COLLOQUIUM MATHEMATICUM

VOL. XXIII 1971 FASC. 1

DIMENSION INEQUALITIES FOR UNIONS AND MAPPINGS OF SEPARABLE METRIC SPACES

BY

A. LELEK (WARSZAWA)

Some inequalities involving the dimension of a space and the dimension of its image under a closed mapping were discovered by Arhangel'skiĭ [8], Freudenthal [17] and Keesling [30]. In this paper we shall show how, in the separable metric case, all of them follow from two inequalities of Vaĭnšteĭn [83]. We shall also obtain new inequalities and describe an example. In addition, the present paper is intended to survey the field outlined by the title, and this is why we shall discuss a number of other related results. Our attention is given to separable metric spaces which, from the topological point of view, coincide with subsets of the Hilbert cube. Because the Hilbert cube is infinite-dimensional, it is natural that some topics of infinite-dimensional topology approach our area and can be treated as infinite-dimensional analogues of dimension inequalities. However, we shall restrict ourselves to the finite-dimensional case which seems to bring root ideas of the matter. The classical book of Hurewicz and Wallman [23] serves as a general reference for readers not acquainted with dimension theory. All mappings throughout are meant to be continuous; all spaces and their images under mappings are assumed to be separable metric and non-empty.

Let us recall two fundamental theorems which used to appear in any course in dimension theory (see [23], p. 28-32). Both of them will be applied in the sequel.

THEOREM 1 (K. Menger and P. S. Urysohn). *If A_i $(i = 1, \ldots, k)$ are subsets of a space, then*

(i) $$\dim \bigcup_{i=1}^{k} A_i \leqslant \sum_{i=1}^{k} \dim A_i + k - 1.$$

Note that sometimes inequality (i) becomes the equality. Actually, each k-dimensional space can be represented as the union of $k+1$ disjoint 0-dimensional subsets(1).

(1) Theorem 1 has several analogues for non-metrizable spaces, and its proper range seems to be the class of hereditarily normal spaces (see [15], p. 306 - 307). See also Adnadževič [3], Egorov and Podstavkin [13], de Groot and Nishiura [19] and [60], Smirnov [76] and [77], Zarelua [85].

THEOREM 2 (W. Hurewicz and L. A. Tumarkin). *If* A_i $(i = 1, 2, \ldots)$ *are closed subsets of a space, then*

$$\dim \bigcup_{i=1}^{\infty} A_i = \operatorname{Sup}\{\dim A_i : i = 1, 2, \ldots\}.$$

A collection $\{A_s : s \in S\}$ of subsets A_s of a space X is called *locally finite* (or *locally countable*) provided each point $x \in X$ has a neighbourhood U_x in X such that U_x meets only finitely many (or countably many, respectively) sets A_s. Since the notion of dimension for separable metric spaces has a local character, the next theorem(²) follows directly from Theorem 2.

THEOREM 3. *If* A_s $(s \in S)$ *are closed subsets of a space such that the collection* $\{A_s : s \in S\}$ *is locally countable, then*

$$\dim \bigcup_{s \in S} A_s = \operatorname{Sup}\{\dim A_s : s \in S\}.$$

In the following theorem(³) we consider subsets indexed with ordinal numbers less than a given fixed ordinal τ which, besides, is quite arbitrary.

THEOREM 4 (M. Katětov and K. Morita). *If* A_a $(a < \tau)$ *are closed subsets of a space and there exist open subsets* G_a $(a < \tau)$ *such that* $A_a \subset G_a$ *for* $a < \tau$ *and the collection* $\{G_a : a < \beta\}$ *is locally finite for* $\beta < \tau$, *then*

$$\dim \bigcup_{a<\tau} A_a = \operatorname{Sup}\{\dim A_a : a < \tau\}. \tag{ii}$$

We now mention a theorem(⁴) showing that, in some cases, formula (ii) remains true without assumption that the sets A_a are closed. The proof of Theorem 5 essentially utilizes Theorem 3 (see [58], p. 18) and is much easier to accomplish, via Theorem 2, provided τ is countable.

THEOREM 5 (K. Nagami). *If* A_a $(a < \tau)$ *are subsets of a space such that*

$$\operatorname{cl} \bigcup_{a<\beta} A_a = \bigcup_{a<\beta} A_a$$

for $\beta < \tau$, *then* (ii) *holds.*

(²) Theorem 3 can be generalized to cover the class of non-separable metric spaces (see [58], p. 17) as well as a class of non-metrizable spaces (ibidem, p. 195). See also Dowker [10], de Groot and Nishiura [19] and [60], Hemmingsen [20], Kimura [32], Lifanov [46], Lokucievskiĭ [47], Pasynkov [63] and [65], Smirnov [76].

(³) Theorem 4 possesses generalizations which cover normal spaces and hereditarily paracompact spaces (see [58], p. 193 and 199). See also Dowker [11], Okuyama [61] and [62], Zarelua [84].

(⁴) Theorem 5 holds for non-separable metric spaces as it has been proved by Nagami [53] and Nagata [57]. See also Arhangel'skiĭ [4], McAuley [51].

Suppose that f is a mapping defined on a space X. Then X admits the decomposition

$$X = \bigcup_{y \in f(X)} f^{-1}(y)$$

into pairwise disjoint closed subsets $f^{-1}(y)$, and Theorems 2-4 motivate introducing the symbol

$$\dim f = \operatorname{Sup}\{\dim f^{-1}(y)\colon y \in f(X)\}.$$

In an attempt, however, to connect $\dim X$ with $\dim f$ one needs to involve $\dim f(X)$ and put a condition on the mapping. We say f to be a *closed mapping* provided, for each closed subset $A \subset X$, the set $f(A)$ is closed in $f(X)$. The mapping f is said to be *finite-dimensional* provided $\dim f < \infty$. We define

$$C_f(k) = \{y \in f(X)\colon k \leqslant \operatorname{card} f^{-1}(y)\},$$

$$D_f(k) = \{y \in f(X)\colon k \leqslant \dim f^{-1}(y)\},$$

for k an integer, and we have the following theorem (5) of Vaĭnšteĭn [83].

THEOREM 6 (I. A. Vaĭnšteĭn). *If f is a finite-dimensional closed mapping of a space X, then*

(iii) $$\dim X \leqslant \operatorname{Max}\{\dim f(X),\, d_f\},$$

(iv) $$\dim f(X) \leqslant \operatorname{Max}\{\dim X,\, \dim C_f(2)+1,\, d_f+1\},$$

where $d_f = -1$ *for* $\dim f = 0$, *and*

$$d_f = \operatorname{Max}\{\dim D_f(k)+k\colon k = 1, \ldots, \dim f\}$$

for $\dim f > 0$.

We shall deduce several dimension inequalities from Theorem 6; we start with a result (6) of Keesling [30] whose special case was discovered earlier by Jung [25].

THEOREM 7 (J. E. Keesling). *If f is a closed mapping of a finite-dimensional space X, then*

(v) $$\dim X \leqslant \dim D_f\big(\dim X - \dim f(X)\big) + \dim f.$$

---

(5) Theorem 6 has been generalized for paracompact spaces by Skljarenko [73] and [74]. Actually, this generalization offers even an inequality stronger than (iv); we shall write it down at the end of this paper (see Theorem 49). Let us mention that although inequalities (iii) and (iv) are not explicitly written in [83], they are clearly equivalent to those from [83].

(6) Theorem 7 in the form of Keesling [30] has been proved also for non-separable metric spaces.

Proof. If $\dim X \leqslant \dim f(X)$, then $D_f(\dim X - \dim f(X)) = f(X)$ and (v) trivially holds. If $\dim X > \dim f(X)$, then $\dim X \leqslant d_f$, by (iii). Let $d_f = \dim D_f(k_0) + k_0$ where $k_0 = 1, \ldots, \dim f$. Since $D_f(k_0) \subset f(X)$, we have

$$\dim D_f(k_0) \leqslant \dim f(X),$$

and thus we obtain the inequalities

$$\dim X - \dim f(X) \leqslant \dim D_f(k_0) + k_0 - \dim f(X) \leqslant k_0,$$

which imply the inclusion $D_f(k_0) \subset D_f(\dim X - \dim f(X))$. But also $k_0 \leqslant \dim f$ and $\dim X \leqslant \dim D_f(k_0) + k_0$, whence (v) follows and Theorem 7 is proved.

The next theorem(7) is a well-known result (see [23], p. 92); it is a simple consequence of inequality (iii) from Theorem 6.

THEOREM 8 (W. Hurewicz). *If f is a closed mapping of a space X, then*

(vi) $$\dim X \leqslant \dim f(X) + \dim f.$$

For mappings of compacta onto polyhedra, some additional information concerning inequality (vi) is given in the work of Boltjanskiĭ and Soltan [9] and [78].

THEOREM 9. *If f is a closed mapping of a space X such that* $\dim f(X) < \dim X$, *then*

(vii) $$\dim f(X) \leqslant \dim C_f(k) + \dim f - 1$$

for $k = 1, 2, \ldots$

Proof. If f is infinite-dimensional, then inequality (vii) is trivial. If f is finite-dimensional, then so is X, by Theorem 6. Thus $D_f(\dim X - \dim f(X)) \subset C_f(k)$ for every $k = 1, 2, \ldots$, and (vii) follows from (v).

Clearly, the condition saying that $\dim f(X) < \dim X$ cannot be omitted in Theorem 9. A symmetrical situation will be discussed in Theorem 18 below. However, for $k = 2$, we have the following easy proposition (to be generalized in Theorem 19).

THEOREM 10. *If f is a closed mapping of a space X, then*

(viii) $$\dim f(X) \leqslant \dim C_f(2) + \dim X + 1.$$

Proof. Let $A = X \setminus f^{-1}(C_f(2))$. Since the mapping f is closed, the partial mapping $f|A$ is a homeomorphism of A onto $f(A)$. Thus $\dim f(A) = \dim A \leqslant \dim X$. But we have $f(X) = C_f(2) \cup f(A)$, and (viii) follows from (i).

(7) Theorem 8 has been extended to cover non-separable metric spaces (see [58], p. 63) as well as non-metrizable spaces (ibidem, p. 216). See also Katětov [26], Pasynkov [64], Šersnev [68], Skljarenko [73], Zarelua [84], Zaremba-Szczepkowicz [88].

Let us mention that the set $C_f(2)$ occurs in a result of Eilenberg [14] concerning embedability in Euclidean spaces. As a corollary to Theorem 6, inequality (iv), we obtain the following theorem.

THEOREM 11. *If f is a finite-dimensional closed mapping of a space X such that* $\dim X < \dim f(X)$ *and* $d_f+1 < \dim f(X)$ (*see Theorem* 6), *then*

$$\dim f(X) \leqslant \dim C_f(2)+1. \tag{ix}$$

We are going to use Theorem 11 in the inductive proof of the next theorem. Theorem 12 itself generalizes Theorem 11 and leads to further results, especially interesting in the case of zero-dimensional mappings (see Theorem 14). In the case of finite-to-one mappings, Theorem 12 was proved by Hurewicz [22] (compare also Theorem 48).

THEOREM 12. *If f is a finite-dimensional closed mapping of a space X such that* $d_f < \dim f(X)$, *then*

$$\dim f(X) \leqslant \dim C_f(k)+k-1 \tag{x}$$

for $k = 1, \ldots, m_f$, *where*

$$m_f = \operatorname{Min}\{\dim f(X)-\dim X+1, \dim f(X)-d_f\}.$$

Proof. Let us observe that $\dim X \leqslant \dim f(X)$, by (iii). Hence $m_f \geqslant 1$. If $\dim X = \dim f(X)$ or $d_f+1 = \dim f(X)$, then $m_f = 1$ and inequality (x) for $k = 1$ is trivial, as we always have $C_f(1) = f(X)$. We therefore can assume that $\dim X < \dim f(X)$ and $d_f+1 < \dim f(X)$. Thus $m_f \geqslant 2$ and, by Theorem 11, we get inequality (ix) which coincides with inequality (x) for $k = 2$. Consequently, if $m_f = 2$, the conclusion of Theorem 12 is already verified. Let us assume that $m_f \geqslant 3$.

We shall find a finite sequence of subsets $X_k \subset X$, where $0 < k < m_f$, such that X_k satisfy the conditions

$$X_{k+1} \subset X_k \text{ and } f(X_{k+1}) \subset f(X_k \setminus X_{k+1}), \tag{1$_k$}$$

$$f_k = f|X_{k+1} \text{ is a closed mapping}, \tag{2$_k$}$$

$$\dim f(X_k) \leqslant \dim f(X_{k+1})+1, \tag{3$_k$}$$

for $k = 1, \ldots, m_f-2$. Put $X_1 = X$. Before defining X_k for $k > 1$, let us make two remarks. First, it follows from $(3)_k$ and from the definition of m_f that

$$\begin{aligned}\dim f(X) &= \dim f(X_1) \leqslant \dim f(X_{k+1})+k \\ &\leqslant \dim f(X_{k+1})+m_f-2 \leqslant \dim f(X_{k+1})+\dim f(X)-\dim X-1,\end{aligned}$$

whence $\dim X_{k+1} \leqslant \dim X < \dim f(X_{k+1})$. Second, by same argument we get

$$\dim f(X) \leqslant \dim f(X_{k+1})+\dim f(X)-d_f-2,$$

whence $d_f+1 < \dim f(X_{k+1})$. However, we have $\dim f_k \leqslant \dim f$ and $D_{f_k}(j) \subset D_f(j)$ for every $j = 1, 2, \ldots$ Thus $d_{f_k} \leqslant d_f$, according to the definition of d_f, and we obtain $d_{f_k}+1 < \dim f(X_{k+1})$. This, together with $(2)_k$, enables us to apply Theorem 11 to the mapping f_k ($k \leqslant m_f-2$) when, in our definition of the sets X_k, we proceed by induction on k. The k-th step will be essentially the same as the first step, so that we shall give a detailed description only of the first step.

To find X_2, we need a decomposition of the set $C_f(2)$. Since the mapping f is closed, the sets

$$B_j = \{y \epsilon f(X)\colon 1/j \leqslant \operatorname{diam} f^{-1}(y)\}$$

are closed in $f(X)$ for $j = 1, 2, \ldots$ And, clearly, the set $C_f(2)$ decomposes into the union $B_1 \cup B_2 \cup \ldots$ Then

$$\dim f(X) \leqslant \dim \bigcup_{j=1}^{\infty} B_j + 1 = \operatorname{Sup}\{\dim B_j\colon j = 1, 2, \ldots\} + 1,$$

by Theorem 11 and Theorem 2, and there exists a positive integer j_0 such that $\dim f(X) \leqslant \dim B_{j_0} + 1$. For each point $p \epsilon f^{-1}(B_{j_0})$, the set

$$A(p) = \{x \epsilon f^{-1}(B_{j_0})\colon \operatorname{dist}(p, x) \leqslant 1/3j_0\}$$

is a closed neighbourhood of p in $f^{-1}(B_{j_0})$. Since f is a closed mapping, so is $f|f^{-1}(B_{j_0})$, and therefore $f(A(p))$ are closed subsets of B_{j_0}. But X being separable metric, there exist points $p_i \epsilon f^{-1}(B_{j_0})$ ($i = 1, 2, \ldots$) such that

$$f^{-1}(B_{j_0}) = \bigcup_{i=1}^{\infty} A(p_i),$$

and thus we obtain

$$\dim f(X) \leqslant \dim \bigcup_{i=1}^{\infty} f(A(p_i)) + 1 = \operatorname{Sup}\{\dim f(A(p_i))\colon i = 1, 2, \ldots\} + 1,$$

by Theorem 2. Consequently, there exists a positive integer i_0 such that $\dim f(X) \leqslant \dim f(A(p_{i_0}))+1$. We define $X_2 = A(p_{i_0})$ and we see condition $(3)_1$ is fulfilled. Since $f|f^{-1}(B_{j_0})$ is a closed mapping, so is $f|A(p_{i_0})$, and condition $(2)_1$ holds. It follows readily from the definitions of B_{j_0} and $A(p)$ that condition $(1)_1$ holds too.

Repeating successively the procedure just described, we construct the sets $X_1, \ldots, X_{n+1}$ (where $n = m_f-2$) which satisfy conditions $(1)_1$-$(1)_n$, $(2)_1$-$(2)_n$, and $(3)_1$-$(3)_n$. Moreover, by $(2)_n$, we can apply Theorem 11 to the mapping f_n and as a result we get the inequality

$$\dim f(X_{n+1}) \leqslant \dim C_{f_n}(2) + 1. \tag{4}$$

If $k = 1, \ldots, n$ and $y \in f(X_{k+1})$, then it follows from conditions $(1)_1$-$(1)_k$ that the set $f^{-1}(y)$ meets each of the sets

$$X_1 \setminus X_2, \ldots, X_k \setminus X_{k+1}, X_{k+1},$$

whence $f(X_{k+1}) \subset C_f(k+1)$. On the other hand, conditions $(3)_1$-$(3)_k$ imply that

$$\dim f(X_1) \leqslant \dim f(X_{k+1}) + k \leqslant \dim C_f(k+1) + k,$$

which means that inequalities (x) are true for $k = 1, \ldots, m_f - 1$. Since $C_{f_n}(2) \subset f(X_{n+1})$, we conclude also that if $y \in C_{f_n}(2)$, then the set $f^{-1}(y)$ has at least one point in common with the set $X_i \setminus X_{i+1}$ for $i = 1, \ldots, n$ and $f^{-1}(y)$ has at least two points in common with the set X_{n+1}. Thus $C_{f_n}(2) \subset C_f(n+2) = C_f(m_f)$ and

$$\dim f(X) \leqslant \dim f(X_{n+1}) + n \leqslant \dim C_{f_n}(2) + n + 1 \leqslant \dim C_f(m_f) + m_f - 1,$$

by (4). Consequently, inequality (x) is true for $k = m_f$, and the proof of Theorem 12 is complete.

THEOREM 13. *If f is a finite-dimensional closed mapping of a space X, then*

(xi) $$\dim f(X) \leqslant \operatorname{Max}\{c_f, d_f\},$$

where $c_f = -1$ *for* $\dim f(X) \leqslant d_f$, *and*

$$c_f = \operatorname{Min}\{\dim C_f(k) + k - 1:\ k = 1, \ldots, m_f\}$$

for $\dim f(X) > d_f$ *(see Theorem* 6 *and Theorem* 12*).*

As a consequence of Theorem 13 we obtain an older result of Freudenthal [17] concerning zero-dimensional mappings.

THEOREM 14 (H. Freudenthal). *If f is a zero-dimensional closed mapping of a space X, then*

(xii) $$\dim f(X) \leqslant \operatorname{Min}\{\dim C_f(k) + k - 1:\ k = 1, \ldots, \dim f(X) - \dim X + 1\}.$$

Proof. By the definition of d_f, we have $d_f = -1$. Hence we get $m_f = \dim f(X) - \dim X + 1$ and (xii) follows from (xi).

We mention a result, due to Vaĭnšteĭn [82], which is stronger than Theorem 14. Suppose f is a closed mapping of a space X and $y \in f(X)$. If $U \subset X$ is open and $f^{-1}(y) \subset U$, then $f(U)$ is a neighbourhood of y in $f(X)$. Let $C'_f(k)$ $(k = 1, 2, \ldots)$ denote the set of all points $y \in f(X)$ such that there exists an open subset $U \subset X$ for which we have

$$k - 1 \leqslant \operatorname{card} U \cap f^{-1}(y)$$

and $f(U)$ is not a neighbourhood of y. Thus if f is closed and $\operatorname{card} f^{-1}(y) \leqslant k - 1$, the point y does not belong to $C'_f(k)$. In other words, we have $C'_f(k) \subset C_f(k)$ for $k = 1, 2, \ldots$

THEOREM 15 (I. A. Vaĭnšteĭn). *If f is a zero-dimensional closed mapping of a space X such that, for every point $y \epsilon f(X)$ satisfying*

$$f^{-1}(y) \subset \operatorname{cl}[X \setminus f^{-1}(y)],$$

there exists an isolated point in $f^{-1}(y)$, then

(xiii) $\dim f(X) \leqslant \operatorname{Min}\{\dim C'_f(k)+k-1:\ k=1,\ldots,\dim f(X)-\dim X+1\}$.

Some ideas related to Theorem 15 are contained in the paper of Každan and Vaĭnšteĭn [27]. The next theorem is a classical result (see [23], p. 93, and [35], p. 97).

THEOREM 16 (W. Hurewicz). *If f is a closed mapping of a space X, then*

(xiv) $$\dim f(X) \leqslant \dim X + \operatorname{Sup}\{\operatorname{card} f^{-1}(y):\ y \epsilon f(X)\} - 1.$$

Proof. Inequality (xiv) is trivial for $\dim f > 0$. If $\dim f = 0$, then we have

$$\dim f(X) \leqslant \dim C_f(\dim f(X) - \dim X + 1) + \dim f(X) - \dim X,$$

by Theorem 14. It follows that the set $C_f(\dim f(X) - \dim X + 1)$ is non-empty, which implies (xiv).

We shall see in inequality (xxxv) of Theorem 38 below that -1 in (xiv) can be replaced by -2 provided the space X fulfils some extra conditions. Also, a slightly stronger version ([8]) of Theorem 16 is the following theorem (see [58], p. 68).

THEOREM 17 (J. Nagata). *If f is a closed mapping of a space X such that $f(X)$ is non-discrete, then*

(xv) $\dim f(X) \leqslant \dim X + \operatorname{Sup}\{\operatorname{card} f^{-1}(y) \cap \operatorname{cl}[X \setminus f^{-1}(y)]:\ y \epsilon f(X)\} - 1$.

Proof. It is not difficult to check that $f|X'$, where

$$X' = \bigcup_{y \epsilon f(X)} f^{-1}(y) \cap \operatorname{cl}[X \setminus f^{-1}(y)],$$

is a closed mapping. Moreover, each point of $f(X) \setminus f(X')$ is isolated in $f(X)$. Since any set consisting of isolated points is open and countable, it follows from Theorem 2 that $\dim f(X) = \dim f(X')$, and applying Theorem 16 to $f|X'$ we obtain inequality (xv).

THEOREM 18. *If f is a closed mapping of a space X such that* $\dim X \leqslant \dim f(X)$, *then*

(xvi) $$\dim f(X) \leqslant \dim C_f(k) + \dim f + k - 1$$

for $k = 1, \ldots, \dim f(X) - \dim X + 1$.

([8]) Theorem 17 as proved by Nagata [58] holds for non-separable metric spaces and has an analogue for non-metrizable spaces (ibidem, p. 218). Thus the same can be said about Theorem 16. See also Keesling [28], Morita [52], Nagami [53] and [54], Skljarenko [70], Zarelua [87].

Proof. If f is zero-dimensional, then Theorem 18 follows from Theorem 14. If f is infinite-dimensional, then inequality (xvi) trivially holds. Thus we can assume that $0 < \dim f < \infty$. Let $d_f = \dim D_f(k_0) + k_0$ where $k_0 = 1, \ldots, \dim f$. We always have $D_f(1) \subset C_f(k)$ for $k = 1, 2, \ldots$ If $\dim f(X) \leqslant d_f$, then

$$\dim f(X) \leqslant \dim D_f(k_0) + k_0 \leqslant \dim D_f(1) + \dim f \leqslant \dim C_f(k) + \dim f,$$

which is even more than inequality (xvi) for every $k = 1, 2, \ldots$ Consequently, we can also assume that $d_f < \dim f(X)$ so that the conditions of Theorem 12 are satisfied. If $k \leqslant m_f$, then (xvi) is a consequence of (x). On the other hand, if

$$m_f < k \leqslant \dim f(X) - \dim X + 1,$$

then $\dim f(X) - d_f = m_f \leqslant k-1$, by the definition of m_f, and we get again

$$\begin{aligned}\dim f(X) &= d_f + \big(\dim f(X) - d_f\big) \leqslant d_f + k - 1 \\ &= \dim D_f(k_0) + k_0 + k - 1 \leqslant \dim C_f(k) + \dim f + k - 1,\end{aligned}$$

which completes the proof of Theorem 18.

Let us now formulate a theorem (see Theorem 19 below, compare Theorem 10) giving, in the separable metric case, a solution to a problem raised by Arhangel'skiĭ [8]. Observe that if $f\colon X \to f(X)$ is a closed mapping, then the set $C_f(k)$ is of type F_σ in $f(X)$, for $k = 1, 2, \ldots$ In our proof of Theorem 12 we have shown this for $k = 2$, and the argument for $k \geqslant 3$ is quite analogical. Thus, by Theorem 2, the set $C_f(k)$ contains a set A such that A is closed in $f(X)$ and $\dim A = \dim C_f(k)$. Consequently, the relative dimension which appears in Arhangel'skiĭ's question coincides, in our case, with the usual dimension.

THEOREM 19. *If f is a closed mapping of a space X, then*

(xvii) $$\dim f(X) \leqslant \dim C_f(k) + \dim X + k - 1$$

for $k = 1, 2, \ldots$

Proof. The inequality $\dim f \leqslant \dim X$ is trivially true. If $\dim f(X) < \dim X$, then we can apply Theorem 9 and (xvii) follows from (vii). If $\dim f(X) \geqslant \dim X$, then we use Theorem 18 to establish (xvii) for $k \leqslant \dim f(X) - \dim X + 1$. Finally, for $k > \dim f(X) - \dim X + 1$, we have

$$\dim f(X) \leqslant \dim X + k - 2 \leqslant \dim C_f(k) + \dim X + k - 1.$$

THEOREM 20. *If f is a closed mapping of a space X such that* $\dim X \neq \dim f(X)$, *then*

(xviii) $$\dim f(X) \leqslant \dim C_f(2) + \dim f + 1.$$

Proof. If $\dim f(X) < \dim X$, then (xviii) follows from Theorem 9. If $\dim f(X) > \dim X$, then we can apply Theorem 18 for $k = 2$.

Remark. The requirement that the mapping is closed has been a standing assumption in Theorems 6-20. It can, however, be replaced by a condition imposed on the space. A space X is said to be an *F_σ-space* (or a *G_δ-space*) provided X is homeomorphic to a set of type F_σ (or type G_δ, respectively) in a compact space. Thus F_σ-spaces coincide with spaces representable as countable unions of compact spaces. Since each mapping of a compact space is closed, it might be checked with the help of Theorem 2 that a good deal of Theorems 6-20 can survive when we take arbitrary mappings of F_σ-spaces to replace closed mappings of arbitrary spaces. We will see in an example below that the analogical replacement by means of G_δ-spaces does not work.

Let X be a space. A *compactification* of X is a compact space hX together with an embedding h of X into hX such that the image $h(X)$ is dense in hX. The *deficiency* $\operatorname{def} X$ of the space X is defined by the formula

$$\operatorname{def} X = \operatorname{Min}\{\dim(hX \setminus h(X)) \colon hX \in C(X)\},$$

where $C(X)$ denotes the collection of all compactifications of X. Thus X is compact if and only if $\operatorname{def} X = -1$. It is known that $\operatorname{def} X \leqslant \dim X$ (see [23], p. 65). Let us write $\operatorname{subcom} X = -1$ provided there exists a compact subset $C \subset X$ such that $\dim C = \dim X$. The inductive definition of the number $\operatorname{subcom} X$ resembles the definition of dimension. Namely, we write $\operatorname{subcom} X \leqslant n$ provided each point of X has arbitrarily small open neighbourhoods U in X such that

$$\operatorname{subcom}(\operatorname{cl} U \setminus U) \leqslant n-1$$

($n \geqslant 0$). It is rather easy to check that $\operatorname{subcom} X \leqslant \operatorname{def} X$ (see [41], p. 224). For $x \in X$, let $C(X, x)$ denote the *component* of X at x, that is the union of all connected subsets of X that contain x. Three theorems which follow have been proved in [41].

THEOREM 21. *If X is a space, then*

$$\text{(xix)} \qquad \dim X \leqslant \operatorname{Sup}\{\dim C(X, x) \colon x \in X\} + \operatorname{subcom} X + 1.$$

THEOREM 22. *If f is a mapping of a space X, then*

$$\text{(xx)} \qquad \dim X \leqslant \dim f(X) + \dim f + \operatorname{subcom} X + 1.$$

THEOREM 23. *If f is a mapping of a space X such that, for every point $y \in f(X)$, the set $f^{-1}(y)$ is locally compact, then*

$$\text{(xxi)} \qquad \dim X \leqslant \dim f(X) + \operatorname{Max}\{\dim f, \operatorname{def} X\}.$$

We recall that the *quasi-component* $Q(X, x)$ of a space X at a point $x \in X$ is the intersection of all closed-open subsets of X that contain x.

The following theorems were proved by Nishiura [59]. In the case of deficiency zero, Theorem 25 has been treated in [39]. Since $C(X, x) \subset Q(X, x)$ for every point $x \in X$, inequality (xix) implies inequality (xxii) and thus Theorem 24 is a direct consequence of Theorem 21.

THEOREM 24 (T. Nishiura). *If X is a space, then*

$$\text{(xxii)} \qquad \dim X \leqslant \operatorname{Sup}\{\dim Q(X, x)\colon x \in X\} + \operatorname{def} X + 1.$$

THEOREM 25 (T. Nishiura). *If X is a space such that every quasi-component $Q(X, x)$ of X is locally compact, then*

$$\text{(xxiii)} \qquad \dim X = \operatorname{Max}\{\operatorname{Sup}\{\dim Q(X, x)\colon x \in X\}, \operatorname{def} X\}.$$

Proof. There exists a mapping g of X into the Cantor set such that $g^{-1}g(x) = Q(X, x)$ for $x \in X$ (see [35], p. 148). Thus we can apply Theorem 23 for the mapping g. By (xxi), we obtain $\dim X \leqslant \operatorname{Max}\{\dim g, \operatorname{def} X\}$. Because the inverse inequality is trivial, we get (xxiii).

Note. In the same manner, inequality (xx) implies inequality (xxii) and thus Theorem 24 is a consequence of Theorem 22 as well.

We say a space X to be *totally disconnected* provided all quasi-components of X are degenerate. Zero-dimensional spaces are totally disconnected, but not conversely (see the example below). It was proved by Mazurkiewicz [50] that if X is a non-compact totally disconnected space, then the dimension of X is equal to the deficiency of X. The latter theorem is now a simple corollary to Theorem 25, and so is the following proposition(9).

THEOREM 26. *If X is a non-compact space such that every quasi-component of X is zero-dimensional and locally compact, then*

$$\text{(xxiv)} \qquad \dim X = \operatorname{def} X.$$

The theorem(10) which follows is due to Arhangel'skiĭ [7].

THEOREM 27 (A. V. Arhangel'skiĭ). *If f is a closed mapping of a totally disconnected space X such that, for every point $y \in f(X)$, the set $f^{-1}(y)$ is compact, then*

$$\text{(xxv)} \qquad \dim X \leqslant \dim f(X).$$

We say a space X to be *lacunar* provided all compact subsets of X have empty interiors. Another estimation of the deficiency is given by the next theorem which was proved in [37].

---

(9) Theorem 26 in the case of totally disconnected spaces with deficiency zero has been extended over non-metrizable spaces by Flachsmeyer [16].

(10) Theorem 27 as given by Arhangel'skiĭ [7] holds for non-metrizable spaces.

THEOREM 28. *If f is a mapping of a space X such that $f(X)$ is a lacunar G_δ-space and, for every point $y \in f(X)$, the set $f^{-1}(y)$ is compact, then*

$$\text{(xxvi)} \qquad \operatorname{Min}\{\dim f^{-1}(y)\colon\, y \in f(X)\} \leqslant \operatorname{def} X.$$

Since the space P of irrationals is lacunar and P is a G_δ-space, it follows from Theorem 28 that $\dim X = \operatorname{def} P \times X$ for every compact space X. Indeed, one has only to apply (xxvi) to the projection f of $P \times X$ onto P. However, a more general theorem holds as proved in [42].

THEOREM 29. *If X and Y are spaces such that X is compact and Y is not locally compact, then*

$$\text{(xxvii)} \qquad \dim X \leqslant \operatorname{def} X \times Y.$$

Another proof of Theorem 29 as well as some further research is done by Aarts [2]. He has proved, among other things, that just no condition is necessary for a space X in order to satisfy inequality (xxvii) provided Y is a lacunar G_δ-space (compare Theorem 28). On the other hand, if Y is not a G_δ-space, then we get a stronger version of Theorem 29. Namely, the deficiency in inequality (xxvii) can then be replaced by the dimension of remainders in completions rather than compactifications (see [1], p. 29).

Now, suppose X is an n-dimensional totally disconnected space, where $n > 0$. Let us take the mapping g of X into the Cantor set such that inverses of points under g coincide with quasi-components of X (see [35], p. 148). In such a case g is one-to-one, and we have

$$\dim X = n > 0 = \dim g(X) + \dim g,$$

which shows that the condition for the mapping to be closed is an essential hypothesis in the Hurewicz theorem (see Theorem 8). The first example of an n-dimensional totally disconnected G_δ-space was found by Mazurkiewicz [48]. We give here another construction, following some idea of Knaster [33]. Our construction was earlier mimeographed in [43].

Example. This will be a subset of the Euclidean $(n+1)$-space R^{n+1}. For every pair of distinct points x and y in R^{n+1}, where

$$x = (x_1, \ldots, x_{n+1}), \qquad y = (y_1, \ldots, y_{n+1}),$$

let $x -_\lambda y$ denote the first non-zero number in the sequence

$$x_1 - y_1, \ldots, x_{n+1} - y_{n+1}.$$

We see that $x -_\lambda y = -(y -_\lambda x)$. Write $x <_\lambda y$ if and only if $x -_\lambda y < 0$. Then $<_\lambda$ is the so-called lexicographic order in the product R^{n+1} of $n+1$ real lines R each carrying the ordinary order. Observe that every compact non-empty subset of R^{n+1} contains a uniquely determined minimum point with respect to the order $<_\lambda$.

Denote $q_0 = (0, 0, \ldots, 0)$, $q_3 = (3, 0, \ldots, 0)$, and let S_c^n be the n-dimensional sphere in R^{n+1} with center q_0 and radius $1+c$, where c is a number in the Cantor ternary set C in R. Let B be the closed $(n+1)$-dimensional ball in R^{n+1} with center q_0 and radius 4. Denote by π the projection

$$\pi\colon \bigcup_{c\in C} S_c^n \to C,$$

defined by $\pi(x) = c$ for $x \in S_c^n$. Let us consider the collection $\boldsymbol{K}$ of all continua joining q_0 and q_3 in B. The set $\boldsymbol{K}$ with the Hausdorff distance is a compact metric space. Let $f\colon C \to \boldsymbol{K}$ be a mapping of C onto $\boldsymbol{K}$. Since each sphere S_c^n cuts R^{n+1} between q_0 and q_3, we have $f(c)\cap S_c^n \neq \emptyset$ for $c \in C$. Let $p(c)$ be the minimum point in $f(c)\cap S_c^n$ with respect to the order $<_\lambda$. We prove that the set

$$X = \{p(c)\colon\ c\in C\}$$

is an n-dimensional totally disconnected G_δ-space.

In fact, the mapping $\pi|X\colon X \to C$ is one-to-one, thus X is totally disconnected. Clearly, X is a boundary set in R^{n+1}, whence $\dim X \leqslant n$. From the Mazurkiewicz theorem (see [35], p. 466) we know that if V is a connected open subset of R^n and $\dim A \leqslant n-2$, then each two points in $V\setminus A$ can be joined by a continuum contained in $V\setminus A$. Therefore, since the interior of the ball B is a connected open subset of R^{n+1} and X contains at least one point of each continuum joining q_0 and q_3 in B, we get $\dim X > (n+1)-2$. Hence $\dim X = n$. By the continuity of f, the set

$$Y = \bigcup_{c\in C} f(c)\cap S_c^n$$

is compact and $X \subset Y$. To prove that X is a G_δ-space, it suffices to show that $Y\setminus X$ is of type F_σ in Y. Consider the sets

$$F_j = \{y\in Y\colon\ 1/j < y -_\lambda p\pi(y)\}$$

for $j = 1, 2, \ldots$ By the definition of the points $p(c)$, we have

$$Y\setminus X \subset F_1\cup F_2\cup\ldots$$

and it remains to prove that $\operatorname{cl} F_j \subset Y\setminus X$ for $j = 1, 2, \ldots$ Fix an index j and take points $y^1, y^2, \ldots$ from F_j which converge to a point $y \in Y$. Denote $c_i = \pi(y^i)$. Thus the numbers $c_1, c_2, \ldots$ converge to the number $c = \pi(y)$, and

$$1/j < y^i -_\lambda p(c_i)$$

for $i = 1, 2, \ldots$ It follows that there are integers k_i such that $1 \leqslant k_i \leqslant n+1$ $(i = 1, 2, \ldots)$ and the coordinates of y^i and $p(c_i)$ satisfy the conditions

$$1/j < y^i_{k_i} - p(c_i)_{k_i}, \qquad y^i_h = p(c_i)_h$$

for $1 \leqslant h < k_i$ and $i = 1, 2, \ldots$ Let us choose positive integers $i_1 < i_2 < \ldots$ and k such that $k_{i_m} = k$ for $m = 1, 2, \ldots$ and the points $p(c_{i_1}), p(c_{i_2}), \ldots$ converge to a point $z \in Y$. Since $y^{i_1}, y^{i_2}, \ldots$ converge to y, we get

$$0 < 1/j \leqslant y_k - z_k, \qquad y_h = z_h$$

for $1 \leqslant h < k$. Thus $z -_\lambda y = z_k - y_k < 0$. Moreover, the numbers c_{i_1}, $c_{i_2}, \ldots$ converge to c, whence $z \in S_c^n$, and also $z \in f(c)$, by the continuity of f. Consequently, we have $z <_\lambda y$ and $z \in f(c) \cap S_c^n$, which gives $y \neq p(c)$. If $c' \in C$ and $c' \neq c$, then

$$\pi(y) = c \neq c' = \pi p(c'),$$

and so $y \neq p(c')$. We have proved that the point y does not belong to X, which completes checking on the properties of the example.

Given a mapping f of a finite-dimensional space X, we define the set D_f' by the formula

$$D_f' = \{y \in f(X) \colon \dim X - \dim_y f(X) \leqslant \dim f^{-1}(y)\},$$

where $\dim_y f(X)$ denotes the local dimension of the space $f(X)$ at the point y. The theorems which follow are due to Keesling [31] and they may constitute local analogues of Theorem 7.

THEOREM 30 (J. E. Keesling). *If f is a closed mapping of a finite-dimensional space X, then*

$$\text{(xxviii)} \qquad \dim X \leqslant \dim D_f' + \dim f + 1.$$

THEOREM 31 (J. E. Keesling). *If f is a mapping of a finite-dimensional compact space X, then*

$$\text{(xxix)} \qquad \dim X \leqslant \dim D_f' + \dim f.$$

As applications of Theorems 30 and 31 we obtain the following theorems which are classical results (see [35], p. 113). Indeed, for f the identity mapping, (xxviii) and (xxix) imply (xxx) and (xxxi), respectively.

THEOREM 32 (K. Menger). *If X is a finite-dimensional space, then*

$$\text{(xxx)} \qquad \dim X \leqslant \dim \{x \in X \colon \dim X = \dim_x X\} + 1.$$

THEOREM 33 (K. Menger). *If X is a finite-dimensional compact space, then*

$$\text{(xxxi)} \qquad \dim X = \dim \{x \in X \colon \dim X = \dim_x X\}.$$

We recall that a *Cantor manifold* is meant to be a finite-dimensional compact space X such that if $A \subset X$ and

$$\dim A < \dim X - 1,$$

then $X \setminus A$ is connected. It follows from inequality (xxix) that, for each mapping f of a compact space, the set D_f' is non-empty. However, in most cases of mappings of Cantor manifolds, the set D_f' turns out to be positive-dimensional. This is our next theorem which was proved in [44]. Weaker results were obtained by Jung [24] and Skljarenko [75].

THEOREM 34. *If f is a non-constant mapping of a Cantor manifold X, then*

$$0 < \dim D_f' \leqslant \dim D_f\big(\dim X - \dim f(X)\big). \tag{xxxii}$$

In the remainder of the survey we discuss some more special concepts related to dimension inequalities. Let hX be a compactification of a space X. We say that hX is a *perfect compactification* provided

$$\mathrm{cl}\, h(\mathrm{cl}\, U \setminus U) = \mathrm{cl}\, h(X \setminus U) \cap \mathrm{cl}\big(hX \setminus \mathrm{cl}\, h(X \setminus U)\big)$$

for every open subset $U \subset X$. It is not difficult to observe that if the space hX is locally connected, then hX is a perfect compactification if and only if, for every connected open subset V of hX, the set $V \cap h(X)$ is connected (compare [72], p. 430). Thus a result from [38], concerning dense subsets of the n-sphere, is generalized in two different directions by the following theorems, the first of which (11) belongs to Skljarenko [71].

THEOREM 35 (E. G. Skljarenko). *If hX and $h'X$ are compactifications of a space X such that hX is a finite-dimensional perfect compactification, every compact subset $C \subset hX \setminus h(X)$ has dimension*

$$\dim C \leqslant \dim hX - 1,$$

and every connected compact subset of $h'X \setminus h'(X)$ is degenerate, then

$$\dim hX \leqslant \dim h'X. \tag{xxxiii}$$

THEOREM 36. *If hX and $h'X$ are compactifications of a space X such that hX is a Cantor manifold, hX is a perfect compactification, and $\dim\big(h'X \setminus h'(X)\big) \leqslant 0$, then* (xxxiii) *holds.*

Proof. There exists a mapping $f\colon hX \to h'X$ such that $fh(x) = h'(x)$ for $x \in X$ (see [72], p. 434). Thus $f(hX) = h'X$. If X is degenerate, (xxxiii) is trivial. If X is non-degenerate, f is non-constant, and the negation of (xxxiii) would imply the inclusion $D_f' \subset h'X \setminus h'(X)$, contrary to (xxxii). We see that in this way Theorem 36 follows from Theorem 34.

Yet another specific situation allows to connect the dimension of the space with the dimension of the mapping. By an *irreducible continuum* we understand a connected compact space X which contains points a, b such that if A is a closed subset of X and $a, b \in A \neq X$, then A is not

(11) Theorem 35 has been proved by Skljarenko [71] also for non-metrizable spaces and their non-metrizable compactifications.

connected. Each irreducible continuum X admits a monotone mapping $f: X \to R$ into the real line such that f is the *finest mapping* in the collection of all such mappings; this means that if $g: X \to R$ is a monotone mapping, then the set $g^{-1}(t)$, for $t \in g(X)$, is the union of some sets $f^{-1}(t)$ (see [35], p. 200). The following result was proved by Mazurkiewicz [49] and a proof was also set forth in [45].

THEOREM 37 (S. Mazurkiewicz). *If f is the finest mapping of an irreducible continuum X such that* $\dim X > 1$, *then*

$$\dim X = \dim f. \tag{xxxiv}$$

A relation weaker than (xxxiv) is a consequence of Theorem 8, namely that $\dim X \leqslant \dim f + 1$. Under some additional conditions, an analogical improvement of Theorem 16 is also possible. This is the following theorem, due to Hurewicz [22]. A special case of Theorem 38 was recently studied by Sieklucki [69].

THEOREM 38 (W. Hurewicz). *If f is a closed mapping of a space X such that* $\dim X < \dim f(X)$ *and, for every closed subset $A \subset X$ satisfying* $\dim A = \dim X$, *the interior of A in X is non-empty, then*

$$\dim f(X) \leqslant \dim X + \operatorname{Sup}\{\operatorname{card} f^{-1}(y)\colon y \in f(X)\} - 2. \tag{xxxv}$$

Let f be a mapping of a space X. We say f to be an *open mapping* provided, for each open subset $A \subset X$, the set $f(A)$ is open in $f(X)$. The openness here is a much more restrictive condition than the closedness. For example, each mapping of a compact space is closed whereas the mapping which takes only two non-isolated points of a space into one point of another is not open. Now, similarly to the definition of the set $C_f(k)$ for k an integer, let us denote by $C_f(\aleph_1)$ the set of all points $y \in f(X)$ such that the set $f^{-1}(y)$ is uncountable. Thus the mapping is countable-to-one if and only if $C_f(\aleph_1) = \emptyset$. We cite two results ([12]) of Arhangel'skiĭ [5]. It seems worth noticing that inequality (xxxvii) might be considered as a strong improvement of inequality (xvii) from Theorem 19.

THEOREM 39 (A. V. Arhangel'skiĭ). *If f is a countable-to-one closed mapping of a space X and there exists a subset $A \subset X$ such that $f(A) = f(X)$ and $f|A$ is an open mapping, then*

$$\dim f(X) = \dim X. \tag{xxxvi}$$

THEOREM 40 (A. V. Arhangel'skiĭ). *If f is a mapping of a compact space X and there exists a subset $A \subset X$ such that $f(A) = f(X)$ and $f|A$ is an open mapping, then*

$$\dim f(X) \leqslant \dim C_f(\aleph_1) + \dim X + 1. \tag{xxxvii}$$

([12]) Theorem 39 has been proved by Arhangel'skiĭ [5] also for non-separable metric spaces. In the case where the mapping itself is open, Theorem 40 has been generalized for non-metrizable spaces by the same author [6].

The behaviour of dimension under open mappings was investigated by Roberts [67] and Hodel [21]. Two theorems which follow([13]) are to be found in those papers.

THEOREM 41 (R. E. Hodel and J. H. Roberts). *If f is an open mapping of a space X such that, for every point $y \in f(X)$, there exists an isolated point in $f^{-1}(y)$, then*

(xxxviii) $$\dim f(X) \leqslant \dim X.$$

THEOREM 42 (R. E. Hodel). *If f is an open mapping of a space X such that, for every point $y \in f(X)$, the set $f^{-1}(y)$ is discrete, then* (xxxvi) *holds.*

Since each countable G_δ-space contains an isolated point, the following results([14]) of Taĭmanov [80] and [81] are corollaries to Theorems 39 and 41, respectively.

THEOREM 43 (A. D. Taĭmanov). *If f is a countable-to-one open closed mapping of a space X, then* (xxxvi) *holds.*

THEOREM 44 (A. D. Taĭmanov). *If f is a countable-to-one open mapping of a space X such that, for every point $y \in f(X)$, the set $f^{-1}(y)$ is a G_δ-space, then* (xxxviii) *holds.*

Related to Theorems 42 and 44 is a recent result ([15]) of Pasynkov [65] dealing with open mappings of G_δ-spaces.

THEOREM 45 (B. A. Pasynkov). *If f is an open mapping of a G_δ-space X such that, for every point $y \in f(X)$, the set $f^{-1}(y)$ is a countable union of discrete subspaces, then* (xxxviii) *holds.*

What is actually done in Theorems 39-41 and 43-45 when combined with Theorem 8 gives various generalizations of a well-known theorem([16]) of Aleksandrov (see [35], p. 115).

THEOREM 46 (P. S. Aleksandrov). *If f is a countable-to-one open mapping of a compact space X, then* (xxxvi) *holds.*

The openness of the mapping in Theorem 42 can be replaced by the closedness if some rather restrictive conditions of another kind are

([13]) Theorems 41 and 42 admit several analogues for non-separable metric spaces; they have been proved by Hodel [21]. In the case of finite-to-one mappings, Theorem 42 has been earlier proved by Nagami [54] for non-separable metric spaces and for a class of non-metrizable spaces. See also Arhangel'skiĭ [5], Keesling [28], Nagami [55] and [56].

([14]) Theorems 43 and 44 possess generalizations that hold for non-separable metric spaces according to Arhangel'skiĭ [5]. In the case of finite-to-one mappings, Theorem 43 has been extended to completely regular spaces by Keesling [29].

([15]) Theorem 45 has been proved by Pasynkov [65] for normal G_δ-spaces (of type G_δ in the Čech-Stone compactification).

([16]) Theorem 46 as a consequence of either Theorem 40 or Theorem 45 which have been extended over non-metrizable range also permits such an extension.

imposed. The following theorem(17) was proved independently by Nagami [55] and Suzuki [79].

THEOREM 47 (K. Nagami and J. Suzuki). *If f is a finite-to-one closed mapping of a space X such that*

$$\operatorname{card} f^{-1}(y) = \operatorname{card} f^{-1}(y')$$

for $y, y' \in f(X)$, then (xxxvi) *holds.*

The definition of $\dim f$ motivates introducing a dual symbol by the formula

$$\operatorname{card} f = \operatorname{Sup}\{\operatorname{card} f^{-1}(y)\colon\ y \in f(X)\},$$

and then inequality (xiv) from the Hurewicz theorem can be rewritten as $\dim f(X) \leqslant \dim X + \operatorname{card} f - 1$ (see Theorem 16). An improvement of the latter inequality, inequality (xxxix) below(18), is due to Zarelua [86].

THEOREM 48 (A. V. Zarelua). *If f is a finite-to-one closed mapping of a space X such that $\operatorname{card} f < \infty$, then*

$$\text{(xxxix)} \qquad \dim f(X) \leqslant \operatorname{Max}\{\dim f^{-1}(C_f(k)) + k - 1\colon\ k = 1, \ldots, \operatorname{card} f\}.$$

The proof of Theorem 48 as given in [87] essentially depends on methods of algebraic topology. On the other hand, there are results concerning dimension inequalities which involve algebraic concepts more explicitly. Let $CH_f(0)$ denote the set of all points $y \in f(X)$ such that the set $f^{-1}(y)$ is not connected, and let $CH_f(k)$ $(k = 1, 2, \ldots)$ denote the set of all points $y \in f(X)$ such that the k-dimensional cohomology group of $f^{-1}(y)$ with integer coefficients is not trivial. Given a finite-dimensional set $Y \subset f(X)$, we define $r\dim Y$ to be the maximum dimension of closed subsets of $f(X)$ which are contained in Y. We have the following theorem(19) of Skljarenko [74].

THEOREM 49 (E. G. Skljarenko). *If f is a finite-dimensional closed mapping of a space X, then*

$$\text{(xl)} \qquad \dim f(X) \leqslant \operatorname{Max}\{\dim X,\ ch_f + 1\},$$

where

$$ch_f = \operatorname{Max}\{r\dim CH_f(k) + k\colon\ k = 0, \ldots, \dim f\}.$$

(17) Theorem 47 remains valid for non-separable metric spaces and, as shown by Nagami [55], a stronger version dealing with boundaries of inverses (compare Theorem 17) is also possible. See also Keesling [28].

(18) Theorem 48 has been proved by Zarelua [87] in a more general setting for paracompact spaces.

(19) Theorem 49 has been proved by Skljarenko [75] also for paracompact spaces.

Since $CH_f(0) \subset C_f(2)$ and $CH_f(k) \subset D_f(k)$ for $k > 0$, inequality (xl) is a stronger result than inequality (iv) from Theorem 6. Another corollary to Theorem 49 is a theorem([20]) of Dyer [12].

THEOREM 50 (E. Dyer). *If f is a mapping of a compact space X such that, for every point $y \in f(X)$, the set $f^{-1}(y)$ is homologically trivial, then* (xxxviii) *holds.*

In the special case where X is an orientable manifold and $f(X)$ is a polyhedron, some conditions implying inequality (xxxviii) were also discovered by Frum-Ketkov [18]. Cohomological dimension analogues of Theorems 8, 16 and 46 were studied by Kuz'minov [36].

Addendum. I give an account of the present status of eight problems which I proposed in papers [37]-[41] and [44]. Problem P 312 was solved in the negative in [42]. A negative solution of P 313 was given by Reichaw-Reichbach [66] (see also [42], p. 535). Affirmative solutions of P 350 and P 373 were given by Jung [24] and Nishiura [59], respectively. Problem P 390 remains open. Jung [24] and Skljarenko [75] answered P 391 in the affirmative. Problem P 469 remains unsettled. Kuperberg's [34] solved P 614 in the affirmative.

Added in proof. It is now proved by Calvin F. K. Jung that the condition of compactness of point-inverses can be removed from Theorem 28. Another Jung's result answers P 469 in a particular case. Two papers of him are submitted to this journal as well as a recent paper of Togo Nishiura who provides a complete affirmative solution of P 469 by means of some closed extensions of mappings.

REFERENCES

[1] J. M. Aarts, *Completeness degree; a generalization of dimension*, Fundamenta Mathematicae 63 (1968), p. 27 - 41.

[2] — *On the dimension of remainders in extensions of product spaces*, Colloquium Mathematicum 22 (1971), p. 233 - 238.

[3] Д. Аднаджевич, *О размерности* dm *топологических пространств*, General Topology and its Relations to Modern Analysis and Algebra II, Proceedings of the Second Prague Topological Symposium, 1966, Prague 1967, p. 22 - 24.

[4] А. В. Архангельский, *О рангах систем множеств и размерности пространств*, Доклады Академии наук СССР 143 (1962), p. 755 - 758.

[5] — *Отображения открытые и близкие к открытым; связи между пространствами*, Труды Московского математического общества 15 (1966), p. 181 - 223.

([20]) Theorem 50 as a consequence of Theorem 49 is also true for paracompact spaces.

[6] — *О факторизации отображений по весу и размерности*, Доклады Академии наук СССР 174 (1967), p. 1243 - 1246.

[7] — *О совершенных отображениях и уплотнениях*, ibidem 176 (1967), p. 983 - 986.

[8] — *On closed maps increasing dimension*, Czechoslovak Mathematical Journal 18 (1968), p. 389 - 391.

[9] В. Г. Болтянский и П. С. Солтан, *Обобщение теоремы Гуревича о размерности прообразов*, Математический сборник 69 (1966), p. 257 - 285.

[10] C. H. Dowker, *Inductive dimension of completely normal spaces*, The Quarterly Journal of Mathematics 4 (1953), p. 267 - 281.

[11] — *Local dimension of normal spaces*, ibidem 6 (1955), p. 101 - 120.

[12] E. Dyer, *Certain transformations which lower dimension*, Annals of Mathematic 63 (1956), p. 15 - 19.

[13] В. Егоров и Ю. Подставкин, *Об одном определении размерности*, Доклады Академии наук СССР 178 (1968), p. 774 - 777.

[14] S. Eilenberg, *Remarque sur un théorème de M. Hurewicz*, Fundamenta Mathematicae 24 (1935), p. 156 - 159.

[15] R. Engelking, *Outline of general topology*, Amsterdam-Warszawa 1968.

[16] J. Flachsmeyer, *Nulldimensionale Räume*, General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the Symposium held in Prague in September, 1961, Prague 1962, p. 152 - 154.

[17] H. Freudenthal, *Über dimensionserhöhende stetige Abbildungen*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse 5 (1932), p. 34 - 38.

[18] Р. Л. Фрум-Кетков, *Гомологические свойства прообразов точек при отображениях многообразий, повышающих размерность*, Доклады Академии наук СССР 122 (1958), p. 349 - 351.

[19] J. de Groot and T. Nishiura, *Inductive compactness as a generalization of semicompactness*, Fundamenta Mathematicae 58 (1966), p. 201 - 218.

[20] E. Hemmingsen, *Some theorems in dimension theory for normal Hausdorff spaces*, Duke Mathematical Journal 13 (1946), p. 495 - 504.

[21] R. E. Hodel, *Open functions and dimension*, ibidem 30 (1963), p. 461 - 467.

[22] W. Hurewicz, *Über dimensionserhöhende stetige Abbildungen*, Journal für die Reine und Angewandte Mathematik 169 (1933), p. 71 - 78.

[23] — and H. Wallman, *Dimension theory*, Princeton 1948.

[24] C. F. K. Jung, *Mappings on manifolds*, Colloquium Mathematicum 17 (1967), p. 53 - 60.

[25] — *Mappings on compact metric spaces*, ibidem 19 (1968), p. 73 - 76.

[26] M. Katětov, *On the dimension of non-separable spaces*, Czechoslovak Mathematical Journal 2 (1952), p. 333 - 368, and 6 (1956), p. 485 - 516.

[27] Я. М. Каждан и И. А. Вайнштейн, *Конечнократные непрерывные отображения, повышающие размерность*, Известия Академии наук СССР, Серия математическая 8 (1944), p. 129 - 138.

[28] J. E. Keesling, *Mappings and dimension in general metric spaces*, Pacific Journal of Mathematics 25 (1968), p. 277 - 288.

[29] — *Open and closed mappings and compactification*, Fundamenta Mathematicae 65 (1969), p. 73 - 81.

[30] — *Closed mappings which lower dimension*, Colloquium Mathematicum 20 (1969), p. 237 - 241.

[31] — *Closed mappings and local dimension*, ibidem 21(1970), p. 75 - 79.

[32] N. Kimura, *On a sum theorem in dimension theory*, Proceedings of the Japan Academy 43 (1967), p. 98 - 102.

[33] B. Knaster, *Sur les coupures biconnexes des espaces euclidiens de dimension $n > 1$ arbitraire*, Математический сборник 19 (1946), p. 9 - 18.

[34] K. Kuperberg and W. Kuperberg, *On weakly zero-dimensional mappings*, Colloquium Mathematicum 22(1971), p. 245 - 248.

[35] K. Kuratowski, *Topology*, vol. II, New York-Warszawa 1968.

[36] В. Кузьминов, *О когомологических размерностях компактов*, Сибирский математический журнал 5 (1964), p. 1282 - 1304.

[37] A. Lelek, *Sur deux genres d'espaces complets*, Colloquium Mathematicum 8 (1961), p. 31 - 34.

[38] — *On compactifications of some subsets of Euclidean spaces*, ibidem 9 (1962), p. 79 - 83.

[39] — *On the dimension of quasi-components in peripherically compact spaces*, ibidem, p. 241 - 244.

[40] — *On mappings that change dimensions of spheres*, ibidem 10 (1963), p. 45 - 48.

[41] — *Dimension and mappings of spaces with finite deficiency*, ibidem 12 (1964), p. 221 - 227.

[42] — *О размерности наростов при компактных расширениях*, Доклады Академии наук СССР 160 (1965), p. 534 - 537.

[43] — *Some problems in metric topology*, Louisiana State University Lecture Notes, Baton Rouge 1966.

[44] — *On mappings of Cantorian manifolds*, Colloquium Mathematicum 17 (1967), p. 315 - 318.

[45] — and D. Zaremba, *Dimensions of irreducible continua and fixations of components in compact spaces*, Fundamenta Mathematicae 51 (1962), p. 81 - 86.

[46] И. К. Лифанов, *Некоторые теоремы относительно размерности* Ind, Доклады Академии наук СССР 177 (1967), p. 33 - 36.

[47] О. В. Локуциевский, *О размерности бикомпактов*, ibidem 67 (1949), p. 217 - 219.

[48] S. Mazurkiewicz, *Sur les problèmes $\varkappa$ et λ de Urysohn*, Fundamenta Mathematicae 10 (1927), p. 311 - 319.

[49] — *Ein Satz über irreduzible Kontinua*, Monatshefte für Mathematik und Physik 37 (1930), p. 163 - 168.

[50] — *Über total zusammenhangslose Mengen*, Fundamenta Mathematicae 22 (1934), p. 267 - 269.

[51] L. F. McAuley, *Conditions for the equality of the inductive dimensions*, Portugaliae Mathematica 24 (1965), p. 21 - 30.

[52] K. Morita, *On closed mappings and dimension*, Proceedings of the Japan Academy 32 (1956), p. 161 - 165.

[53] K. Nagami, *Some theorems in dimension theory for non-separable spaces*, Journal of the Mathematical Society of Japan 9 (1957), p. 80 - 92.

[54] — *Finite-to-one closed mappings and dimension*, Proceedings of the Japan Academy 34 (1958), p. 503 - 506, and 35 (1959), p. 437 - 439.

[55] — *Mappings of finite order and dimension theory*, Science Council of Japan, Japanese Journal of Mathematics 30 (1960), p. 25 - 54.

[56] — *Mappings defined on 0-dimensional spaces and dimension*, Journal of the Mathematical Society of Japan 14 (1962), p. 101 - 118.

[57] J. Nagata, *Two remarks on dimension theory for metric spaces*, Proceedings of the Japan Academy 36 (1960), p. 53 - 54.

[58] — *Modern dimension theory*, Amsterdam 1965.

[59] T. Nishiura, *On the dimension of semi-compact spaces and their quasi-components*, Colloquium Mathematicum 12 (1964), p. 7 - 10.

[60] — *Inductive invariants and dimension theory*, Fundamenta Mathematicae 59 (1966), p. 243 - 262.

[61] A. Okuyama, *On spaces with some kinds of open coverings*, Memoirs of the Osaka University of the Liberal Arts and Education, Series B 10 (1961), p. 1 - 4.

[62] — *On cohomological dimension for paracompact spaces*, Proceedings of the Japan Academy 38 (1962), p. 489 - 494, and p. 655 - 659.

[63] Б. А. Пасынков, *О змеевидных бикомпактах*, Czechoslovak Mathematical Journal 13 (1963), p. 473 - 476.

[64] — *О формуле В. Гуревича*, Вестник Московского университета, Серия I, математика, механика, 4 (1965), p. 3 - 5.

[65] — *Об открытых отображениях*, Доклады Академии наук СССР 175 (1967), p. 292 - 295.

[66] M. Reichaw-Reichbach, *A note on absolute G_δ-spaces*, Proceedings of the American Mathematical Society 14 (1963), p. 806 - 808.

[67] J. H. Roberts, *Open transformations and dimension*, Bulletin of the American Mathematical Society 53 (1947), p. 176 - 178.

[68] М. Л. Шерснев, *Характерцстика размерности метрического пространства при помощи размерностных свойств его отображений в эвклидовы пространства*, Математический сборник 60 (1963), p. 207 - 218.

[69] K. Sieklucki, *A generalization of a theorem of S. Mazurkiewicz concerning Peano functions*, Annales Societatis Mathematicae Polonae, Series I: Commentationes Mathematicae 12 (1969), p. 251 - 253.

[70] Е. Г. Скляренко, *Две теоремы о бесконечномерных пространствах*, Доклады Академии наук СССР 143 (1962), p. 1053 - 1056.

[71] — *О совершенных бикомпактных расширениях*, ibidem 146 (1962), p. 1031-1034.

[72] — *Некоторые вопросы теории бикомпактных расширений*, Известия Академии наук СССР, Серия математическая 26 (1962), p. 427 - 452.

[73] — *Теорема об отображениях, понижающих размерность*, Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques 10 (1962), p. 429 - 432.

[74] — *О теореме Вьеториса-Бегла*, Доклады Академии наук СССР 149 (1963), p. 264 - 267.

[75] — *О некоторых приложениях теории пучков в общей топологии*, Успехи математических наук 19 (1964), No. 6, p. 47 - 70.

[76] Ю. М. Смирнов, *Некоторые соотношения в теории размерности*, Математический сборник 29 (1951), p. 157 - 172.

[77] — *О метрической размерности в смысле П. С. Александрова*, Известия Академии наук СССР, Серия математическая 20 (1956), p. 679 - 684.

[78] П. С. Солтан, *О размерности прообразов при отображении компактов в полиэдры*, Доклады Академии наук СССР 130 (1960), p. 510 - 513.

[79] J. Suzuki, *Note on a theorem for dimension*, Proceedings of the Japan Academy 35 (1959), p. 201 - 202.

[80] А. Д. Тайманов, *Об открытых образах борелевских множеств*, Математический сборник 37 (1955), p. 293 - 300.

[81] — *О замкнутых отображениях II*, ibidem 52 (1960), p. 579 - 588.

[82] И. А. Вайнштейн, *О повышающих размерность отображениях*, Доклады Академии наук СССР 67 (1949), p. 9 - 12.

[83] — *Об одномерных отображениях*, ibidem 83 (1952), p. 175 - 178.

[84] А. В. Зарелуа, *О теореме Гуревича*, Математический сборник 60 (1963), p. 17 - 28.

[85] — *О равенстве размерностей*, ibidem 62 (1963), p. 295 - 319.

[86] — *О конечнократных отображениях*, General Topology and its Relations to Modern Analysis and Algebra II, Proceedings of the Second Prague Topological Symposium, 1966, Prague 1967, p. 359 - 360.

[87] — *О конечнократных отображениях*, Доклады Академии наук СССР 172 (1967), p. 775 - 778.

[88] D. Zaremba-Szczepkowicz, *On orthogonal mappings and their dimensions*, Fundamenta Mathematicae 57 (1965), p. 205 - 213.

Reçu par la Rédaction le 3. 12. 1969