a_3^2 + 4a_2^2a_3 - a_2^4$$

and

$$C_4 = 20a_5 - 32a_2a_4 - 18a_3^2 + 48a_2a_3 - 16a_2^4$$

of $g(z) = zf'(z)/f(z)$, $|z| < 1$, and $h(z) = 1 + zf''(z)/f'(z)$, $|z| < 1$, respectively, for b_1 sufficiently close to 1. Actually, these classical functionals (cf. [3]) gave a reason to consider a more general functional B. Let

$$A = [\tfrac{1}{2}(5 - p + \tfrac{1}{2}p^2) - (1-p)q - r]^2,$$

$$C = \tfrac{1}{2}(11 - 6p + 2p^2 - p^3) - (1-p)^2 q - 2(1-p)r - 4s,$$

$$t(b_1) = A[(\log b_1^{-1})^{-1} - \tfrac{3}{2} + q]^{-1} + C, \quad \exp[-1/(\tfrac{3}{2} - q)] < b_1 \leqslant 1, \quad q < \tfrac{3}{2},$$

$$T(b_1) = (1 - b_1)^{-2}[(2 - \tfrac{1}{2}p)^2 b_1^2 - (1 - \tfrac{1}{2}p)^2].$$

Obviously, we put $T(1) = +\infty$ for $p < 3$ and $T(1) = -\infty$ for $p \geqslant 3$. The main result of this paper is the following

THEOREM 1. *If p, q, r, s, b_1 are real numbers which satisfy the conditions*

$$q \leqslant \tfrac{3}{2}, \quad A \leqslant (\tfrac{3}{2} - q)C, \quad \exp[-1/(\tfrac{3}{2} - q)] \leqslant b_1 \leqslant 1, \quad t(b_1) \leqslant T(b_1),$$

and f belongs to $S(b_1)$, then the corresponding B does not exceed

$$B^* = \tfrac{1}{2}(1-b_1^4).$$

The estimate is sharp for every p, q, r, s, b_1. All the extremal functions are given by the formula

$$f_c^*(z) = e^{-ic}P^{-1}\big(b_1 P(e^{ic}z)\big), \quad P(z) = z/(1-z^4)^{1/2}, \quad |z|<1, \ -\pi < c \leqslant \pi.$$

Remark 1. The function t is strictly decreasing for $A \neq 0$ and constant for $A = 0$. The function T, restricted to the interval $0 \leqslant b_1 \leqslant 1$, has exactly one extremum (minimum) at

$$b_1^* = (1-\tfrac{1}{2}p)^2/(2-\tfrac{1}{2}p)^2$$

provided that $p < 3$ and is strictly decreasing otherwise. Therefore

$$t(b_1) \geqslant t(1) = C \quad \text{and} \quad T(b_1) \leqslant T(0) = -(1-\tfrac{1}{2}p)^2$$

for

$$\exp[-1/(\tfrac{3}{2}-q)] < b_1 \leqslant \min(b_1^*, 1).$$

On the other hand, since $q < \frac{3}{2}$ and $A \geqslant 0$, $C > (\frac{3}{2}-q)^{-1}A \geqslant 0$. Consequently

$$t(b_1) > T(b_1) \quad \text{for } \exp[-1/(\tfrac{3}{2}-q)] < b_1 \leqslant \min(b_1^*, 1).$$

Thus, given p, q, r and s, Theorem 1 leads to a very simple set of numbers b_1 such that the estimate $B \leqslant B^*$ holds; this set is the interval $b^* \leqslant b_1 \leqslant 1$, where b^* is the only solution of the equation $T(b^*) = t(b^*)$, $b^* > b_1^*$, i.e.

$$\exp\left[-\frac{t(b^*)-C}{(\frac{3}{2}-q)t(b^*)+A-(\frac{3}{2}-q)C}\right]$$
$$= \frac{t(b^*)-[(3-p)t(b^*)+(1-\frac{1}{2}p)^2(2-\frac{1}{2}p)^2]^{1/2}}{t(b^*)-(2-\frac{1}{2}p)^2}, \quad b^* > b_1^*.$$

It is easily seen that b^* is always positive. If $b^* = 1$, the interval in question reduces to the point $b_1 = 1$; if $b^* > 1$, it is the empty set.

Remark 2. The condition $A < (\frac{3}{2}-q)C$ may be rewritten in the equivalent form

$$(\tfrac{3}{2}-q)s$$
$$\leqslant \tfrac{1}{8}(4-4p-p^3-\tfrac{1}{8}p^4)-(3-3p+p^2-\tfrac{1}{4}p^3)q+\tfrac{1}{2}(1+p+\tfrac{1}{4}p^2)r-\tfrac{1}{4}r^2.$$

The authors confine themselves to calculating b^* effectively in three cases of a considerable theoretical importance.

THEOREM 2. *The following sharp estimates hold:*

$$|a_5| \leqslant \tfrac{1}{2}(1-b_1^4), \quad 0.698 \leqslant b_1 \leqslant 1,$$

$$|B_4| \leqslant 2(1-b_1^4), \qquad 0.603 \leqslant b_1 \leqslant 1,$$
$$|C_4| \leqslant 10(1-b_1^4), \qquad 0.460 \leqslant b_1 \leqslant 1.$$

The estimate $|a_5| \leqslant \frac{1}{2}(1-b_1^4)$, $0.75 \leqslant b_1 \leqslant 1$, has been established in [4].

Finally, the authors prove in the case $n = 5$ a conjecture posed in conclusion of [3]:

THEOREM 3. *In the Euclidean (p, q, r, s)-space there is a neighbourhood D of $(0, 0, 0, 0)$ such that if the point (p, q, r, s) and a function f corresponding to (a_2, a_3, a_4, a_5) belong to D and $S(b_1)$, respectively, then $B \leqslant B^*$ for b_1 in an interval $b(p, q, r, s) \leqslant b_1 \leqslant 1$, $0 < b(p, q, r, s) < 1$.*

2. Proof of Theorem 1. The proof is analogous to that given in [4] for $B = |a_5|$.

Consider the power series

$$\log\frac{f(z)-f(z_0)}{z-z_0} = \sum_{m,k=0}^{\infty} A_{mk} z_0^k z^m$$

and

$$-\log[1-\overline{f(z_0)}f(z)] = \sum_{m,k=1}^{\infty} B_{mk} \bar{z}_0^k z^m$$

in the bicylinder $|z| < 1$, $|z_0| < 1$. The fact that f belongs to $S(b_1)$ guarantees evidently that these power series converge in that bicylinder and, conversely, the fact that these power series converge there and that $A_{00} = \log b_1$ guarantees that f belongs to $S(b_1)$. The numbers A_{mk} and B_{mk} are polynomials in $b_1, \bar{b}_1, b_2, \bar{b}_2, \ldots$; they form a symmetric and hermitean matrix, respectively. If x_0 is a real number while $x_1, \ldots, x_n$ are complex numbers (in particular, real), then the following inequalities hold (the so-called *generalized Grunsky-Nehari inequalities*):

$$\mathrm{Re}\sum_{m,k=0}^{n} A_{mk}x_m x_k + \sum_{m,k=1}^{n} B_{mk}x_m\bar{x}_k \leqslant \sum_{m=1}^{n}(1/m)|x_m|^2. \tag{1}$$

We let $n = 2$ and apply (1) to a function f of $S(b_1)$ for which we assume

$$a_5 - pa_2a_4 - qa_3^2 + ra_2^2a_3 - sa_2^4 > 0; \tag{2}$$

this normalization can always be achieved by a properly chosen rotation. Hence we get

$$\begin{aligned}\mathrm{Re}(A_{00}x_0^2 + A_{11}x_1^2 + A_{22}x_2^2 + 2A_{01}x_0x_1 + 2A_{02}x_0x_2 + 2A_{12}x_1x_2)&\\ \leqslant (1-B_{11})|x_1|^2 + (\tfrac{1}{2}-B_{22})|x_2|^2 - 2\,\mathrm{Re}(B_{12}x_1\bar{x}_2)&.\end{aligned} \tag{3}$$

Direct calculation gives

$$A_{00} = \log b_1, \qquad A_{01} = a_2, \qquad A_{02} = a_3 - \tfrac{1}{2}a_2^2,$$

$$A_{11} = a_3 - a_2^2, \qquad A_{12} = a_4 - 2a_2a_3 + a_2^3,$$

$$A_{22} = a_5 - 2a_2a_4 - \tfrac{3}{2}a_3^2 + 4a_2^2a_3 - \tfrac{3}{2}a_2^4$$

and

$$B_{11} = b_1^2, \qquad B_{12} = b_1^2\bar{a}_2,$$

$$B_{22} = b_1^2|a_2|^2 + \tfrac{1}{2}b_1^4.$$

If we insert these expressions in (3), we obtain

$$\begin{aligned}(4)\quad & \log b_1 x_0^2 + \operatorname{Re}[(a_3 - a_2^2)x_1^2 + (a_5 - 2a_2a_4 - \tfrac{3}{2}a_3^2 + 4a_2^2a_3 - \tfrac{3}{2}a_2^4)x_2^2 + \\ & \quad + 2a_2x_0x_1 + 2(a_3 - \tfrac{1}{2}a_2^2)x_0x_2 + 2(a_4 - 2a_2a_3 + a_2^3)x_1x_2] \\ & \qquad \leqslant (1 - b_1^2)|x_1|^2 + (\tfrac{1}{2} - b_1^2|a_2|^2 - \tfrac{1}{2}b_1^4)|x_2|^2 - 2\operatorname{Re}(b_1^2\bar{a}_2x_1\bar{x}_2).\end{aligned}$$

Inequality (4) simplifies considerably for the particular choice $x_1 = \check{x}_1 a_2$, $\check{x}_1 = |\check{x}_1|$, $x_2 = |x_2| \neq 0$, and reduces to

$$\begin{aligned}& \operatorname{Re} a_5 - \tfrac{1}{2}(1 - b_1^4) \\ & \quad \leqslant \operatorname{Re}[2(1 - \check{x}_1x_2^{-1})a_2a_4 + \tfrac{3}{2}a_3^2 - (4 - 4\check{x}_1x_2^{-1} + \check{x}_1^2x_2^{-2})a_2^2a_3 + \\ & \quad + (\tfrac{3}{2} - 2\check{x}_1x_2^{-1} + \check{x}_1^2x_2^{-2})a_2^4] + [(1 - b_1^2)\check{x}_1^2x_2^{-2} - 2b_1^2\check{x}_1x_2^{-1} - b_1^2]|a_2|^2 + \\ & \quad + \log b_1^{-1}x_0^2x_2^{-2} - 2\operatorname{Re}[a_3 + (\check{x}_1x_2^{-1} - \tfrac{1}{2})a_2^2]x_0x_2^{-1}.\end{aligned}$$

Now we choose $\check{x}_1$ so that $2(1 - \check{x}_1x_2^{-1}) = p$, i.e. $\check{x}_1 = (1 - \tfrac{1}{2}p)x_2$. Hence, by (2), we get

$$\begin{aligned}(5)\quad B - B^* &= \operatorname{Re}(a_5 - pa_2a_4 - qa_3^2 + ra_2^2a_3 - sa_2^4) - \tfrac{1}{2}(1 - b_1^4) \\ & \leqslant \operatorname{Re}\{(\tfrac{3}{2} - q)a_3^2 - [(1 + \tfrac{1}{2}p)^2 - r]a_2^2a_3 + [(\tfrac{1}{2}p)^2 + \tfrac{1}{2} - s]a_2^4\} + \\ & \quad + [(1 - \tfrac{1}{2}p)^2 - (2 - \tfrac{1}{2}p)^2b_1^2]|a_2|^2 + \log b_1^{-1}x_0^2x_2^{-2} - \\ & \quad - 2\operatorname{Re}[a_3 + \tfrac{1}{2}(1 - p)a_2^2]x_0x_2^{-1}.\end{aligned}$$

We minimize then the right-hand side of (5) by choosing

$$x_0x_2^{-1} = (\log b_1^{-1})^{-1}\operatorname{Re}[a_3 + \tfrac{1}{2}(1 - p)a_2^2].$$

Consequently, choosing $A^{1/2}$ so that $A^{1/2} = \tfrac{5}{2}$ for $p = q = r = 0$, we have

$$\begin{aligned}& B - B^* \\ & \leqslant \operatorname{Re}\{(\tfrac{3}{2} - q)[a_3 + \tfrac{1}{2}(1 - p)a_2^2]^2 - \\ & \quad - [\tfrac{1}{2}(5 - p + \tfrac{1}{2}p^2) - (1 - p)q - r]a_2^2[a_3 + \tfrac{1}{2}(1 - p)a_2^2] + \\ & \quad + \tfrac{1}{4}[\tfrac{1}{2}(11 - 6p + 2p^2 - p^3) - (1 - p)^2q - 2(1 - p)r - 4s]a_2^4\} + \\ & \quad + [(1 - \tfrac{1}{2}p)^2 - (2 - \tfrac{1}{2}p)^2b_1^2]|a_2|^2 - (\log b_1^{-1})^{-1}\operatorname{Re}^2[a_3 + \tfrac{1}{2}(1 - p)a_2^2] \\ & = \operatorname{Re}\{(\tfrac{3}{2} - q)[a_3 + \tfrac{1}{2}(1 - p)a_2^2]^2 - A^{1/2}a_2^2[a_3 + \tfrac{1}{2}(1 - p)a_2^2] + \tfrac{1}{4}Ca_2^4\} + \\ & \quad + [(1 - \tfrac{1}{2}p)^2 - (2 - \tfrac{1}{2}p)^2b_1^2]|a_2|^2 - (\log b_1^{-1})^{-1}\operatorname{Re}^2[a_3 + \tfrac{1}{2}(1 - p)a_2^2].\end{aligned}$$

We further introduce the notation $a_3+\frac{1}{2}(1-p)a_2^2 = u+iv$, $a_2^2 = U+$ $+iV$. Therefore we arrive at the estimate

$$(6)\quad B-B^* \leqslant (\tfrac{3}{2}-q)(u^2-v^2)-A^{1/2}(uU-vV)+\tfrac{1}{4}C(U^2-V^2)+ \\ +[(1-\tfrac{1}{2}p)^2-(2-\tfrac{1}{2}p)^2b_1^2]|a_2|^2-(\log b_1^{-1})^{-1}u^2.$$

Suppose first that in the conditions under which Theorem 1 is asserted to hold, the first three signes $\leqslant$ are replaced with $<$, respectively. The remaining cases will follow from our particular result by a simple passing to the corresponding limits.

Now we observe that $(\frac{3}{2}-q)v^2-A^{1/2}vV+\frac{1}{4}CV^2 \geqslant 0$, since, by our hypotheses, $q<\frac{3}{2}$ and $A<(\frac{3}{2}-q)C$. Hence we may replace (6) by the simpler inequality

$$\begin{aligned} &B-B^* \\ &\leqslant [\tfrac{3}{2}-q-(\log b_1^{-1})^{-1}]u^2-A^{1/2}uU+\tfrac{1}{4}CU^2+[(1-\tfrac{1}{2}p)^2-(2-\tfrac{1}{2}p)^2b_1^2]|a_2|^2 \\ &= -[(\log b_1^{-1})^{-1}-\tfrac{3}{2}+q]\{u+\tfrac{1}{2}A^{1/2}[(\log b_1^{-1})^{-1}-\tfrac{3}{2}+q]^{-1}U\}^2+ \\ &\quad +\tfrac{1}{4}\{A[(\log b_1^{-1})^{-1}-\tfrac{3}{2}+q]^{-1}+C\}U^2+[(1-\tfrac{1}{2}p)^2-(2-\tfrac{1}{2}p)^2b_1^2]|a_2|^2, \end{aligned}$$

where, by our hypotheses, $b_1>\exp[-1/(\frac{3}{2}-q)]$, i.e., $(\log b_1^{-1})^{-1}-\frac{3}{2}+$ $+q>0$. Therefore

$$(7)\quad B-B^* \leqslant \tfrac{1}{4}\{A[(\log b_1^{-1})^{-1}-\tfrac{3}{2}+q]^{-1}+C\}U^2+ \\ +[(1-\tfrac{1}{2}p)^2-(2-\tfrac{1}{2}p)^2b_1^2]|a_2|^2.$$

On the other hand, we have $U\leqslant |a_2|^2$ and $|a_2|\leqslant 2(1-b_1)$ (cf., e.g., [3], formula (6)). Consequently, for $b_1\neq 1$, (7) becomes

$$B-B^* \leqslant |a_2|^2\big(\{A[(\log b_1^{-1})^{-1}-\tfrac{3}{2}+q]^{-1}+C\}(1-b_1)^2+ \\ +[(1-\tfrac{1}{2}p)^2-(2-\tfrac{1}{2}p)^2b_1^2]\big),$$

i.e.

$$(8)\quad B-B^* \leqslant |a_2|^2(1-b_1)^2[t(b_1)-T(b_1)].$$

Since, by our hypotheses, $t(b_1)\leqslant T(b_1)$, we get $B-B^*\leqslant 0$, as desired.

Finally we notice that, since (2) gives no loss of generality, in order to demonstrate that $B=B^*$ can only hold for $f=f_c^*$, it is sufficient to show that $B=B^*$ with the additional condition (2) can only hold for $f=f_0^*$. To this end we observe first that if

$$q<\tfrac{3}{2},\quad A<(\tfrac{3}{2}-q)C,\quad \exp[-1/(\tfrac{3}{2}-q)]<b_1<1,\quad t(b_1)<T(b_1),$$

then the expression $(1-b_1)^2[t(b_1)-T(b_1)]$ in (8) is negative. Consequently, (8) yields that $B-B^*$ can only vanish for $a_2=0$. Hence we must seek for the extremum functions in the subclass of $S(b_1)$ with $a_2=0$. Here, by (2), we assume that $a_5-pa_2a_4-qa_3^2+ra_2^2a_3-sa_2^4>0$, i.e. $a_5-qa_3^2>0$.

Now we shall prove that if f belongs to $S(b_1)$, $q < \frac{3}{2}$, $A < (\frac{3}{2}-q)C$, $\exp[-1/(\frac{3}{2}-q)] < b_1 < 1$, $a_2 = 0$ and $a_5 - qa_3^2 > 0$, then $f = f_c^*$.

The proof is completely analogous to that given in [7] (p. 16-17) in the case where $q = 0$. On the other hand, for any f_c^*, $-\pi < c \leqslant \pi$, we have $B = B^*$, as it can easily be verified by direct calculation. Thus the proof is completed.

3. Proof of Theorem 2. Consider first $|a_5|$. We have $p = q = r = s = 0$, whence

$$A = \tfrac{25}{4}, \quad C = \tfrac{11}{2}, \quad t(b_1) = 25[4(\log b_1^{-1})^{-1}-6]^{-1}+\tfrac{11}{2},$$
$$T(b_1) = (1-b_1)^{-2}(4b_1^2-1).$$

Therefore

$$b_1 = \exp\left[-\frac{2t(b_1)-11}{3t(b_1)-4}\right] = \frac{T(b_1)-\eta[3T(b_1)+4]^{1/2}}{T(b_1)-4}, \quad \eta = 1 \text{ or } -1.$$

By Theorem 1 and Remark 1, $|a_5| \leqslant \frac{1}{2}(1-b_1^4)$ for $b^* \leqslant b_1 \leqslant 1$, where b^* is the only solution of the equation

$$\exp\left[-\frac{2t(b^*)-11}{3t(b^*)-4}\right] = \frac{t(b^*)-[3t(b^*)+4]^{1/2}}{t(b^*)-4} \tag{9}$$

considered in the interval $\exp(-\frac{2}{3}) < b^* \leqslant 1$. Since, by [2] (p. 65),

$$0.6977 < \exp\left(-\frac{2\cdot 10.39-11}{3\cdot 10.39-4}\right) < 0.6978$$

and, on the other hand,

$$0.6978 < \frac{10.39-(3\cdot 10.39+4)^{1/2}}{10.39-4} < 0.6980,$$

we have $0.697 < b^* < 0.698$. Thus, by (9), we have found the best possible value of b^* for our method.

Similarly, for $|B_4|$ we get $A = \frac{25}{16}$, $C = 2$, and $0.603 \leqslant b_1 \leqslant 1$, while for $|C_4|$ we obtain $A = \frac{144}{625}$, $C = \frac{71}{125}$, and $0.460 \leqslant b_1 \leqslant 1$, as desired.

4. Proof of Theorem 3. Theorem 3 is an immediate consequence of Theorems 1 and 2 and of the fact that $(p, q, r, s, b_1) = (0, 0, 0, 0, b^*)$, where b^* is the same as in Section 3, is a regular point for the equation

$$\exp\left[-\frac{t(b_1)-C}{(\frac{3}{2}-q)t(b_1)+A-(\frac{3}{2}-q)C}\right]$$
$$= \frac{t(b_1)-[(3-p)t(b_1)+(1-\frac{1}{2}p)^2(2-\frac{1}{2}p)^2]^{1/2}}{t(b_1)-(2-\frac{1}{2}p)^2}$$

which is itself a straightforward analogue of (9) for (p, q, r, s, b_1) close to $(0, 0, 0, 0, b^*)$ (cf. Remark 1).

5. Conclusions. In view of the results obtained in [5], [3] and in the present paper it seems to the authors natural to pose the following two conjectures:

CONJECTURE 1 (**P 800**). For every integer n, $n \geqslant 2$, there is an interval

$$b_n^* \leqslant b_1 \leqslant 1, \quad 0 < b_n^* < 1,$$

such that if f belongs to $S(b_1)$, then the n-th coefficient B_n of the corresponding function g satisfies the inequality

$$|B_n| \leqslant 2(1-b_1^{n-1}).$$

The estimate is sharp for every b_1, and all the extremal functions are given by the formula

$$f_{n,c}(z) = e^{-ic} P_n^{-1}\big(b_1 P_n(e^{ic} z)\big), \quad P_n(z) = z/(1-z^{n-1})^{2/(n-1)} \tag{10}$$

$$|z| < 1, \quad -\pi < c \leqslant \pi.$$

CONJECTURE 2 (**P 801**). For every integer n, $n \geqslant 2$, there is an interval

$$b_n^{**} \leqslant b_1 \leqslant 1, \quad 0 < b_n^{**} < 1,$$

such that if f belongs to $S(b_1)$, then the n-th coefficient C_n of the corresponding function h satisfies the inequality

$$|C_n| \leqslant 2n(1-b_1^{n-1}).$$

The estimate is sharp for every b_1 and all the extremal functions are given by formula (10).

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