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TWO INEQUALITIES FOR STARLIKE FUNCTIONS

BY

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Suppose the function f belongs to the class S; that is, f is normalized ($f(0) = 0$ and $f'(0) = 1$), analytic and univalent in the unit disk $D = \{z\colon |z| < 1\}$. Set

$$\varphi_f(z) = z\frac{f'(z)}{f(z)} \quad \text{and} \quad \psi_f(z) = 1+z\frac{f''(z)}{f'(z)}.$$

The analytic function f is said to be *starlike* in D if the image domain $f(D)$ is star-shaped with respect to the origin; that is, f is univalent in D and $\arg f(re^{i\theta})$ is a non-decreasing function of θ for $0 < r < 1$. The last two conditions are equivalent to the inequality $\mathrm{Re}\,\varphi_f(z) \geqslant 0$ $(|z| < 1)$. The class of all starlike functions $f \in S$ will be denoted by S^*.

The analytic function f is said to be *convex* in D if the image domain $f(D)$ is convex; that is, f is univalent in D, and if $\Theta(re^{i\theta})$ denotes the argument of the vector tangent to the image curve of $|z| = r$, then Θ is a non-decreasing function of θ for $0 < r < 1$. Thus the function f is convex if and only if $\mathrm{Re}\,\psi_f(z) \geqslant 0$ $(|z| < 1)$.

It is well known that if a function is convex, then it is starlike of order $1/2$ ([1], p. 44, and [4]). In other words, the inequality $\mathrm{Re}\,\psi_f(z) \geqslant 0$ $(|z| < 1)$ implies the inequality $\mathrm{Re}\,\varphi_f(z) \geqslant 1/2$ $(|z| < 1)$. This leads to the topic of this note: whether there exist further inequalities linking the quantities $\mathrm{Re}\,\varphi_f(z)$ and $\mathrm{Re}\,\psi_f(z)$.

Our tool is the following result due to Robertson ([2] and [3]):

LEMMA. *If $F(u, v)$ is analytic in the v-plane and in the half-plane $\mathrm{Re}\,u > 0$, if $P(z)$ is analytic with positive real part in D, and if $P(0) = 1$, then, on the circle $|z| = r < 1$, the minimum*

$$\min_{P}\min_{|z|=r}\mathrm{Re}\,F\big(P(z), zP'(z)\big)$$

is attained only for a function $P = P_0$ of the form

$$P_0(z) = \frac{1+a}{2}\,\frac{1+ze^{it}}{1-ze^{it}}+\frac{1-a}{2}\,\frac{1+ze^{-it}}{1-ze^{-it}},$$

where $-1 \leqslant a \leqslant 1$ *and* $0 \leqslant t \leqslant 2\pi$. *These extremal functions* P_0 *can also be described by the equation*

$$\frac{P_0(z)-1}{P_0(z)+1} = z\frac{b-z}{1-\bar{b}z} \quad (b = \cos\theta + ai\sin\theta).$$

THEOREM 1. *Suppose* $f \in S^*$ *and* $0 < k < \infty$. *The relation*

$$|\operatorname{Re}\psi_f(z) - \operatorname{Re}\varphi_f(z)| \leqslant k$$

holds in the disk

$$|z| \leqslant \frac{\sqrt{k^2+1}-1}{k}.$$

The equality occurs for the Koebe function.

COROLLARY. *If* $f \in S^*$ *and* $|z| \leqslant r$, *then*

$$|\operatorname{Re}\psi_f(z) - \operatorname{Re}\varphi_f(z)| \leqslant \frac{2r}{1-r^2}.$$

Proof. We prove first that the inequality

$$\operatorname{Re}\psi_f(z) \leqslant \operatorname{Re}\varphi_f(z) + k \tag{1}$$

holds for $|z| \leqslant \left(\sqrt{k^2+1}-1\right)/k$. Since f belongs to S^*, the function $P(z) = \varphi_f(z)$ satisfies the conditions of Robertson's lemma. Set $u = P(z)$ and $v = zP'(z)$. Then inequality (1) is equivalent to the inequality

$$\operatorname{Re}\left(k - \frac{v}{u}\right) \geqslant 0. \tag{2}$$

Since the function $F(u, v) = k - v/u$ satisfies the conditions of Robertson's lemma, the minimum

$$\min_{P}\min_{|z|=r}\operatorname{Re}F\left(P(z), zP'(z)\right)$$

is attained only for a function $P = P_0$, as described in the Lemma.

Set $w = w(z) = \left(P_0(z)-1\right)/\left(P_0(z)+1\right)$. Then

$$u = P_0(z) = \frac{1+w}{1-w} \quad \text{and} \quad v = zP_0'(z) = \frac{2zw'}{(1-w)^2}.$$

Thus inequality (2) holds if and only if

$$\operatorname{Re}\frac{zw'}{1-w^2} \leqslant \frac{k}{2}. \tag{3}$$

Since

$$\operatorname{Re}\frac{zw'}{1-w^2} \leqslant \left|\frac{zw'}{1-w^2}\right| \leqslant \frac{|zw'-w|+|w|}{1-|w|^2},$$

inequality (3) will hold if the last quantity does not exceed $k/2$. But

$$w = z\frac{b-z}{1-\bar{b}z},$$

so that

$$zw'-w = -(1-|b|^2)z^2(1-\bar{b}z)^{-2} \quad \text{and} \quad |zw'-w| = \frac{|z|^2-|w|^2}{1-|z|^2}.$$

If we write $|z| = r$ and $|(b-z)/(1-\bar{b}z)| = x$, we see that (3) holds if

$$\frac{r^2(1-x^2)}{1-r^2}+rx \leqslant \frac{k}{2}(1-r^2x^2),$$

that is,

$$r^2x^2\big(k(1-r^2)-2\big)+2rx(1-r^2)+2r^2-k(1-r^2) \leqslant 0. \tag{4}$$

Consider the left-hand side of (4) as a function of x, say $h(x)$. It is easy to see that $h(1) \leqslant 0$ if and only if

$$r \leqslant \frac{1}{k}\big(\sqrt{k^2+1}-1\big).$$

A straightforward computation now shows that this last condition insures inequality (4) for all x $(0 \leqslant x \leqslant 1)$.

The equality in (4) occurs if $x = 1$ and $rk = \sqrt{k^2+1}-1$. But $x = 1$ only if $|b| = 1$, so that $w(z) = e^{it}z$, and the extremal function is a Koebe function. It is easy to verify that the equality also occurs in (1), namely for the function $f(z) = z(1-z)^{-2}$ at the point

$$z = \frac{1}{k}\big(\sqrt{k^2+1}-1\big).$$

To complete the proof of Theorem 1, we need to show that

$$\operatorname{Re}\varphi_f(z)-k \leqslant \operatorname{Re}\psi_f(z) \qquad \Big(|z| \leqslant \frac{\sqrt{k^2+1}-1}{k}\Big).$$

This is done with the same method, but using the function $F(u, v) = k+v/u$. The equality occurs again for the Koebe function, this time at the point

$$z = -\frac{1}{k}\big(\sqrt{k^2+1}-1\big).$$

THEOREM 2. *Suppose $f \in S^*$. For each $k > 1$ there exists a radius $r(k) > 0$ such that the relation*

$$\operatorname{Re}\psi_f(z) \leqslant k \operatorname{Re}\varphi_f(z) \tag{5}$$

holds in the disk $|z| \leqslant r(k)$. For $k = 2$, we have the explicit expression

$$r(k) = \sqrt{\frac{k-1}{k+1}}$$

and, for $k \geqslant 3$, the radius $r(k)$ satisfies the condition

$$1 - \frac{1}{\sqrt{k-2}} \leqslant r(k)^2 \leqslant \frac{k-1}{k+1}.$$

Proof. We use the same method as in the proof of Theorem 1. Again we set $u = P(z) = \varphi_f(z)$ and $v = zP'(z)$. Then inequality (5) is equivalent to the inequality

$$\operatorname{Re}\left((k-1)u - \frac{v}{u}\right) \geqslant 0. \tag{6}$$

The function $F(u, v) = (k-1)u - v/u$ satisfies the conditions of Robertson's lemma, so that we know the structure of an extremal function. Setting again $P(z) = \big(1+w(z)\big)/\big(1-w(z)\big)$, we see that inequality (6) holds if and only if

$$\operatorname{Re} K(z) \geqslant 0, \qquad \text{where } K(z) = \frac{1}{1-w(z)^2}\left[(k-1)\big(1+w(z)\big)^2 - 2zw'(z)\right].$$

But $\operatorname{Re} K(z) \geqslant 0$ if and only if $|K(z)-1|/|K(z)+1| \leqslant 1$, so that inequality (6) is equivalent to the condition

$$\begin{aligned} &|k(1+w)^2 - 2(zw'-w) - 4w - 2| \\ &\qquad\qquad \leqslant |k(1+w)^2 - 2(zw'-w) - 4w - 2w^2|. \end{aligned} \tag{7}$$

A simple calculation shows that the inequality $|A-2| \leqslant |A-2w^2|$ holds if and only if $1-|w|^4 \leqslant \operatorname{Re}(A - A\overline{w}^2)$. Hence (7) is equivalent to the condition

$$1-|w|^4 \leqslant k(1-|w|^4) + 2(k-2)(1-|w|^2)\operatorname{Re} w - 2\operatorname{Re}[(zw'-w)(1-\overline{w}^2)].$$

In the last summand, we replace the real part by its absolute value and obtain the stronger inequality

$$1+|w|^2 \leqslant k(1+|w|^2) + 2(k-2)\operatorname{Re} w - 2\,\frac{|z|^2-|w|^2}{1-|z|^2}\,\frac{|1-w^2|}{1-|w|^2}. \tag{8}$$

Inequality (8) holds for $z = 0$. Hence there exists a positive $r(k)$ such that (8) holds for all z ($|z| \leqslant r(k)$) and all w ($|w| \leqslant |z|$).

The functions

$$f_\beta(z) = \frac{z}{1+z^2}\left(\frac{1+iz}{1-iz}\right)^\beta \quad (0<\beta<1)$$

belong to S^* and have the properties that

$$\varphi_{f_\beta}(i\beta) = 1 \quad \text{and} \quad \psi_{f_\beta}(i\beta) = \frac{1+\beta^2}{1-\beta^2}.$$

This shows that, for all $k>1$, $r(k)$ can at most have the value $\sqrt{(k-1)/(k+1)}$.

For $k=2$, we replace $|1-w^2|$ by $1+|w|^2$ in (8), and we use the estimate $(r^2-|w|^2)/(1-|w|^2)\leqslant r^2$, where $|z|=r$, to obtain the stronger inequality

$$0\leqslant\left(k-1-\frac{2r^2}{1-r^2}\right)(1+|w|^2),$$

which is satisfied if $r^2\leqslant(k-1)/(k+1)$.

For $k\neq 2$, we compute the minimum of the right-hand side of (8), assuming w to have a fixed absolute value but a variable argument. For $k\geqslant 3$ and $r^2\leqslant 1-(k-2)^{-1/2}$, this minimum turns out to be

$$k(1+|w|^2)-2(k-2)|w|-2\frac{r^2-|w|^2}{1-r^2}.$$

Hence (8) holds if

$$(9)\qquad 0\leqslant k-1-\frac{2r^2}{1-r^2}-2(k-2)|w|+\left(k-1+\frac{2}{1-r^2}\right)|w|^2.$$

The right-hand side of (9) is a quadratic equation in $|w|$ that is non--negative for $|w|=0$ if $r^2\leqslant(k-1)/(k+1)$ and that has no real root if

$$r^2\leqslant\frac{2\sqrt{k-1}-1}{2\sqrt{k-1}+1}.$$

But since

$$1-(k-2)^{-1/2}<\frac{2\sqrt{k-1}-1}{2\sqrt{k-1}+1},$$

Theorem 2 is established.

It would be interesting to know (**P 964**) whether inequality (5) holds for

$$|z|\leqslant\sqrt{\frac{k-1}{k+1}}\quad \text{for all } k>1.$$

Finally, the fact that convexity implies starlikeness of order 1/2 leads to the question whether there exists a number $c > 0$ such that every normalized analytic function f with $\operatorname{Re}\psi_f(z) \geqslant -c$ ($|z| < 1$) is starlike (of order 0) in D. We answer this question in the negative.

THEOREM 3. *For each number $c > 0$ there exists a normalized analytic function f_c such that $\operatorname{Re}\psi_{f_c}(z) \geqslant -c$ for every $z \in D$ and $\operatorname{Re}\varphi_{f_c}(z) < 0$ for some $z \in D$.*

Proof. The function $w_c(z) = [1+(2c+1)z]/(1-z)$ maps D onto the half-plane $\operatorname{Re} w > -c$. If we set

$$f_c(z) = \frac{1}{2c+1} \frac{1-(1-z)^{2c+1}}{(1-z)^{2c+1}},$$

it is easy to see that f_c is analytic and normalized in D and that $\psi_{f_c}(z) = w_c(z)$. A computation shows that

$$\varphi_{f_c}(z) = (2c+1) \frac{z}{1-z} \frac{1}{1-(1-z)^{2c+1}},$$

and that $\operatorname{Re}\varphi_{f_c}(z) \geqslant 0$ if and only if

$$\operatorname{Re}\left(z - |z|^2 - z(1-\bar{z})^{2c+2}\right) \geqslant 0. \tag{10}$$

But inequality (10) can be violated by some $z \in D$, regardless of the choice of c.

REFERENCES

[1] A. Marx, *Untersuchungen über schlichte Abbildungen*, Mathematische Annalen 107 (1932), p. 40-67.

[2] M. S. Robertson, *Extremal problems for analytic functions with positive real part and applications*, Transactions of the American Mathematical Society 106 (1963), p. 236-253.

[3] — *Some radius of convexity problems*, Michigan Mathematical Journal 10 (1963), p. 231-236.

[4] E. Stohhäcker, *Beiträge zur Theorie der schlichten Funktionen*, Mathematische Zeitschrift 37 (1933), p. 356-380.

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