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NORM INEQUALITIES FOR INTEGRAL OPERATORS ON CONES

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1. Introduction. A non-empty open subset $V \subset \mathbf{R}^n$ is called a *cone* if and only if the following conditions hold:

(i) if $x \in V$ and $\lambda > 0$, then $\lambda x \in V$,
(ii) if $x, y \in V$, then $x + y \in V$,
(iii) if $x \in \overline{V}$ then $-x \notin \overline{V}$.

Condition (iii) is equivalent to

(iii)′ V does not contain a line.

The *dual* V^* of V is defined as $V^* = \{x \in \mathbf{R}^n : x \cdot y > 0\,,\ \forall y \in \overline{V}\,,\ y \neq 0\}$. Clearly, V^* is also a cone. It is well known that $V^{**} = V$ (see [1] and [5]). The cone V is *self-dual* if and only if $V = V^*$.

A cone V defines a partial ordering in $\mathbf{R}^n$. We write $x \underset{V}{<} y$ if and only if $y - x \in V$. The *cone interval* $\langle a, b\rangle$ is then given by $\langle a, b\rangle = \{x \in V : a \underset{V}{<} x \underset{V}{<} b\}$. The *characteristic function* ϕ_V of the cone V is defined by

$$\phi_V(x) = \int_{V^*} e^{-x\cdot y}\, dy\,, \quad x \in V\,. \tag{1}$$

Also, the function Δ_V is defined by

$$\Delta_V(x) = \int_{\langle 0,x\rangle} dy\,, \quad x \in V\,, \tag{2}$$

i.e., $\Delta_V(x)$ is the measure of the cone interval $\langle 0, x\rangle$. Since V is an open set, any matrix A which maps V onto V is regular. The group of all matrices A which map V onto V is called the *automorphism group* of V and denoted by $G(V)$. V is called *homogeneous* iff $G(V)$ is transitive, i.e. for every $x, y \in V$ there exists $A \in G(V)$ such that $y = Ax$. Throughout this paper we will consider only homogeneous cones. We will therefore write "cone" for "homogeneous cone".

Denote by $|A|$ the absolute value of the determinant of the matrix A. A function $f : V \to \mathbf{R}^+$ is said to be *V-homogeneous of order δ* if $f(Ax) = |A|^\delta f(x)$ for all $A \in G(V)$. It can be shown that ϕ_V is V-homogeneous of order -1 and Δ_V is V-homogeneous of order 1. Note that if f is V-homogeneous of any order then f is either identically 0 on V or $f(x) \neq 0$ for all $x \in V$.

The following well-known theorem is an important tool in the proofs of the norm inequalities below. We therefore include a short proof.

THEOREM 1 [4]. *If V is a homogeneous cone, then all V-homogeneous functions are, up to a multiplicative constant, powers of Δ_V.*

Proof. Assume first that f is V-homogeneous of order 0. For any $x, y \in V$, there exists $A \in G(V)$ so that $Ax = y$. Hence, $f(y) = f(Ax) = f(x)$. So, $f(x) = c$. Now if f is V-homogeneous of order δ, then f/Δ_V^δ is V-homogeneous of order 0. Thus, $f(x)/\Delta_V^\delta(x) = c$. ∎

The key to the study of homogeneous cones is the "$*$-function". It is defined by $x^* = -\operatorname{grad}\log\phi_V(x)$ and has the following remarkable properties (see [2], [6], and [7]):

$$(Ax)^* = (A^t)^{-1}x^* \quad \text{for every } A \in G(V)\,, \tag{3}$$

$$\phi_V(x)\cdot\phi_{V^*}(x^*) = c\,, \tag{4}$$

$$(x^*)^* = x\,, \tag{5}$$

$$|\partial_x x^*| = c\phi_V^2(x) = c\phi_{V^*}^{-2}(x^*)\,, \tag{6}$$

where c is a constant depending on V, and $|\partial_x x^*|$ is the absolute value of the Jacobian determinant of the transformation $x \to x^*$. From (3) above and from $V^{**} = V$, we see that $A \in G(V) \iff A^t \in G(V^*)$.

Let $G(V \to V^*)$ be the group of matrices mapping V onto V^*. A homogeneous cone is called a *domain of positivity* (D.P.) if there is $S \in G(V \to V^*)$ so that S is symmetric and positive definite, i.e., for each $x \in \mathbf{R}^n$, $x \neq 0$, $x^t S x > 0$.

Clearly, self-dual cones are D.P. It is shown in [6] that if one such S exists, then there are many more: if V is a D.P., then for every $x \in V$, $K(x) = -(\partial_x x^*)$ is symmetric and positive definite, mapping V onto V^*.

It is known that if V is a D.P., then

$$0 \underset{V}{<} x \underset{V}{<} z \iff 0 \underset{V^*}{<} z^* \underset{V^*}{<} x^*\,. \tag{7}$$

For completeness, we give a proof based on Corollary 3.9 in [6]. The corollary states that, for all $x, y \in V$,

$$(x+y)^* = x^* - K(x)(x^* + y^*)^*\,. \tag{8}$$

Hence,

$$x^* - z^* = K(x)(x^* + (z-x)^*)^* .$$

To prove $x \underset{V}{\leq} z \iff z^* \underset{V^*}{\leq} x^*$, suppose $z - x \in V$. Then

$$z^* = x^* - K(x)(x^* + (z-x)^*)^* \underset{V^*}{\leq} x^* .$$

The converse follows from $V^{**} = V$.

In this paper, we will be considering integral operators of the form

$$Kf(x) = \int_V k(x,y)f(y)\,dy, \quad x \in V, \tag{9}$$

where $k : V \times V \to \mathbf{R}^+$ and $f : V \to \mathbf{R}^+$. The kernel k is said to be *$V \times V$-homogeneous of order β* iff

$$k(Ax, Ay) = |A|^\beta k(x,y) \quad \text{for all } A \in G(V).$$

A number of important integral operators are of the type described above. Hardy's inequality can be considered as an L^p-boundedness result concerning an integral operator on $\mathbf{R}^+$. The Hardy integral operator has a natural generalization to $\mathbf{R}^n$ which is of the form given in (9). In a recent paper, [4], the following theorem is proved:

THEOREM 2 (Ostrogorski). *Let V be a homogeneous, self-dual cone in $\mathbf{R}^n$ and $1 \leq p < \infty$. Assume that k is a $V \times V$-homogeneous kernel of order 0 and $k(x,y) = k(y^*, x^*)$ for all $x, y \in V$. If the integral $K\Delta_V^\alpha(x)$ is convergent (for some $\alpha \in \mathbf{R}$), then*

$$\int_V (Kf(x))^p \Delta_V^{\gamma-p}(x)\,dx \leq c \int_V f^p(x)\Delta_V^\gamma(x)\,dx \tag{10}$$

where $\gamma = -\alpha p - 1$.

As special cases of K on self-dual cones, two operators are considered in [4]: *Hardy's operator*, defined by

$$Hf(x) = \int_{\langle 0,x\rangle} f(y)\,dy, \tag{11}$$

and *Laplace's operator*, defined by

$$Lf(x) = \int_V e^{-x^* \cdot y} f(y)\,dy. \tag{12}$$

Now let $\Sigma = \{x \in V : |x| = 1\}$, $\sigma_0(V) = \inf\{\alpha \in \mathbf{R} : \int_\Sigma \Delta_V^\alpha(t)\,dt < \infty\}$ and $\sigma(V) = \max\{-1, \sigma_0\}$. If $\alpha > \sigma(V)$, then both $H\Delta_V^\alpha(x) < \infty$ and

$L\Delta_V^\alpha(x) < \infty$. Therefore, if $\gamma < -\sigma(V)p - 1$, then

$$\int_V (Hf(x))^p \Delta_V^{\gamma-p}(x)\,dx \le c \int_V f^p(x)\Delta_V^\gamma(x)\,dx\,, \tag{13}$$

$$\int_V (Lf(x))^p \Delta_V^{\gamma-p}(x)\,dx \le c \int_V f^p(x)\Delta_V^\gamma(x)\,dx\,. \tag{14}$$

Note that in $\mathbf{R}^2$, self-dual cones are sectors with a 90° opening. The restriction to such cones seems unnatural and indeed we can prove the results above under more general conditions. Thus, we prove the norm inequality for Hardy's operator on domains of positivity, and the norm inequality for Laplace's operator on general homogeneous cones in $\mathbf{R}^n$. Further, we allow more general kernels k, which enables us to include in our discussion the fractional integral operators of Riemann–Liouville and of Weyl. Finally, we prove all results as $[L^p, L^q]$ inequalities, $1 \le p \le q \le \infty$.

Throughout the paper c stands for a generic constant which does not depend on the function f.

2. Norm inequalities on cones

THEOREM 3. *Let V be a homogeneous cone in $\mathbf{R}^n$ and $1 \le p \le q < \infty$. Assume that the kernel $k : V \times V \to \mathbf{R}^+$ is $V \times V$-homogeneous of order β. If for some $\delta, \gamma \in \mathbf{R}$,*

$$K\Delta_V^\delta(x) = \int_V k(x,y)\Delta_V^\delta(y)\,dy < \infty\,, \tag{15}$$

$$\int_V k^{q/p}(x,y)\Delta_V^{\gamma-q+(\delta+\beta+1)q/p'}(x)\,dx < \infty\,, \tag{16}$$

then

$$\Big(\int_V \Delta_V^{\gamma-q}(x)(Kf(x))^q\,dx\Big)^{1/q} \le c\Big(\int_V f^p(x)\Delta_V^{\beta p+(\gamma+1)p/q-1}(x)\,dx\Big)^{1/p}. \tag{17}$$

Proof. Using Hölder's inequality, we have

$$\int_V \Delta_V^{\gamma-q}(x)(Kf(x))^q\,dx$$
$$= \int_V \Delta_V^{\gamma-q}(x)\Big(\int_V k^{1/p}(x,y)f(y)\Delta_V^{-\delta/p'}(y)k^{1/p'}(x,y)\Delta_V^{\delta/p'}(y)\,dy\Big)^q\,dx$$

$$\le \int_V \Delta_V^{\gamma-q}(x)\Big(\int_V k(x,y)f^p(y)\Delta_V^{-\delta(p-1)}(y)\,dy\Big)^{q/p} \times \Big(\int_V k(x,y)\Delta_V^{\delta}(y)\,dy\Big)^{q/p'}\,dx$$

where $1/p+1/p'=1$. Now, $K\Delta_V^{\delta}(x)$ is finite and for every $A \in G(V)$,

$$K\Delta_V^{\delta}(Ax) = \int_V k(Ax,y)\Delta_V^{\delta}(y)\,dy = \int_V k(Ax,Au)\Delta_V^{\delta}(Au)|A|\,du$$
$$= |A|^{\beta+\delta+1}\int_V k(x,u)\Delta_V^{\delta}(u)\,du$$

so that $K\Delta_V^{\delta}$ is homogeneous of order $\delta+\beta+1$. Hence,

$$\int_V k(x,y)\Delta_V^{\delta}(y)\,dy = c\Delta_V^{\delta+\beta+1}(x)\,.$$

We therefore have

$$\int_V \Delta_V^{\gamma-q}(x)(Kf(x))^q\,dx$$
$$\le c\int_V \Delta_V^{\gamma-q+(\delta+\beta+1)q/p'}(x)\Big(\int_V k(x,y)f^p(y)\Delta_V^{-\delta(p-1)}(y)\,dy\Big)^{q/p}\,dx\,.$$

Since $q/p \ge 1$, we can apply Minkowski's integral inequality to the last integral. Thus,

$$\int_V \Delta_V^{\gamma-q}(x)(Kf(x))^q\,dx$$
$$\le c\Big(\int_V f^p(y)\Delta_V^{-\delta(p-1)}(y) \times \Big(\int_V k^{q/p}(x,y)\Delta_V^{\gamma-q+(\delta+\beta+1)q/p'}(x)\,dx\Big)^{p/q}\,dy\Big)^{q/p}.$$

Again, the innermost integral is finite for each y, and is V-homogeneous of order $\gamma+\beta q+\delta q/p'-q/p+1$. It therefore equals $c\Delta_V^{\gamma+\beta q+\delta q/p'-q/p+1}(y)$. Substituting then yields

$$\int_V \Delta_V^{\gamma-q}(x)(Kf(x))^q\,dx \le c\Big(\int_V f^p(y)\Delta_V^{(\gamma+1)p/q+\beta p-1}(y)\,dy\Big)^{q/p}.$$

Raising both sides to $(1/q)$th power gives the result. ∎

Note that if $V = V^*$, $\beta = 0$, and $p = q$, we obtain (10).

The following theorem corresponds to the case $q = \infty$ in Theorem 3.

THEOREM 4. *Let V be a homogeneous cone and $1 \le p < \infty$. Assume that the kernel $k : V \times V \to \mathbf{R}^+$ is $V \times V$-homogeneous of order β. Assume also that*

$$K\Delta_V^\delta(x) = \int_V k(x,y)\Delta_V^\delta(y)\,dy < \infty \quad \textit{for some } \delta\,, \tag{18}$$

$$g(y) = \operatorname*{ess\,sup}_{x\in V} k(x,y)\Delta_V^{(\delta+\beta)(p-1)-1}(x) < \infty\,. \tag{19}$$

Then

$$\operatorname*{ess\,sup}_{x\in V} \Delta_V^{-1}(x)Kf(x) \le c\Big(\int_V f^p(x)\Delta_V^{\beta p-1}(x)\,dx\Big)^{1/p}. \tag{20}$$

Proof. For every $A \in G(V)$,

$$\begin{aligned} g(Ay) &= \operatorname*{ess\,sup}_{x\in V} k(x,Ay)\Delta_V^{(\delta+\beta)(p-1)-1}(x) \\ &= \operatorname*{ess\,sup}_{x\in V} k(Ax,Ay)\Delta_V^{(\delta+\beta)(p-1)-1}(Ax) \\ &= |A|^{\delta(p-1)+\beta p-1}\operatorname*{ess\,sup}_{x\in V} k(x,y)\Delta_V^{(\delta+\beta)(p-1)-1}(x)\,. \end{aligned}$$

So, g is V-homogeneous of order $\delta(p-1)+\beta p-1$ and therefore,

$$g(y) = c\Delta_V^{\delta(p-1)+\beta p-1}(y)\,.$$

Now, by Hölder's inequality,

$$\begin{aligned} &\Delta_V^{-1}(x)Kf(x) \\ &= \Delta_V^{-1}(x)\int_V k^{1/p}(x,y)\Delta_V^{-\delta/p'}(y)f(y)k^{1/p'}(x,y)\Delta_V^{\delta/p'}(y)\,dy \\ &\le \Delta_V^{-1}(x)\Big(\int_V k(x,y)\Delta_V^{-\delta(p-1)}(y)f^p(y)\,dy\Big)^{1/p}\Big(\int_V k(x,y)\Delta_V^{\delta}(y)\,dy\Big)^{1/p'} \\ &= c\Delta_V^{(\delta+\beta+1)1/p'-1}(x)\Big(\int_V k(x,y)\Delta_V^{-\delta(p-1)}(y)f^p(y)\,dy\Big)^{1/p} \\ &= c\Big(\int_V k(x,y)\Delta_V^{-p+(\delta+\beta+1)(p-1)}(x)\Delta_V^{-\delta(p-1)}(y)f^p(y)\,dy\Big)^{1/p} \\ &\le c\Big(\int_V \Delta_V^{\delta(p-1)+\beta p-1}(y)\Delta_V^{-\delta(p-1)}(y)f^p(y)\,dy\Big)^{1/p} \end{aligned}$$

$$= c\Big(\int_V \Delta_V^{\beta p-1}(y) f^p(y)\,dy\Big)^{1/p}. \ \blacksquare$$

The conclusion of Theorem 4 can also be proved under somewhat different conditions.

THEOREM 5. *Let V be a homogeneous cone and $1 \le p < \infty$. Assume that the kernel k is $V \times V$-homogeneous of order β. Assume also that*

$$g(x) = \Big(\int_V k^{p'}(x,y)\Delta_V^{(1-\beta)p'-1}(y)\,dy\Big)^{1/p'} < \infty. \tag{21}$$

Then

$$\operatorname*{ess\,sup}_{x\in V} \Delta_V^{-1}(x) Kf(x) \le c\Big(\int_V f^p(y)\Delta_V^{\beta p-1}(y)\,dy\Big)^{1/p}. \tag{22}$$

Proof. Applying Hölder's inequality, we get

$$\Delta_V^{-1}(x)Kf(x) = \Delta_V^{-1}(x)\int_V k(x,y)\Delta_V^{-\beta+1/p}(y) f(y)\Delta_V^{\beta-1/p}(y)\,dy$$

$$\le \Delta_V^{-1}(x)\Big(\int_V f^p(y)\Delta_V^{\beta p-1}(y)\,dy\Big)^{1/p}\Big(\int_V k^{p'}(x,y)\Delta_V^{(1-\beta)p'-1}(y)\,dy\Big)^{1/p'}.$$

Since $g(x) < \infty$ and is V-homogeneous of order 1, we have $g(x) = c\Delta_V(x)$, and the conclusion follows from (21). $\blacksquare$

The following theorem corresponds to the case $p = \infty$ and $q = \infty$ in Theorem 3.

THEOREM 6. *Let V be a homogeneous cone. Assume that the kernel k is $V \times V$-homogeneous of order β. Assume also that*

$$K\Delta_V^{\delta}(x) = \int_V k(x,y)\Delta_V^{\delta}(y)\,dy < \infty \quad \textit{for some } \delta. \tag{23}$$

Then

$$\operatorname*{ess\,sup}_{x\in V} \Delta_V^{-1-\delta-\beta}(x) Kf(x) \le c \operatorname*{ess\,sup}_{x\in V} f(x)\Delta_V^{-\delta}(x). \tag{24}$$

Proof.

$$\Delta_V^{-1-\delta-\beta}(x)Kf(x) = \Delta_V^{-1-\delta-\beta}(x)\int_V k(x,y)\Delta_V^{\delta}(y) f(y)\Delta_V^{-\delta}(y)\,dy$$

$$\le \Delta_V^{-1-\delta-\beta}(x)\Big(\int_V k(x,y)\Delta_V^{\delta}(y)\,dy\Big)\operatorname*{ess\,sup}_{y\in V} f(y)\Delta_V^{-\delta}(y)$$

$$= c \operatorname*{ess\,sup}_{y\in V} f(y)\Delta_V^{-\delta}(y). \ \blacksquare$$

Another reasonable generalization of Ostrogorski's theorem is to consider kernels $k : V^* \times V \to \mathbf{R}^+$. The kernel $k : V^* \times V \to \mathbf{R}^+$ is said to be $V^* \times V$-*homogeneous of order* β iff $k((A^t)^{-1}x, Ay) = |A|^\beta k(x,y)$ for all $A \in G(V)$. The following discussion shows that all results concerning such kernels can be deduced from results on kernels $V \times V \to \mathbf{R}^+$.

DEFINITION 1. We define an operator S mapping measurable functions on V onto measurable functions on V^* by $Sf(x) = f(x^*)$ where $x \in V^*$.

THEOREM 7.

$$\int_{V^*} (Sf(x))^q \Delta_{V^*}^\delta(x)\, dx = c \int_V f^q(y) \Delta_V^{-\delta-2}(y)\, dy. \tag{25}$$

Proof. Using (4) and (6), we obtain

$$\int_{V^*} (Sf(x))^q \Delta_{V^*}^\delta(x)\, dx = c \int_V f^q(x^*) \Delta_V^{-\delta}(x^*) |\partial_{x^*} x|\, dx^*$$

$$= c \int_V f^q(x^*) \Delta_V^{-\delta}(x^*) \Delta_V^{-2}(x^*)\, dx^* = c \int_V f^q(y) \Delta_V^{-\delta-2}(y)\, dy. \ \blacksquare$$

Applying Theorem 7, we obtain the following versions of Theorems 3–6 for $V^* \times V$ operators.

THEOREM 8. *Let V be a homogeneous cone in $\mathbf{R}^n$ and $1 \le p \le q < \infty$. Assume that the kernel k is $V^* \times V$-homogeneous of order β. If*

$$\int_V k(x,y) \Delta_V^\delta(y)\, dy < \infty, \tag{26}$$

$$\int_{V^*} k^{q/p}(x,y) \Delta_{V^*}^{-\gamma+q-2-(\delta+\beta+1)q/p'}(x)\, dx < \infty, \tag{27}$$

then

$$\Big(\int_{V^*} (Kf(x))^q \Delta_{V^*}^{-\gamma+q-2}(x)\, dx \Big)^{1/q} \le c \Big(\int_V f^p(y) \Delta_V^{\beta p+(\gamma+1)p/q-1}(y)\, dy \Big)^{1/p}. \tag{28}$$

Proof. We will apply Theorem 3. Define $\widetilde{k} : V \times V \to \mathbf{R}^+$ by

$$\widetilde{k}(u,y) = k(u^*, y).$$

Clearly, $\widetilde{k}$ is $V \times V$-homogeneous of order β if and only if k is $V^* \times V$-

homogeneous of order β. Then

$$SKf(u) = \int_V \tilde{k}(u,y)f(y)\,dy\,.$$

By hypothesis,

$$\int_V \tilde{k}(u,y)\Delta_V^{\delta}(y)\,dy = \int_V k(u^*,y)\Delta_V^{\delta}(y)\,dy < \infty\,.$$

Also, by Theorem 7 and (27),

$$\begin{aligned}
&\int_V \tilde{k}^{q/p}(u,y)\Delta_V^{\gamma-q+(\delta+\beta+1)q/p'}(u)\,du \\
&\quad = c\int_{V^*} (S\tilde{k})^{q/p}(x,y)\Delta_{V^*}^{-\gamma+q-2-(\delta+\beta+1)q/p'}(x)\,dx \\
&\quad = c\int_{V^*} k^{q/p}(x,y)\Delta_{V^*}^{-\gamma+q-2-(\delta+\beta+1)q/p'}(x)\,dx < \infty\,.
\end{aligned}$$

Note that $S(\int_V \tilde{k}(u,y)f(y)\,dy) = Kf(u)$. So by Theorems 3 and 7,

$$\begin{aligned}
&\Big(\int_{V^*} (Kf(u))^q\Delta_{V^*}^{-\gamma+q-2}(u)\,du\Big)^{1/q} \\
&\quad = c\Big(\int_V \Big(\int_V \tilde{k}(u,y)f(y)\,dy\Big)^q \Delta_V^{\gamma-q}(u)\,du\Big)^{1/q} \\
&\quad \le c\Big(\int_V f^p(x)\Delta_V^{\beta p+(\gamma+1)p/q-1}(x)\,dx\Big)^{1/p}\,. \blacksquare
\end{aligned}$$

The following theorem corresponds to the case $q=\infty$ in Theorem 8.

THEOREM 9. *Let V be a homogeneous cone and $1 \le p < \infty$. Assume that the kernel $k : V^* \times V \to \mathbf{R}^+$ is $V^* \times V$-homogeneous of order β. If*

$$\int k(x,y)\Delta_V^{\delta}(y)\,dy < \infty \quad \textit{for some } \delta\,, \tag{29}$$

$$\operatorname*{ess\,sup}_{x\in V^*} k(x,y)\Delta_{V^*}^{-(\delta+\beta)(p-1)+1}(x) < \infty\,, \tag{30}$$

then

$$\operatorname*{ess\,sup}_{x\in V^*} \Delta_{V^*}(x)Kf(x) \le c\Big(\int_V f^p(y)\Delta_V^{\beta p-1}(y)\,dy\Big)^{1/p}\,. \tag{31}$$

The conclusion of Theorem 9 also follows from different conditions.

THEOREM 10. *Let V be a homogeneous cone and $1 \le p < \infty$. Assume that the kernel k is $V^* \times V$-homogeneous of order β. If*

$$\int_V k^{p'}(x,y)\Delta_V^{(1-\beta)p'-1}(y)\,dy < \infty, \tag{32}$$

then

$$\operatorname*{ess\,sup}_{x\in V^*} \Delta_{V^*}(x)Kf(x) \le c\Big(\int_V f^p(y)\Delta_V^{\beta p-1}(y)\,dy\Big)^{1/p}. \tag{33}$$

THEOREM 11. *Let V be a homogeneous cone. Assume that the kernel $k : V^* \times V \to \mathbf{R}^+$ is $V^* \times V$-homogeneous of order β. If*

$$\int_V k(x,y)\Delta_V^{\delta}(y)\,dy < \infty \quad \textit{for some } \delta, \tag{34}$$

then

$$\operatorname*{ess\,sup}_{x\in V^*} \Delta_{V^*}^{1+\delta+\beta}(x)Kf(x) \le c \operatorname*{ess\,sup}_{y\in V} f(y)\Delta_V^{-\delta}(y). \tag{35}$$

3. Applications. As a first application, we consider norm inequalities in $\mathbf{R}^n$ for the Riemann–Liouville and Weyl fractional integral operators. See [3] for some related recent results.

The *Riemann–Liouville operator* is defined by

$$R_r f(x) = \frac{1}{\Gamma(r)} \int_{\langle 0,x\rangle} \Delta_V^{r-1}(x-t)f(t)\,dt, \quad x \in V. \tag{36}$$

The *Weyl operator* is defined by

$$W_r f(x) = \frac{1}{\Gamma(r)} \int_{\langle x,\infty\rangle} \Delta_V^{r-1}(t-x)f(t)\,dt, \quad x \in V, \tag{37}$$

where $r \ge 1$.

Note that the Weyl operator is the dual of the Riemann–Liouville operator: for any non-negative measurable functions f and g defined on V,

$$\int_V (R_r g)\cdot f = \int_V g \cdot (W_r f). \tag{38}$$

We also note that the cases $r = 1$ in Theorems 12 and 13 below provide generalizations of the well-known Hardy's inequalities from $\mathbf{R}$ to $\mathbf{R}^n$.

THEOREM 12 (Riemann–Liouville's Inequalities). *Let V be a domain of positivity in $\mathbf{R}^n$. If $1 \le p \le q < \infty$ and $\gamma < -\sigma(V)q/p' - \sigma(V^*) + q(1/p -$*

$r+1)-2$, *then*

$$\left(\int_V \Delta_V^{\gamma-q}(x)(R_r f(x))^q\,dx\right)^{1/q} \le c\left(\int_V f^p(x)\Delta_V^{(r-1)p+(\gamma+1)p/q-1}(x)\,dx\right)^{1/p}. \tag{39}$$

If $1\le p<\infty$ *and* $\alpha<2-(1+\sigma(V))/p'-r$, *then*

$$\operatorname*{ess\,sup}_{x\in V}\Delta_V^{-1+\alpha}(x)R_r(f\Delta_V^{-\alpha})(x) \le c\left(\int_V f^p(x)\Delta_V^{(r-1)p-1}(x)\,dx\right)^{1/p}. \tag{40}$$

If $\alpha<1-r-\sigma(V)$, *then*

$$\operatorname*{ess\,sup}_{x\in V}\Delta_V^{-1+\alpha}(x)R_r(f\Delta_V^{-\alpha})(x)\le c\operatorname*{ess\,sup}_{x\in V} f(x)\Delta_V^{r-1}(x). \tag{41}$$

Proof. We prove (39) as an application of Theorem 3. The kernel

$$k(x,t)=\frac{1}{\Gamma(r)}\Delta_V^{r-1}(x-t)\chi_{\langle 0,x\rangle}(t)$$

is $V\times V$-homogeneous of order $r-1$.

Now, since $x\underset{V}{\le} y$ implies $\Delta_V(x)<\Delta_V(y)$ for all $x,y\in V$, we have,

$$\begin{aligned}R_r\Delta_V^{\delta}(x)&=\frac{1}{\Gamma(r)}\int_{\langle 0,x\rangle}\Delta_V^{r-1}(x-t)\Delta_V^{\delta}(t)\,dt\\&\le\frac{1}{\Gamma(r)}\int_{\langle 0,x\rangle}\Delta_V^{r-1}(x)\Delta_V^{\delta}(t)\,dt=\frac{\Delta_V^{r-1}(x)}{\Gamma(r)}\int_{\langle 0,x\rangle}\Delta_V^{\delta}(t)\,dt\end{aligned}$$

and the last integral is finite if $\delta>\sigma(V)$. Moreover, since V is a D.P., $t\underset{V}{\le} x$ iff $x^*\underset{V^*}{\le} t^*$, see (7). Therefore,

$$\begin{aligned}&\Gamma^{q/p}(r)\int_V k^{q/p}(x,t)\Delta_V^{\gamma-q+(\delta+r)q/p'}(x)\,dx\\&=\int_{\langle t,\infty\rangle}\Delta_V^{(r-1)q/p}(x-t)\Delta_V^{\gamma-q+(\delta+r)q/p'}(x)\,dx\\&\le\int_{\langle t,\infty\rangle}\Delta_V^{(r-1)q/p+\gamma-q+(\delta+r)q/p'}(x)\,dx\\&=c\int_{\langle 0,t^*\rangle}\Delta_{V^*}^{-(r-1)q/p-\gamma+q-(\delta+r)q/p'-2}(x^*)\,dx^*.\end{aligned}$$

Again, the integral on the right is finite if

$$\delta' = -(r-1)q/p - \gamma + q - (\delta + r)q/p' - 2 > \sigma(V^*).$$

Now, the condition on γ in the hypothesis ensures the existence of $\delta \in \mathbf{R}$ such that $\delta > \sigma(V)$ and $\delta' > \sigma(V^*)$. By applying Theorem 3, the proof is complete.

To prove (40), we apply Theorem 5 to the kernel

$$k(x,t) = \frac{1}{\Gamma(r)}\chi_{\langle 0,x\rangle}(t)\Delta_V^{\alpha}(x)\Delta_V^{-\alpha}(t)\Delta_V^{r-1}(x-t). \tag{42}$$

Clearly, k is $V \times V$-homogeneous of order $r-1$. Also,

$$\int_{\langle 0,x\rangle} \Delta_V^{\alpha p'}(x)\Delta_V^{-\alpha p'}(t)\Delta_V^{(r-1)p'}(x-t)\Delta_V^{(2-r)p'-1}(t)\,dt$$

$$\leq \Delta_V^{\alpha p'+(r-1)p'}(x) \int_{\langle 0,x\rangle} \Delta_V^{(2-r-\alpha)p'-1}(t)\,dt,$$

and the last integral is finite since $\alpha < 2-(1+\sigma(V))/p' - r$. The conditions of Theorem 5 are satisfied and we get (40).

To prove (41), we apply Theorem 6 to the kernel $k(x,t)$ of (42). Let $\delta = 1-r$ in (23). We have

$$\frac{1}{\Gamma(r)}\int_{\langle 0,x\rangle} \Delta_V^{\alpha}(x)\Delta_V^{-\alpha}(t)\Delta_V^{r-1}(x-t)\Delta_V^{1-r}(t)\,dt$$

$$\leq \frac{\Delta_V^{\alpha+r-1}(x)}{\Gamma(r)} \int_{\langle 0,x\rangle} \Delta_V^{1-r-\alpha}(t)\,dt.$$

Now, since $\alpha < 1 - r - \sigma(V)$, the last integral is finite, and the proof is complete. ■

THEOREM 13 (Weyl's Inequalities). *Let V be a domain of positivity in $\mathbf{R}^n$. If $1 \leq p \leq q < \infty$ and $\gamma > \sigma(V) + \sigma(V^*)q/p' + 2q - q/p$, then*

$$\left(\int_V \Delta_V^{\gamma-q}(x)(W_r f(x))^q\,dx\right)^{1/q} \tag{43}$$

$$\leq c\left(\int_V f^p(x)\Delta_V^{(r-1)p+(\gamma+1)p/q-1}(x)\,dx\right)^{1/p}.$$

If $1 \leq p < \infty$ and $\alpha > \sigma(V^*)/p' + 2 - 1/p$, then

$$\operatorname*{ess\,sup}_{x\in V} \Delta_V^{-1+\alpha}(x)W_r(f\Delta_V^{-\alpha})(x) \tag{44}$$

$$\le c\Big(\int\limits_V f^p(x)\Delta_V^{(r-1)p-1}(x)\,dx\Big)^{1/p}.$$

If $\alpha > 2 + \sigma(V^*)$, *then*

$$\operatorname*{ess\,sup}_{x\in V} \Delta_V^{-1+\alpha}(x) W_r(f\Delta^{-\alpha})(x) \le c \operatorname*{ess\,sup}_{x\in V} f(x)\Delta_V^{r-1}(x)\,. \tag{45}$$

Proof. We will use duality of the Weyl operator and the Riemann–Liouville operator. Let g be any non-negative function defined on V with $\int_V g^{q'}(x)\,dx = 1$, where $1/q' + 1/q = 1$. Applying Hölder's inequality we have

$$\begin{aligned}
&\int\limits_V \Delta_V^{\gamma/q-1}(x)g(x)W_r f(x)\,dx = \int\limits_V f(x)R_r(g\Delta_V^{\gamma/q-1})(x)\,dx\\
&\le \Big(\int\limits_V f^p(x)\Delta_V^{(r-1)p+(\gamma+1)p/q-1}(x)\,dx\Big)^{1/p}\\
&\quad\times\Big(\int\limits_V \Delta_V^{-(r-1)p'-(\gamma+1)p'/q+p'-1}(x)(R_r(g\Delta_V^{\gamma/q-1})(x))^{p'}\,dx\Big)^{1/p'}.
\end{aligned}$$

We now apply Theorem 12 to the last integral. Note that $q \ge p$ iff $p' \ge q'$; also $\gamma > \sigma(V) + \sigma(V^*)q/p' + 2q - q/p$ implies that

$$\begin{aligned}
&(-(r-1)p' - (\gamma+1)p'/q + p' - 1) + p'\\
&\qquad < -\sigma(V)p'/q - \sigma(V^*) + p'(1/q' - r + 1) - 2\,.
\end{aligned}$$

Therefore, using (39), we see that the last integral is majorized by

$$\begin{aligned}
&c\Big(\int\limits_V (g(x)\Delta_V^{\gamma/q-1}(x))^{q'}\Delta_V^{2q'-(\gamma+1)q'/q-1}(x)\,dx\Big)^{1/q'}\\
&\qquad = c\Big(\int\limits_V g^{q'}(x)\,dx\Big)^{1/q'} = c\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int\limits_V \Delta_V^{\gamma/q-1}(x)g(x)W_r f(x)\,dx\\
&\qquad \le c\Big(\int\limits_V f^p(x)\Delta_V^{(r-1)p+(\gamma+1)p/q-1}(x)\,dx\Big)^{1/p}.
\end{aligned}$$

Taking supremum over all $g \ge 0$ so that $\int_V g^{q'}(x)\,dx = 1$, we get (43).

To prove (44), let g be any non-negative function on V so that

$\int_V g(x)\,dx = 1$. We have

$$\int_V \Delta_V^{-1+\alpha}(x)g(x)W_r(f\Delta_V^{-\alpha})(x)\,dx = \int_V f(x)\Delta_V^{-\alpha}(x)R_r(g\Delta_V^{-1+\alpha})(x)\,dx$$
$$\leq \Big(\int_V f^p(x)\Delta_V^{(r-1)p-1}(x)\,dx\Big)^{1/p}$$
$$\times \Big(\int_V \Delta_V^{-(r-1)p'+(p'-1)-\alpha p'}(x)(R_r(g\Delta_V^{-1+\alpha})(x))^{p'}\,dx\Big)^{1/p'}.$$

Now, the condition $\alpha > \sigma(V^*)/p' + 2 - 1/p$ implies that

$$-(r-1)p' + (p'-1) - \alpha p' + p' < -\sigma(V^*) + p'(1-r+1) - 2$$

and hence using (39), the last integral is majorized by

$$c\int_V g(x)\Delta_V^{-1+\alpha}(x)\Delta_V^{(r-1)+(-(r-1)p'+(p'-1)-\alpha p'+p'+1)/p'-1}(x)\,dx$$
$$= c\int_V g(x)\,dx = c.$$

Therefore,

$$\int_V \Delta_V^{-1+\alpha}(x)g(x)W_r(f\Delta_V^{-\alpha})(x)\,dx \leq c\Big(\int_V f^p(x)\Delta_V^{(r-1)p-1}(x)\,dx\Big)^{1/p}.$$

Taking supremum over all $g \geq 0$ so that $\int_V g(x)\,dx = 1$ completes the proof of (44).

Finally, to prove (45), let g be such that $\int_V g(x)\,dx = 1$. We have

$$\int_V \Delta_V^{-1+\alpha}(x)g(x)W_r(f\Delta_V^{-\alpha})(x)\,dx = \int_V f(x)\Delta_V^{-\alpha}(x)R_r(g\Delta_V^{-1+\alpha})(x)\,dx$$
$$\leq \operatorname*{ess\,sup}_{x\in V}(f(x)\Delta_V^{r-1}(x))\int_V \Delta_V^{-\alpha-r+1}(x)R_r(g\Delta_V^{-1+\alpha})(x)\,dx.$$

By Fubini's theorem

$$\int_V \Delta_V^{-\alpha-r+1}(x)R_r(g\Delta_V^{-1+\alpha})(x)\,dx$$
$$= \frac{1}{\Gamma(r)}\int_V g(t)\Delta_V^{-1+\alpha}(t)\Big(\int_{\langle t,\infty\rangle} \Delta_V^{r-1}(x-t)\Delta_V^{-\alpha-r+1}(x)\,dx\Big)\,dt.$$

Now, since $\alpha > 2 + \sigma(V^*)$, it can be shown that the integral in parentheses is finite and V-homogeneous of order $-\alpha+1$; it therefore equals $c\Delta_V^{-\alpha+1}(t)$.

Hence, the last expression is a constant, and therefore

$$\int_V \Delta_V^{-1+\alpha}(x)g(x)W_r(f\Delta_V^{-\alpha})(x)\,dx \le c \operatorname*{ess\,sup}_{x\in V} f(x)\Delta_V^{r-1}(x)\,.$$

Taking supremum over all $g \ge 0$ with $\int_V g^{q'}(x)\,dx = 1$ gives us (45). ∎

Note that Theorem 13 could also have been proved directly.

We remark that Riemann–Liouville's and Weyl's inequalities were proved for domains of positivity. Laplace's inequalities, however, will be proved in the more general setting of homogeneous cones.

THEOREM 14 (Laplace's Inequalities). *Let V be a homogeneous cone in $\mathbf{R}^n$. If $1 \le p \le q < \infty$ and $\gamma < -\sigma(V)q/p' - \sigma(V^*) + q/p - 2$, then*

$$\left(\int_V \Delta_V^{\gamma-q}(x)(Lf(x))^q\,dx\right)^{1/q} \le c\left(\int_V f^p(x)\Delta_V^{(\gamma+1)p/q-1}(x)\,dx\right)^{1/p}. \tag{46}$$

If $1 \le p < \infty$ and $\alpha < 1/p - \sigma(V)/p'$, then

$$\operatorname*{ess\,sup}_{x\in V} \Delta_V^{-1+\alpha}(x)L(f\Delta_V^{-\alpha})(x) \le c\left(\int_V f^p(x)\Delta_V^{-1}(x)\,dx\right)^{1/p}. \tag{47}$$

If $\alpha < -\sigma(V)$, then

$$\operatorname*{ess\,sup}_{x\in V} \Delta_V^{-1+\alpha}(x)L(f\Delta_V^{-\alpha})(x) \le c \operatorname*{ess\,sup}_{x\in V} f(x)\,. \tag{48}$$

Proof. We will consider the kernel $k(x,y) = e^{-x^*\cdot y}$. Since $(Ax)^* = (A^t)^{-1}x^*$, k is $V \times V$-homogeneous of order 0. We will apply Theorem 3. To verify (15) we note that it is shown in [4] that if $\delta > \sigma(V)$, then

$$\int_V e^{-x^*\cdot y}\Delta_V^\delta(y)\,dy < \infty\,.$$

To verify (16) we note

$$\int_V e^{-(x^*\cdot y)q/p}\Delta_V^{\gamma-q+(\delta+1)q/p'}(x)\,dx = c\int_{V^*} e^{-(x^*\cdot y)q/p}\Delta_{V^*}^{-\gamma+q-(\delta+1)q/p'-2}(x^*)\,dx^*\,.$$

We therefore need δ which satisfies both

$$\sigma(V) < \delta \quad \text{and} \quad -\gamma + q - (\delta+1)q/p' - 2 > \sigma(V^*)\,.$$

Our condition on γ implies the existence of such δ, and (46) is proved.

To prove (47), we verify that the kernel

$$k(x,y) = e^{-x^*\cdot y}\Delta_V^{\alpha}(x)\Delta_{V^*}^{-\alpha}(y) \tag{49}$$

satisfies (21). The kernel k is $V \times V$-homogeneous of order 0 and

$$\int_V e^{-(x^*\cdot y)p'}\Delta_V^{\alpha p'}(x)\Delta_{V^*}^{-\alpha p'}(y)\Delta_V^{p'-1}(y)\,dy$$

$$= \Delta_V^{\alpha p'}(x)\int_V e^{-(x^*\cdot y)p'}\Delta_{V^*}^{-\alpha p'+p'-1}(y)\,dy,$$

and the last integral is finite since $\alpha < 1/p - \sigma(V)/p'$. The result now follows from Theorem 5.

Finally, to prove (48), we verify that the kernel (49) satisfies the condition (23) with $\delta = 0$. Thus,

$$\int_V e^{-x^*\cdot y}\Delta_V^{\alpha}(x)\Delta_{V^*}^{-\alpha}(y)\,dy = \Delta_V^{\alpha}(x)\int_V e^{-x^*\cdot y}\Delta_{V^*}^{-\alpha}(y)\,dy.$$

The latter integral is finite since $-\alpha > \sigma(V)$ and the result follows from Theorem 6. ∎

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