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LITTLEWOOD–PALEY THEORY AND ε-FAMILIES OF OPERATORS

BY

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1. Introduction. In analysis we use a number of different function and distribution spaces. For instance, in classical harmonic analysis we frequently use Lebesgue and Hardy spaces, in partial differential equations the Sobolev spaces are natural, and in approximation theory Lipschitz and Besov spaces are important. It has been known for a long time that most of these spaces share a common underlying structure. More precisely, it is well known (see [8], [19], [21], and [24]) that by using Littlewood–Paley theory most distribution spaces on $\mathbf{R}^n$ can be characterized through the action of appropriate families of convolution operators. Here we shall study analogous characterizations in terms of more general families of operators.

We note that, classically, the Fourier transform is the basic natural tool for studying Littlewood–Paley characterizations of distribution spaces, since the operators involved commute with translations. However, in our more general situation, this tool is not available. Instead we shall rely heavily on Calderón–Zygmund operator theory; this is especially reflected in the definitions of the families of operators we consider. We note that these families have previously been studied, primarily in the context of L^2, by Christ and Journé [2].

A fundamental result in Calderón–Zygmund theory is the celebrated $T1$-theorem of David and Journé. A major part of this paper is devoted to the study of versions of this theorem for general classes of operators and spaces.

A brief description of the contents of this paper now follows. In Section 2 we give results on which our paper depends: 1) A description of the basic scales of spaces we study, the Besov and Triebel–Lizorkin spaces; and their characterization by a family of convolution type operators, and

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also by developments as sums of smooth atoms. 2) An introduction to the notions of almost diagonal matrices, almost diagonal operators, and the notions of smooth molecules of the first and second kind. 3) A presentation of basic notions of Calderón–Zygmund operators and of the Weak Boundedness Property. In Section 3 we present two versions of the $T1$-theorem for Besov spaces of order zero. In Section 4 we consider characterizations of the Triebel–Lizorkin and Besov spaces in terms of families of operators (ϵ-families) that are not necessarily convolution operators.

2. Preliminary results. In this section we give the definitions of the Besov and Triebel–Lizorkin spaces, the definitions and properties of almost diagonal matrices and operators, and the definitions and basic results for Calderón–Zygmund operators in the forms that we will use.

Most of our notation will be introduced as we proceed; but we will make a few brief comments here. $\mathcal{S}(\mathbf{R}^n) = \mathcal{S}$ is the Schwartz space of rapidly decreasing test functions and $\mathcal{S}'$ is its dual, the space of tempered distributions. $\mathcal{D}(\mathbf{R}^n) = \mathcal{D}$ is the Schwartz space of compactly supported test functions, and $\mathcal{D}'$ is its dual, the space of Schwartz distributions. The Fourier transform is defined in the usual way for $f \in \mathcal{S}'$ and is denoted by $\widehat{f}$. Below we will choose a function φ in $\mathcal{S}$. For $\nu \in \mathbf{Z}$ we set $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$. Whenever Q is used as an index, as in $\sum_Q$, $\{\cdot\}_Q$, or $\sup_Q$, this means that Q varies over all dyadic cubes in $\mathbf{R}^n$. The side length of a cube Q is denoted by $\ell(Q)$ and x_Q denotes the "lower left corner" of Q. The set aQ, $a > 0$, is the cube concentric with Q and with side length $a\ell(Q)$. We only consider cubes with sides parallel to the axes. For the function φ that we choose in Section 2.1 we let

$$\varphi_Q(x) = 2^{\nu/2}\varphi(2^\nu x - k)$$

when

$$Q = Q_{\nu k} = \{\, x : 2^{-\nu}k_i \le x_i \le 2^{-\nu}(k_i + 1),\ i = 1, \dots, n\}.$$

For f a distribution and η a test function we set

$$\langle f, \eta\rangle = f(\overline{\eta})\,.$$

As is usual, the letter c will denote constants, often different from place to place.

2.1. *Definitions of the Besov and Triebel–Lizorkin spaces.* The spaces we are going to study belong to two basic scales. To define these we choose a function φ with the following properties:

$$\varphi \in \mathcal{S}(\mathbf{R}^n)\,, \tag{1}$$

$$\operatorname{supp}\widehat{\varphi} \subset \{\, \xi \in \mathbf{R}^n : 1/2 \le |\xi| \le 2\,\}\,, \tag{2}$$

$$|\widehat{\varphi}(\xi)| \ge c > 0 \quad \text{if } 3/5 \le |\xi| \le 5/3\,, \tag{3}$$

$$\sum_{\nu} |\widehat{\varphi_\nu}(\xi)|^2 = 1 \quad \text{if } \xi \neq 0 . \tag{4}$$

Remark. Under these four conditions it is easy to see that the representation

$$f = \sum_{\nu} \langle f, \varphi_\nu \rangle \varphi_\nu$$

holds in a variety of senses. For example, it holds in the sense of convergence in L^2 and in $\mathcal{S}'(\mathbf{R}^n)/\mathcal{P}$ (tempered distributions modulo polynomials). See [12, §2] and the references given there for details.

The Triebel–Lizorkin space $\dot{F}_p^{\alpha,q}$, $\alpha \in \mathbf{R}$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$, is the collection of all $f \in \mathcal{S}'(\mathbf{R}^n)/\mathcal{P}$ such that

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \Big\| \Big(\sum_{\nu \in \mathbf{Z}} (2^{\nu\alpha} |\varphi_\nu * f|)^q \Big)^{1/q} \Big\|_{L^p} < \infty . \tag{5}$$

For $q = \infty$, (5) is given the usual interpretation. When $p = \infty$ this definition requires a modification:

$$\|f\|_{\dot{F}_\infty^{\alpha,q}} = \sup_P \Big(|P|^{-1} \int_P \sum_{\nu = -\log_2 \ell(P)}^{\infty} (2^{\nu\alpha} |\varphi_\nu * f(x)|)^q \, dx \Big)^{1/q} \tag{6}$$

We refer the reader to [12] for a discussion of $\|\cdot\|_{\dot{F}_\infty^{\alpha,q}}$. If we interchange the order of summation and integration in these definitions we obtain the space $\dot{B}_p^{\alpha,q}$, $\alpha \in \mathbf{R}$, and $1 \leq p, q \leq \infty$. This is the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \Big(\sum_{\nu \in \mathbf{Z}} (2^{\nu\alpha} \|\varphi_\nu * f\|_{L^p})^q \Big)^{1/q} . \tag{7}$$

The definitions of the Triebel–Lizorkin spaces and the Besov spaces are independent of the choice of the function φ that satisfies (1)–(4). For this and for other standard facts about these spaces the reader is referred to [24] and [12].

For our study of the boundedness properties of operators on the Besov and Triebel–Lizorkin spaces we will use atomic decompositions of these spaces. Throughout this paper the parameters α, p, and q will be restricted to the range $|\alpha| < 1$, $1 \leq p, q \leq \infty$, and this allows us to greatly simplify the notion of a "smooth molecule" that follows shortly. In general, one can consider the range $\alpha \in \mathbf{R}$, $0 < p, q \leq \infty$; but not in this paper. With this convention in mind we say that $\{a_Q\}_Q$ is a family of *smooth atoms* if for each dyadic cube Q, in $\mathbf{R}^n$, the function a_Q satisfies:

$$a_Q \in \mathcal{D} , \tag{8}$$

$$\operatorname{supp} a_Q \subset 3Q , \tag{9}$$

$$\int a_Q(x)\,dx = 0\,, \tag{10}$$

$$|\partial^\gamma a_Q(x)| \leq |Q|^{-1/2-\gamma/n} \qquad \text{for } 0 \leq |\gamma| \leq 1\,. \tag{11}$$

It is known (see [10, 12]) that each $f \in \dot{F}_p^{\alpha,q}$ has a smooth atomic decomposition

$$f = \sum_Q s_Q a_Q\,,$$

where $\{a_Q\}_Q$ is a family of smooth atoms and the sequence $s = \{s_Q\}_Q$ satisfies

$$\|s\|_{\dot{f}_p^{\alpha,q}} = \left\| \Big(\sum_Q (|Q|^{-\alpha/n}|s_Q|\tilde{\chi}_Q)^q\Big)^{1/q} \right\|_{L^p} < \infty\,, \tag{12}$$

where $\tilde{\chi}_Q$ is the "L^2-normalized characteristic function" of the cube Q: $\tilde{\chi}_Q(x) = |Q|^{-1/2}\chi_Q(x)$. Furthermore,

$$\|f\|_{\dot{F}_p^{\alpha,q}} \sim \inf\left\{ \|s\|_{\dot{f}_p^{\alpha,q}} : f = \sum_Q s_Q a_Q \right\}. \tag{13}$$

The usual caveats go with this definition; in particular, it must be modified if $p = \infty$:

$$\|f\|_{\dot{f}_\infty^{\alpha,q}} = \sup_P \Big(|P|^{-1} \int_P \sum_{Q \subset P} (|Q|^{-\alpha/n}|s_Q|\tilde{\chi}_Q(x))^q\,dx \Big)^{1/q}. \tag{14}$$

Elements of the Besov space $\dot{B}_p^{\alpha,q}$ also have smooth atomic decompositions. The corresponding sequence space is $\dot{b}_p^{\alpha,q}$ with norm given by

$$\|s\|_{\dot{b}_p^{\alpha,q}} = \Big(\sum_{\nu \in \mathbf{Z}} \Big(2^{\nu\alpha} \Big\| \sum_{\ell(Q)=2^{-\nu}} |s_Q|\tilde{\chi}_Q(\cdot) \Big\|_{L^p} \Big)^q \Big)^{1/q}. \tag{15}$$

In analogy with (13) we have

$$\|f\|_{\dot{B}_p^{\alpha,q}} \sim \inf\left\{ \|s\|_{\dot{b}_p^{\alpha,q}} : f = \sum_Q s_Q a_Q \right\}. \tag{16}$$

See [10] for details.

2.2. *Almost diagonal operators and matrices.* The class of Calderón–Zygmund operators is closely related to a class of matrices studied in [12]. Keeping in mind that $1 \leq p, q \leq \infty$ we let

$$\omega_\delta(t) = (1+t)^{-(n+\delta)}\,, \tag{17}$$

$$\Omega_{QP}(\delta,\alpha) = \left(\frac{\ell(Q)}{\ell(P)}\right)^\alpha \omega_\delta\left(\frac{|x_Q - x_P|}{\ell(Q) \vee \ell(P)}\right) \left(\frac{\ell(Q)}{\ell(P)} \wedge \frac{\ell(P)}{\ell(Q)}\right)^{(n+\delta)/2}. \tag{18}$$

(We use the conventions: $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$.) We say that a matrix $\{A_{QP}\}$ is *almost diagonal* (for $\dot{F}_p^{\alpha,q}$ or $\dot{B}_p^{\alpha,q}$) if there is a

$\delta > 0$ such that

$$|A_{QP}| \le c_\delta \Omega_{QP}(\delta, \alpha) \tag{19}$$

for all dyadic cubes, Q and P. An almost diagonal matrix induces a bounded operator on the sequence spaces $\dot{f}_p^{\alpha,q}$ and $\dot{b}_p^{\alpha,q}$ with norms depending only on the values of α, p, q, δ, and c_δ.

Remark. Complete results for the Triebel–Lizorkin spaces can be found in [12]. For those familiar with interpolation theory the result for Besov spaces follows from the result for Triebel–Lizorkin spaces from the reiteration theorem. A more elementary route is to notice that the proof for the Triebel–Lizorkin spaces simplifies to give a proof for the Besov spaces.

Furthermore, almost diagonal matrices are closed under composition and hence form an algebra. See [12, §2, §3, and Appendix D]. It is particularly important to note their Theorem D.2 which gives this result in a quantitative form. Let A_1 and A_2 be almost diagonal matrices, and let δ_i and c_{δ_i} be the constants for these matrices; then $A = A_1 A_2$ is almost diagonal with constants δ and c_δ. If $\delta_1 \neq \delta_2$ we can take $\delta = \delta_1 \wedge \delta_2$ while if they are equal we can take δ to be any value less than their common value. The value of c_δ is determined only by the constants explicitly given.

A notion almost as basic as that of a smooth atom is that of a smooth molecule. In [12, §3] the notion of a family of smooth molecules is introduced, which is a system $\{m_Q\}_Q$ that satisfies the four conditions (3.3)–(3.6) in that paper. We will call these *smooth molecules of the first kind*. They introduce a second kind of family (not named in their paper) which satisfy four related conditions: (3.7)–(3.10). We call these *smooth molecules of the second kind*. Lemma 3.8 in their paper can be rephrased to say that if $\{\eta_Q\}_Q$ and $\{\theta_Q\}_Q$ are families of smooth molecules of the first and second kind, respectively, then $\{\langle\theta_Q, \eta_P\rangle\}_{QP}$ is an almost diagonal matrix with constants δ and c_δ independent of the choice of the families of smooth molecules, for all admissible values of α, p, and q.

It is easy to check that if $\{m_Q\}_Q$ satisfies the following three conditions:

$$\int m_Q(x)\, dx = 0\,, \tag{20}$$

$$|m_Q(x)| \le |Q|^{-1/2}(1 + |x - x_Q|/|\ell(Q)|)^{-(n+\epsilon)}\,, \tag{21}$$

$$\begin{aligned} &|m_Q(x) - m_Q(y)| \\ &\quad \le |Q|^{-1/2-\epsilon/n}|x-y|^\epsilon \sup_{|z|\le|x-y|} (1 + |x - z - x_Q|/\ell(Q))^{-(n+\epsilon)}\,, \end{aligned} \tag{22}$$

then $\{m_Q\}$ is a family of smooth molecules of both the first and the second kind, provided $|\alpha| < \epsilon \le 1$ and $1 \le p, q \le \infty$.

PROPOSITION 2.1. *If* $\{\eta_Q\}_Q$ *and* $\{\theta_Q\}_Q$ *both satisfy conditions* (20)–(22)

then

$$\{\langle \theta_Q, \eta_P \rangle\}_{QP}$$

and its adjoint are almost diagonal matrices for $|\alpha| < \epsilon \le 1$ *and* $1 \le p, q \le \infty$, *with the constant* c_δ *independent of the choice of the families.*

Proof. In view of our discussion above this follows from Lemma 3.8 of [12]. ∎

Remark. The following facts are easily verified.

- If $\{a_Q\}_Q$ is a family of smooth atoms, i.e., satisfies conditions (8)–(11), then $a_Q = cm_Q$ where $\{m_Q\}_Q$ satisfies (20)–(22) with $\epsilon = 1$.
- If φ is the function we chose to define our spaces, i.e., satisfies (1)–(4), then $\varphi_Q = cm_Q$ where $\{m_Q\}_Q$ satisfies (20)–(22) with $\epsilon = 1$.

This is immediate from the definitions.

We say that a continuous linear operator $T : \mathcal{D} \to \mathcal{D}'$ is *almost diagonal* if its associated matrix $\{\langle T\varphi_P, \varphi_Q \rangle\}_{QP}$ is almost diagonal. This definition does not depend on the particular choice of φ. In fact, we have the following characterization:

LEMMA 2.2. *Suppose that* $\{a_Q\}_Q$ *and* $\{b_Q\}_Q$ *are families of smooth atoms. Then an operator* T *is almost diagonal if and only if there are* $\delta > 0$ *and* $c_\delta > 0$ *such that*

$$|\langle Tb_P, a_Q \rangle| \le c_\delta \Omega_{QP}(\delta, \alpha),$$

where δ *and* c_δ *do not depend on the choice of the families of smooth atoms.*

Proof. This is an easy consequence of the fact that the class of almost diagonal matrices is closed under composition. To see this we assume first that T is almost diagonal. Let $A_{QP} = \langle a_P, \varphi_Q \rangle$ and $B_{QP} = \langle b_P, \varphi_Q \rangle$. It follows from Proposition 2.1, and the remark that follows it, that $A = \{A_{QP}\}$ and $B = \{B_{QP}\}$ as well as their adjoints are both almost diagonal with constants that are independent of the choice of families of smooth atoms. We have

$$a_Q = \sum_R A_{RQ} \varphi_R, \quad Tb_P = \sum_S B_{SP} T\varphi_S,$$

$$\langle Tb_P, a_Q \rangle = \sum_R \sum_S \overline{A_{RQ}} B_{SP} \langle T\varphi_S, \varphi_R \rangle .$$

The double sum on the right hand side represents the composition of three almost diagonal matrices and this establishes one direction.

For the other direction we note (see the proof of 9.12 in [12]) that the function φ has a smooth atomic decomposition:

$$\varphi = \sum_{k\in\mathbf{Z}^n} \frac{C}{(1+|k|)^L} a_{Q_{0k}} \tag{23}$$

for any (fixed) sufficiently large value of L with a value of C that depends on L. Let $B = \{B_{QP}\}$ where

$$B_{QP} = \begin{cases} c/(1+|x_Q - x_P|/\ell(Q))^L & \text{if } \ell(P) = \ell(Q), \\ 0 & \text{if } \ell(P) \neq \ell(Q). \end{cases}$$

Then by the expression for φ in (23) we can write

$$\varphi_Q = \sum_R B_{RQ} a_R,$$

where the smooth atoms a_R are obtained from the $a_{Q_{0k}}$ by dilation and translation. Similarly,

$$T\varphi_P = \sum_S B_{SP} T b_S.$$

Hence,

$$\langle T\varphi_P, \varphi_Q\rangle = \sum_R \sum_S \overline{B_{RQ}} B_{SP} \langle T b_S, a_R\rangle .$$

This is again the composition of three almost diagonal matrices and this completes the proof. ∎

2.3. *CZO's and the weak boundedness property.* The class of Calderón–Zygmund operators is defined as follows. Suppose that T is a continuous linear operator from $\mathcal{D}(\mathbf{R}^n)$ to $\mathcal{D}'(\mathbf{R}^n)$. By the Schwartz kernel theorem there is a distribution K in $\mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n)$ such that

$$\langle T\bar{\theta}, \eta\rangle = \langle K, \eta \otimes \theta\rangle, \qquad \theta, \eta \in \mathcal{D}.$$

The distribution K is called the kernel of T. We say that K is a *Calderón–Zygmund kernel* if its restriction to the set $\{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n : x \neq y\}$ is a continuous function $K(x,y)$ that satisfies:

$$|K(x,y)| \le c\frac{1}{|x-y|^n} \quad \text{for all } x \neq y, \tag{24}$$

$$|K(x,y) - K(x,y')| \le c\frac{|y-y'|^\epsilon}{|x-y|^{n+\epsilon}} \quad \text{whenever } 2|y-y'| \le |x-y|, \tag{25}$$

$$|K(x,y) - K(x',y)| \le c\frac{|x-x'|^\epsilon}{|x-y|^{n+\epsilon}} \quad \text{whenever } 2|x-x'| \le |x-y|, \tag{26}$$

for some constant c and an ϵ, $0 < \epsilon \le 1$. In this case we say that T is a *Calderón–Zygmund operator*, and write $T \in$ CZO. If we wish to emphasize the "Lipschitz condition of order ϵ" we write $T \in \mathrm{CZO}(\epsilon)$. A consequence is that if $T \in \mathrm{CZO}$, $\theta, \eta \in \mathcal{D}$, and $\operatorname{supp}\theta \cap \operatorname{supp}\eta = \emptyset$ then

$$\langle T\overline{\theta}, \eta\rangle = \iint_{\mathbf{R}^n \times \mathbf{R}^n} K(x,y)\overline{\theta}(x)\overline{\eta}(y)\,dx\,dy.$$

In a similar vein, if $T \in \mathrm{CZO}$, $\theta \in \mathcal{D}$, and $x \neq \operatorname{supp}\theta$ then

$$T\theta(x) = \int_{\mathbf{R}^n} K(x,y)\theta(y)\,dy.$$

The reader is warned that several slight variants of the definitions used here are also commonly used.

This class of operators, which is a generalization of the singular integral operators of the now classical Calderón–Zygmund theory [1], was first systematically studied by Coifman and Meyer [3]. Their work, and that of many others, culminated in the celebrated "$T1$-theorem" of David and Journé [6]. In order to state this result we need to recall another definition. Suppose θ is a function that is defined on $\mathbf{R}^n$, $z \in \mathbf{R}^n$, and $t > 0$. Set

$$\theta_t^z(x) = t^{-n}\theta((x-z)/t)$$

for all x in $\mathbf{R}^n$. We say that a linear and continuous operator $T : \mathcal{D} \to \mathcal{D}'$ satisfies the *weak boundedness property*, and write $T \in \mathrm{WBP}$, if for each bounded subset $\mathcal{B}$ of $\mathcal{D}$ there is a constant $c = c(\mathcal{B})$ such that for all $\theta, \eta \in \mathcal{B}$,

$$|\langle T(\theta_t^z), \eta_t^z\rangle| \leq ct^{-n} \quad \text{for all } z \text{ and } t. \tag{27}$$

Notice that if T is bounded on L^p for any p, $1 \leq p \leq \infty$, then $T \in \mathrm{WBP}$. It is often convenient to use a stronger condition than (27), since it may be easier to establish. To wit: For all $\theta, \eta \in \mathcal{D}$ with supports contained in balls of radius $t > 0$,

$$|\langle T\theta, \eta\rangle| \leq ct^n(\|\theta\|_{L^\infty} + t\|\nabla\theta\|_{L^\infty})(\|\eta\|_{L^\infty} + t\|\nabla\eta\|_{L^\infty}). \tag{28}$$

We are now in a position to state the theorem of David and Journé.

PROPOSITION 2.3 ($T1$-Theorem). *Suppose that $T \in \mathrm{CZO}$. Then T is bounded on L^2 if and only if $T \in \mathrm{WBP}$, $T1 \in \mathrm{BMO}$, and $T^*1 \in \mathrm{BMO}$.*

David and Journé showed in [6] that the proof of this theorem can be reduced to the case: $T1 = 0$, $T^*1 = 0$, and the reader is referred to their paper for details and further references. The reader is also referred to the opening pages of [9] where the authors of that paper show how to make sense of $T1$ and T^*1 as linear functionals on $\mathcal{D}_0 = \{\psi \in \mathcal{D} : \int \psi\,dx = 0\}$. In this paper we will consider various versions of the "reduced form" of the $T1$-theorem.

Our approach for proving boundedness results is, to a large extent, standard and depends on two technical lemmas. To state these properly we need a slight extension of the class CZO=CZO(ϵ). We say that $T \in \mathrm{CZO}_x =$

$\mathrm{CZO}_x(\epsilon)$ if it satisfies all the conditions for a CZO except for (25). That is, we only require smoothness in the first variable of the kernel K.

PROPOSITION 2.4 (Meyer). *Suppose $T \in \mathrm{CZO}_x \cap \mathrm{WBP}$ and $T1 = 0$. Then T maps $\mathcal{D}$ into L^∞ and there exists a constant c such that if the support of $\theta \in \mathcal{D}$ is contained in a ball of radius $t > 0$ then*

$$\|T(\theta)\|_{L^\infty} \leq c(\|\theta\|_{L^\infty} + t\|\nabla\theta\|_{L^\infty})\,. \tag{29}$$

This result appears in [17]. It also appears in [23, Lemma 4.1.4] in a form that is important for later applications. A trivial but crucial corollary of this last proposition is:

COROLLARY 2.5. *Suppose $\{a_Q\}_Q$ is a family of smooth atoms and T satisfies the conditions of Proposition 2.4. Then for all Q*

$$\|Ta_Q\|_{L^\infty} \leq c|Q|^{-1/2} \tag{30}$$

where c is a constant that is independent of the choice of the family of smooth atoms.

Proof. The result is immediate from conditions (9) and (11) for smooth atoms. ∎

It is well understood that under the conditions of Proposition 2.4 the function $T\theta$, $\theta \in \mathcal{D}$, can be defined pointwise. (See the discussion prior to Lemma 4.1.22 in [23].) With $T\theta(x)$ defined in this sense we obtain:

PROPOSITION 2.6 (Meyer). *Suppose $T \in \mathrm{CZO}_x \cap \mathrm{WBP}$ and $T1 = 0$. Suppose further that x and x' are distinct points in $\mathbf{R}^n$, $\Phi \in \mathcal{D}$ has support in $\{\, z : |z - x'| \leq 4|x - x'|\}$, and $\Phi(z) = 1$ when $|z - x'| \leq 2|x - x'|$. Let $\Psi = 1 - \Phi$. Then for all $\theta \in \mathcal{D}$*

$$\begin{aligned} T\theta(x) - T\theta(x') = {} & \int K(x,y)(\theta(y) - \theta(x))\Phi(y)\,dy \\ & - \int K(x',y)(\theta(y) - \theta(x))\Phi(y)\,dy \\ & + \int (K(x,y) - K(x',y))(\theta(y) - \theta(x'))\Psi(y)\,dy \\ & + (\theta(x) - \theta(x'))T\Phi(x)\,, \end{aligned}$$

where the various integrals are all absolutely convergent.

Proposition 2.4 is proved by Meyer in [17], but the reader is referred to the treatment by Torres [23, Lemma 4.1.22] for a formulation and proof that we will need in the following section.

3. Calderón–Zygmund operators

3.1. *The classical case.* The boundedness on the Besov spaces, $\dot{B}_p^{\alpha,q}$, under the conditions of the following theorem was proven by Lemarié [15].

THEOREM 3.1. *Suppose* $1 \le p,q \le \infty$ *and* $0 < \alpha < \epsilon \le 1$. *If* $T \in \mathrm{CZO}_x(\epsilon) \cap \mathrm{WBP}$ *and* $T1 = 0$ *then* T *is an almost diagonal operator on* $\dot{F}_p^{\alpha,q}$.

Proof. As indicated above it will suffice to show that $\{\langle Tb_P, a_Q\rangle\}$ is almost diagonal where $\{a_Q\}_Q$ and $\{b_Q\}_Q$ are families of smooth atoms. We also may assume that $x_P = 0$. In the proof that follows δ is a positive number that is less than $\min\{\epsilon, 2\alpha, 2(\epsilon - \alpha)\}$.

Suppose $\ell(Q) \ge \ell(P)$ and that P is far from Q: $6\sqrt{n}Q \cap 3P = \emptyset$. Then b_P and a_Q have disjoint supports. Furthermore, whenever $x \in 3Q$ and $y \in 3P$ then $2|x - x_Q| \le |x_Q - y|$. Using the fact that a_Q has a vanishing moment, the size estimates for the atoms, and the smoothness estimate for the kernel we obtain

$$
\begin{aligned}
|\langle Tb_P, a_Q\rangle| &= \Big| \int_{3Q} Tb_P(x) a_Q(x)\, dx \Big| \\
&= \Big| \int_{3Q} (Tb_P(x) - Tb_P(x_Q)) a_Q(x)\, dx \Big| \\
&= \Big| \int_{3Q} \int_{3P} (K(x,y) - K(x_Q,y)) b_P(y) \overline{a_Q(x)}\, dy\, dx \Big| \\
&\le \int_{3Q} \int_{3P} |K(x,y) - K(x_Q,y)|\, |b_P(y)|\, |a_Q(x)|\, dy\, dx \\
&\le c \int_{3Q} \int_{3P} \frac{|x - x_Q|}{|y - x_Q|^{n+\epsilon}} \frac{1}{|P|^{1/2}} \frac{1}{|Q|^{1/2}}\, dy\, dx \\
&\le c \frac{(\ell(Q))^{\epsilon}}{|x_Q|^{n+\epsilon}} |P|^{1/2} |Q|^{1/2} = c(\ell(Q))^{n/2+\epsilon} (\ell(P))^{n/2} / |x_Q|^{n+\epsilon} \\
&\le c \left(\frac{\ell(Q)}{\ell(P)} \right)^{\alpha - (n+\delta)/2} \frac{1}{(|x_Q|/\ell(Q))^{n+\epsilon}} \sim \Omega_{QP}(\delta, \alpha),
\end{aligned}
$$

provided $\delta < 2\alpha \wedge \epsilon$.

Now suppose $\ell(Q) < \ell(P)$ and P is far from Q: $6\sqrt{n}P \cap 3Q = \emptyset$. Then, as above,

$$
\begin{aligned}
|\langle Tb_P, a_Q\rangle| &\le c(\ell(Q))^{n/2+\epsilon} (\ell(P))^{n/2} / |x_Q|^{n+\epsilon} \\
&= \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2+\epsilon} \frac{1}{(|x_Q|/\ell(P))^{n+\epsilon}} \\
&\le c \left(\frac{\ell(Q)}{\ell(P)} \right)^{\alpha + (n+\delta)/2} \frac{1}{(|x_Q|/\ell(Q))^{n+\delta}} \sim \Omega_{QP}(\delta, \alpha)
\end{aligned}
$$

provided $\delta < 2(\epsilon - \alpha) \wedge \epsilon$.

Next we consider the case where $\ell(Q) \geq \ell(P)$ and P is close to Q: $6\sqrt{n}Q \cap 3P \neq \emptyset$. Then

$$|\langle Tb_P, a_Q\rangle| \leq \Big|\int_{3Q\setminus 6P} Tb_Q(x)\overline{a_Q(x)}\,dx\Big| + \Big|\int_{6P} Tb_Q(x)\overline{a_Q(x)}\,dx\Big|$$
$$= I + II\,.$$

By Corollary 2.5

$$II \leq c|P|^{-1/2}|Q|^{-1/2}|6P| \leq c(|P|/|Q|)^{1/2}\,.$$

For the first term we have

$$I = \Big|\int_{3Q\setminus 6P}\int_{3P} K(x,y)b_P(y)a_Q(x)\,dy\,dx\Big|$$
$$\leq \int_{3Q\setminus 6P}\int_{3P}\frac{1}{|x-y|^n}\frac{1}{|Q|^{1/2}}\frac{1}{|P|^{1/2}}dy\,dx \leq c\Big(\frac{|P|}{|Q|}\Big)^{1/2}\log\frac{|Q|}{|P|}\,.$$

Thus,

$$|\langle Tb_P, a_Q\rangle| \leq c(\ell(P)/\ell(Q))^{n/2}\log(\ell(P)/\ell(Q))$$
$$\leq c(\ell(P)/\ell(Q))^{\alpha-(n+\delta)/2} \sim \Omega_{QP}(\delta,\alpha)\,,$$

provided $\delta < 2\alpha$.

For the final case, $\ell(Q) < \ell(P)$ and P is close to Q: $6\sqrt{n}P \cap 3Q \neq \emptyset$. We use the fact that a_Q has moment zero:

$$\langle Tb_P, a_Q\rangle = \int_{3Q}\Big\{Tb_P(x) - |3Q|^{-1}\int_{3Q} Tb_P(z)\,dz\Big\}\overline{a_Q(x)}\,dx \tag{31}$$
$$= \int_{3Q}|3Q|^{-1}\int_{3Q}\{Tb_P(x) - Tb_P(z)\}\,dz\,\overline{a_Q(x)}\,dx\,.$$

Take a $\Phi \in \mathcal{D}$ that has support in $\{y : |y| \leq 4\}$ and $\Phi(y) = 1$ when $|y| \leq 2$. Fix a pair $x, z \in \mathbf{R}^n$, $x \neq z$ and set $\widetilde{\Phi}(y) = \Phi(|y-z|/|x-z|)$ and $\widetilde{\Psi} = 1 - \widetilde{\Phi}$. We use Proposition 2.6:

$$\begin{aligned} Tb_P(x) - Tb_P(z) = &\int K(x,y)(b_P(y) - b_P(x))\widetilde{\Phi}(y)\,dy \\ &- \int K(z,y)(b_P(y) - b_P(z))\widetilde{\Phi}(y)\,dy \\ &+ \int (K(x,y) - K(z,y))(b_P(y) - b_P(z))\widetilde{\Psi}(y)\,dy \\ &+ (b_P(x) - b_P(z))T\widetilde{\Phi}(x) \\ = &\, I + II + III + IV\,. \end{aligned}$$

Recall that $\operatorname{supp}\widetilde{\Phi} \subset \{y : |y-z| < 4|x-z|\}$. Thus if $y \in \operatorname{supp}\widetilde{\Phi}$ and $x, z \in 3Q$, it follows that

$$|x-y| \leq |x-z| + |y-z| \leq 5|x-z| \leq 15\sqrt{n}\ell(Q)\,.$$

We use the smoothness estimate for atoms:

$$|I| \leq c \int\limits_{|x-y|\leq 15\sqrt{n}\ell(Q)} \frac{1}{|x-y|^n} \frac{|x-y|}{|P|^{1/n+1/2}}\,dy$$

$$= c\frac{1}{|P|^{1/n+1/2}} \int\limits_{|x-y|\leq 15\sqrt{n}\ell(Q)} \frac{1}{|x-y|^{n-1}}\,dy \leq c\frac{\ell(Q)}{\ell(P)}\frac{1}{|P|^{1/2}}\,.$$

By an analogous argument

$$|II| \leq c\frac{\ell(Q)}{\ell(P)}\frac{1}{|P|^{1/2}}\,.$$

We use the smoothness estimate for atoms and Proposition 2.4 to obtain

$$|IV| \leq c\frac{|x-z|}{|P|^{1/n+1/2}} \leq c\frac{\ell(Q)}{\ell(P)}\frac{1}{|P|^{1/2}}\,.$$

We turn to the estimate of III. We may assume, without loss of generality, that $0 < \epsilon < 1$. Notice that $\operatorname{supp}\widetilde{\Psi} \subset \{\, y : |y-z| > 2|x-z|\}$. We break the integral into two parts. Set

$$A_1 = \{\, y : 2|x-z| \leq |y-z| \leq 6\sqrt{n}\ell(P)\}\,,$$
$$A_2 = \{\, y : |y-z| > 6\sqrt{n}\ell(P)\}\,.$$

The integral over A_2 is dominated by

$$c\int\limits_{A_2} \frac{|x-z|^\epsilon}{|y-z|^{n+\epsilon}}\frac{1}{|P|^{1/2}}\,dy < c\left(\frac{\ell(Q)}{\ell(P)}\right)^\epsilon \frac{1}{|P|^{1/2}}\,.$$

The integral over A_1 is dominated by

$$c\int\limits_{A_1} \frac{|x-z|^\epsilon}{|y-z|^{n+\epsilon}}\frac{|y-z|}{|P|^{1/2+1/n}}\,dy = c\frac{|x-z|^\epsilon}{|P|^{1/2+1/n}} \int\limits_{|y-z|\leq 6\sqrt{n}\ell(P)} \frac{1}{|y-z|^{n+\epsilon-1}}\,dy$$

$$\leq c(\ell(Q))^\epsilon(\ell(P))^{1-\epsilon}\frac{1}{|P|^{1/2+1/n}} = c\left(\frac{\ell(Q)}{\ell(P)}\right)^\epsilon \frac{1}{|P|^{1/2}}\,.$$

Assembling the estimates we see that if $x, z \in 3Q$ then

$$|Tb_P(x) - Tb_P(z)| \leq c\left(\frac{\ell(Q)}{\ell(P)}\right)^\epsilon \frac{1}{|P|^{1/2}}\,.$$

Put this into (31) to get

$$|\langle Tb_P, a_Q\rangle| \leq c|3Q|\frac{1}{|Q|^{1/2}}\left(\frac{\ell(Q)}{\ell(P)}\right)^\epsilon \frac{1}{|P|^{1/2}}$$

$$= c\left(\frac{\ell(Q)}{\ell(P)}\right)^{n/2+\epsilon} \leq c\left(\frac{\ell(Q)}{\ell(P)}\right)^{\alpha+(n+\delta)/2} \sim \Omega_{QP}(\delta,\alpha)\,,$$

provided $\delta < 2(\epsilon - \alpha)$. This completes the proof. ∎

Remark. We note that when $\alpha \neq 0$ almost diagonal operators do not map smooth atoms into smooth molecules. This was observed by Torres [23] in the discussion that follows Theorem 4.2.32.

Remark. As we observed earlier, almost diagonal operators preserve the Besov spaces and so our proof also establishes the result of Lemarié referred to before the statement of Theorem 3.1.

3.2. *Meyer's version.* Meyer [18] considered the following variant of the class of Calderón–Zygmund operators. As a substitute for the size condition (24) we now require a pair of conditions:

$$\int_{r\le|x-y|\le 2r} |K(x,y)|\,dy \le c \qquad \text{for all } x \in \mathbf{R}^n, \tag{32}$$

$$\int_{r\le|x-y|\le 2r} |K(x,y)|\,dx \le c \qquad \text{for all } y \in \mathbf{R}^n, \tag{33}$$

for all $r > 0$. In place of conditions (25) and (26) we assume that there is a sequence of nonnegative numbers, $\{\epsilon(k)\}$, $k = 1, 2, \ldots$, such that

$$\int_{2^k r\le|x-y|\le 2^{k+1} r} |K(x+u,y+v) - K(x,y)|\,dy \le \epsilon(k) \tag{34}$$

for all $x \in \mathbf{R}^n$ and $k \ge 1$, and

$$\int_{2^k r\le|x-y|\le 2^{k+1} r} |K(x+u,y+v) - K(x,y)|\,dx \le \epsilon(k) \tag{35}$$

for $y \in \mathbf{R}^n$, $k \ge 1$, and $|u| + |v| \le r$ and for all $r > 0$.

Remark. Central to our approach is the need that the conclusions of Propositions 2.4 and 2.6 hold for the operators we are considering. So let us suppose that T is an operator with a kernel that satisfies the four conditions (32)–(35). In order to be able to make sense of the condition $T1 = 0$ we need the integrability of $T^*\theta$ away from the support of $\theta \in \mathcal{D}_0$, as in the opening pages of [9]. The argument there is easily adjusted and this conclusion follows from (33) provided $\sum_k \epsilon(k) < \infty$. If we follow the proofs of Torres [23, Lemmas 4.1.4 and 4.1.22], we see that the conclusions of Propositions 2.4 and 2.6 do hold for operators satisfying Meyer's conditions providing only that $\{\epsilon(k)\}$ is summable.

The following proposition was proved in [18]:

PROPOSITION 3.2. *Suppose T is an operator with a kernel that is continuous off the diagonal in $\mathbf{R}^n \times \mathbf{R}^n$ and satisfies* (32)–(35), *and that* $T \in$ WBP,

$T1 = 0$, $T^*1 = 0$. *If*

$$\sum_{k=1}^{\infty} k\epsilon(k) < \infty \tag{36}$$

then T is bounded on $\dot{F}_1^{0,1} = \dot{B}_1^{0,1}$ and on its dual $\dot{F}_\infty^{0,\infty} = \dot{B}_\infty^{0,\infty}$.

Our next theorem shows that by requiring slightly more we get boundedness on all Triebel–Lizorkin spaces $\dot{F}_p^{0,q}$, $1 \le p, q \le \infty$.

THEOREM 3.3. *Suppose T is an operator with a kernel that is continuous off the diagonal in $\mathbf{R}^n \times \mathbf{R}^n$ and satisfies* (32)–(35), *and that $T \in$* WBP, $T1 = 0$, $T^*1 = 0$. *If*

$$\sum_{k=1}^{\infty} k^2\epsilon(k) < \infty \tag{37}$$

then T is bounded on $\dot{F}_p^{0,q}$ for all p and q, $1 \le p, q \le \infty$.

Proof. It will suffice to show that the matrix $\{\langle T\varphi_P, \varphi_Q\rangle\}_{QP}$ is bounded on the sequence spaces $\dot{f}_p^{0,q}$. Recall that an almost diagonal matrix is bounded on these spaces. If we examine the proof of Lemma 2.2 we see that $\{\langle T\varphi_P, \varphi_Q\rangle\}_{QP}$ is bounded on $\dot{f}_p^{0,q}$ if and only if $\{\langle Tb_P, a_Q\rangle\}_{QP}$ is bounded on $\dot{f}_p^{0,q}$ whenever $\{b_Q\}_Q$ and $\{a_Q\}_Q$ are families of smooth atoms.

In Corollary 10.3 of [12] Frazier and Jawerth give necessary and sufficient conditions for a matrix $A = \{A_{QP}\}$ to be bounded on all $\dot{f}_p^{0,q}$, $1 \le p, q \le \infty$; namely, A must be bounded on the "four corners": $\dot{f}_1^{0,1}$, $\dot{f}_\infty^{0,\infty}$, $\dot{f}_1^{0,\infty}$, and $\dot{f}_\infty^{0,1}$. This much is trivial, but more importantly they quantify these conditions in (10.2)–(10.5):

(38) A is bounded on $\dot{f}_1^{0,1}$ iff $\sup_P \sum_Q |A_{QP}|(|P|/|Q|)^{1/2} < \infty$,

(39) A is bounded on $\dot{f}_\infty^{0,\infty}$ iff $\sup_Q \sum_P |A_{QP}|(|Q|/|P|)^{1/2} < \infty$,

(40) A is bounded on $\dot{f}_1^{0,\infty}$ iff

$$\sup_{P_0} |P_0|^{-1} \left\| \left\{ \sum_{P \subset P_0} A_{QP}|P|^{1/2} \right\}_Q \right\|_{\dot{f}_1^{0,\infty}} < \infty,$$

(41) A is bounded on $\dot{f}_\infty^{0,1}$ iff

$$\sup_{Q_0} |Q_0|^{-1} \left\| \left\{ \sum_{Q \subset Q_0} A_{QP}|Q|^{1/2} \right\}_P \right\|_{\dot{f}_1^{0,\infty}} < \infty.$$

From Meyer's result (Proposition 3.2) we know that A is bounded on $\dot{f}_1^{0,1}$ and $\dot{f}_\infty^{0,\infty}$ so we only need to check (40) and (41). By the symmetry of these conditions and the symmetry of the assumptions on T and T^* it will suffice to show that (40) is satisfied. That is, it will suffice to show that there is a constant c such that

$$\sup_{P_0} |P_0|^{-1} \Big\| \Big\{ \sum_{P\subset P_0} A_{QP}|P|^{1/2} \Big\}_Q \Big\|_{\dot f_1^{0,\infty}} \le c \tag{42}$$

for all dyadic cubes P_0.

The next reduction follows the argument of the first part of the proof of Theorem 10.3 in [12]. To prove (42) we split the dyadic cubes into disjoint families: $\mathcal{A} = \{ Q : Q \subset 25\sqrt{n}P_0\}$ and $\mathcal{B} = \{ Q : Q \not\subset 25\sqrt{n}P_0\}$.

For $\mathcal{A}$ the estimate follows from (39):

$$\begin{aligned}
&|P_0|^{-1} \Big\| \Big\{ \sum_{P\subset P_0} |\langle Tb_P, a_Q\rangle|\,|P|^{1/2} \Big\}_{Q\in\mathcal{A}} \Big\|_{\dot f_1^{0,\infty}} \\
&= |P_0|^{-1} \int_{25\sqrt{n}P_0} \sup_Q \Big(\sum_{P\subset P_0} |\langle Tb_P, a_Q\rangle|\,|P|^{1/2}\tilde\chi_Q(x)\Big)\,dx \\
&\le c \sup_Q \sum_{P\subset P_0} |\langle Tb_P, a_Q\rangle| (|P|/|Q|)^{1/2} \\
&\le c \sup_Q \sum_P \langle Tb_P, a_Q\rangle| (|P|/|Q|)^{1/2} .
\end{aligned}$$

For $\mathcal{B}$ we use the embedding $\dot f_1^{0,1} \hookrightarrow \dot f_1^{0,\infty}$:

$$\begin{aligned}
&|P_0|^{-1} \Big\| \Big\{ \sum_{P\subset P_0} |\langle Tb_P, a_Q\rangle|\,|P|^{1/2} \Big\}_{Q\in\mathcal{B}} \Big\|_{\dot f_1^{0,\infty}} \\
&\le |P_0|^{-1} \Big\| \Big\{ \sum_{P\subset P_0} |\langle Tb_P, a_Q\rangle|\,|P|^{1/2} \Big\}_{Q\in\mathcal{B}} \Big\|_{\dot f_1^{0,1}} \\
&= |P_0|^{-1} \sum_{P\subset P_0} \sum_{Q\not\subset 25\sqrt{n}P_0} |\langle Tb_P, a_Q\rangle|\,|P|^{1/2}|Q|^{1/2} .
\end{aligned}$$

To complete the proof we show that there is a constant c such that

$$\sum_{P\subset P_0} \sum_{Q\not\subset 25\sqrt{n}P_0} |\langle Tb_P, a_Q\rangle|\,|P|^{1/2}|Q|^{1/2} \le c|P_0| \tag{43}$$

for all dyadic cubes P_0, where c is independent of P_0 and the choice of the families of smooth atoms.

In order to establish (43) we start by fixing a cube P_0, $\ell(P_0) = 2^{-\mu_0}$. By translation invariance we may assume that $x_{P_0} = 0$. For each fixed $P \subset P_0$, $\ell(P) = 2^{-\mu}$, we divide $\mathcal{B}$ into three mutually disjoint families:

$$\begin{aligned}
\mathcal{Q}_1 &= \{ Q \not\subset 25\sqrt{n}P_0 : \ell(P) \le \ell(Q) \le \ell(P_0)\} , \\
\mathcal{Q}_2 &= \{ Q \not\subset 25\sqrt{n}P_0 : \ell(Q) < \ell(P) \le \ell(P_0)\} , \\
\mathcal{Q}_3 &= \{ Q \not\subset 25\sqrt{n}P_0 : \ell(P) \le \ell(P_0) < \ell(Q)\} .
\end{aligned}$$

For each $Q \in \mathcal{Q}_j$ we set $\ell(Q) = 2^{-\nu}$. The strategy in each case is to fix P_0 and $P \subset P_0$ and then to estimate

$$\sum_{Q \in \mathcal{Q}_j} |\langle Tb_P, a_Q \rangle| \, |P|^{1/2} |Q|^{1/2} .$$

Once we have that estimate, we sum over $P \subset P_0$, and notice that when one sums over such P's of a fixed length, the sum of the measures of the P's is the measure of P_0.

We start with $\mathcal{Q}_1$. Fix P_0 and $P \subset P_0$. Observe that $\mu_0 \le \nu \le \mu$. Since $|a_P| \le |Q|^{-1/2}$ and $\operatorname{supp} a_Q \subset 3Q$,

$$|\langle Tb_P, a_Q \rangle| \le \int |Tb_P(x)| \, |a_Q(x)| \, dx \le |Q|^{-1/2} \int_{3Q} |Tb_P(x)| \, dx .$$

Sum over $Q \in \mathcal{Q}_1$ of fixed length, and observe that since $\ell(Q) \le \ell(P_0)$ and $Q \not\subset 25\sqrt{n}P_0$ it follows that $3Q \cap 19\sqrt{n}P_0 = \emptyset$. This yields

$$\sum_{\substack{Q \in \mathcal{Q}_1 \\ \ell(Q) = 2^{-\nu}}} |\langle Tb_P, a_Q \rangle| \le 3^n |Q|^{-1/2} \int_{\mathbf{R}^n \setminus 19\sqrt{n}P_0} |Tb_P(x)| \, dx .$$

There are $\mu - \mu_0 + 1$ admissible values of ν so

$$\sum_{Q \in \mathcal{Q}_1} |\langle Tb_P, a_Q \rangle| \, |P|^{1/2} |Q|^{1/2} \le c(\mu - \mu_0 + 1) |P|^{1/2} \int_{\mathbf{R}^n \setminus 19\sqrt{n}P_0} |Tb_P(x)| \, dx . \tag{44}$$

Use $\int b_P(y) \, dy = 0$. Then for $x \notin 3P$,

$$\begin{aligned} |Tb_P(x)| &= \left| \int_{y \in 3P} \left\{ K(x,y) - |3P|^{-1} \int_{z \in 3P} K(x,z) \, dz \right\} b_P(y) \, dy \right| \\ &= |3P|^{-1} \left| \int_{y \in 3P} \int_{u \in 3P - 3P} (K(x,y) - K(x,y+u)) b_P(y) \, du \, dy \right| \\ &= |3P|^{-1} |P|^{-1/2} \int_{y \in 3P} \int_{|u| \le 3\sqrt{n}\ell(P)} |K(x,y) - K(x,y+u)| \, du \, dy . \end{aligned} \tag{45}$$

Use (45) in the right hand side of (44), and then use (35) to get the bound

$$c \frac{\mu - \mu_0 + 1}{|P|} \int_{3P} \int_{|u| \le 3\sqrt{n}\ell(P)} \int_{\mathbf{R}^n \setminus 19\sqrt{n}P_0} |K(x,y) - K(x,y+u)| \, dx \, du \, dy .$$

For any $y \in 3P$,

$$\mathbf{R}^n \setminus 19\sqrt{n}\ell(P) \subset \bigcup_{k = \mu - \mu_0 + 1}^{\infty} \{ x : 2^k 3\sqrt{n}\ell(P) \le |x - y| \le 2^{k+1} 3\sqrt{n}\ell(P) \} .$$

So for $y \in 3P$, $|u| \le 3\sqrt{n}\ell(P)$,

$$\int_{\mathbf{R}^n \setminus 19\sqrt{n}P_0} |K(x,y) - K(x,y+u)|\, dx \le \sum_{k=\mu-\mu_0+1}^{\infty} \epsilon(k)\,.$$

Thus,

$$\sum_{Q \in \mathcal{Q}_1} |\langle Tb_P, a_Q\rangle|\, |P|^{1/2}|Q|^{1/2} \le c\frac{\mu-\mu_0+1}{|P|}|3P||P| \sum_{k=\mu-\mu_0+1}^{\infty} \epsilon(k)$$

$$= c(\mu-\mu_0+1)|P| \sum_{k=\mu-\mu_0+1}^{\infty} \epsilon(k)\,.$$

Sum over $P \subset P_0$ of fixed length, $\ell(P) = 2^{-\mu}$, and then over $\mu \ge \mu_0$, to get

(46) $$\sum_{P \subset P_0} \sum_{Q \in \mathcal{Q}_1} |\langle Tb_P, a_Q\rangle|\, |P|^{1/2}|Q|^{1/2}$$

$$\le c|P_0| \sum_{\mu=\mu_0}^{\infty} (\mu-\mu_0+1) \sum_{k=\mu-\mu_0+1}^{\infty} \epsilon(k) = c|P_0| \sum_{k=1}^{\infty} \binom{k+1}{2} \epsilon(k)\,.$$

We now consider $\mathcal{Q}_2$, $\ell(Q) < \ell(P) \le \ell(P_0)$, $\mu_0 \le \mu < \nu$. Use $\int a_Q(x)\,dx = 0$, estimates on the size of a_Q and b_P and the fact that $3P \cap 3Q = \emptyset$. Then

$$\begin{aligned}
&|\langle Tb_P, a_Q\rangle| \\
&= \Big| \int_{3Q} \Big\{Tb_P(x) - |3Q|^{-1} \int_{3Q} Tb_P(z)\,dz\Big\} \overline{a_Q(x)}\,dx \Big| \\
&= \Big| \int_{3Q} |3Q|^{-1} \int_{3Q} (Tb_P(x) - Tb_P(z))\,dz\, \overline{a_Q(x)}\,dx \Big| \\
&\le |3Q|^{-1}|Q|^{-1/2} \int_{3Q} \int_{3Q} |Tb_P(x) - Tb_P(z)|\,dz\,dx \\
&\le |3Q|^{-1}|Q|^{-1/2}|P|^{-1/2} \int_{3Q} \int_{3Q} \int_{3P} |K(x,y) - K(z,y)|\,dy\,dz\,dx \\
&\le |3Q|^{-1}|Q|^{-1/2}|P|^{-1/2} \int_{3Q} \int_{|u| \le 3\sqrt{n}\ell(P)} \int_{3P} |K(x,y) - K(x+u,y)|\,dy\,du\,dx\,.
\end{aligned}$$

Collect cubes Q of fixed length $2^{-\nu}$:

$$\sum_{\substack{Q \in \mathcal{Q}_2 \\ \ell(Q) = 2^{-\nu}}} |\langle Tb_P, a_Q\rangle|\, |P|^{1/2}|Q|^{1/2}$$

$$\le 3^n|3Q|^{-1} \int_{3P} \int_{|u| \le 3\sqrt{n}\ell(Q)} \int_{\mathbf{R}^n \setminus 19\sqrt{n}P_0} |K(x,y) - K(x+u,y)|\,dx\,du\,dy\,.$$

Dominate the integral on $\mathbf{R}^n \setminus 19\sqrt{n}P_0$ by the integral over the shells

$$\{x : 2^k 3\sqrt{n}\ell(Q) \le |x-y| \le 2^{k+1}3\sqrt{n}\ell(Q)\}, \qquad k \ge \nu - \mu_0 + 1,$$

and use (35) to obtain

$$\sum_{\substack{Q\in\mathcal{Q}_2\\ \ell(Q)=2^{-\nu}}} |\langle Tb_P, a_Q\rangle|\,|P|^{1/2}|Q|^{1/2} \le c|P| \sum_{k=\nu-\mu_0+1}^{\infty} \epsilon(k),$$

$$\sum_{Q\in\mathcal{Q}_2} |\langle Tb_P, a_Q\rangle|\,|P|^{1/2}|Q|^{1/2} \le c|P| \sum_{\nu\ge\mu} \sum_{k=\nu-\mu_0+1}^{\infty} \epsilon(k)$$
$$= c|P| \sum_{k=\mu-\mu_0+1} (k+1-(\mu-\mu_0+1))\epsilon(k).$$

Now sum over all $P \subset P_0$ of fixed length $\ell(P) = 2^{-\mu}$ and then over $\mu \ge \mu_0$ to get

$$\sum_{P\subset P_0} \sum_{Q\in\mathcal{Q}_2} |\langle Tb_P, a_Q\rangle|\,|P|^{1/2}|Q|^{1/2} \tag{47}$$
$$\le c|P_0| \sum_{\mu\ge\mu_0} \sum_{k=\mu-\mu_0+1}^{\infty} (k+1-(\mu-\mu_0+1))\epsilon(k)$$
$$= c|P_0| \sum_{k=1}^{\infty} \binom{k+1}{2} \epsilon(k).$$

Finally, we consider $\mathcal{Q}_3$. We have $\ell(P) \le \ell(P_0) < \ell(Q)$, $\nu < \mu_0 \le \mu$. Recall the remark at the beginning of this subsection. We see that under the conditions of this theorem Tb_P is integrable. From this and the fact that $T^*1 = 0$ it is not hard to see that $\int Tb_P(x)\,dx = 0$. Use (9) and (11) to obtain

$$|\langle Tb_P, a_Q\rangle| \le \int |Tb_P(x)|\,|a_Q(x)|\,dx \le |Q|^{-1/2} \int_{3Q} |Tb_P(x)|\,dx.$$

Use (9), (11), and $\int Tb_P(x)\,dx = 0$ to get

$$|\langle Tb_P, a_Q\rangle| \le \int |Tb_P(x)|\,|a_Q(x) - a_Q(x_P)|\,dx \tag{48}$$
$$\le |Q|^{-1/2} \int_{3Q} \frac{|x-x_P|}{\ell(Q)}\,dx$$

provided $x_P \notin 3Q$ (in which case $a_Q(x_P) = 0$). If $x_P \in 3Q$ then

$$|\langle Tb_P, a_Q\rangle| \le |Q|^{-1/2} \int_{\mathbf{R}^n} \frac{|x-x_P|}{\ell(Q)}\,dx. \tag{49}$$

Sum over $Q \in \mathcal{Q}_3$ of fixed length $\ell(Q) = 2^{-\nu}$. For at most 3^n of these cubes (49) holds and for all the others we have (48). Since the cubes of fixed length are pairwise disjoint we have

$$\sum_{\substack{Q\in\mathcal{Q}_3\\ \ell(Q)=2^{-\nu}}} |\langle Tb_P, a_Q\rangle|\,|P|^{1/2}|Q|^{1/2}$$

$$\leq 2\cdot 3^n|P|^{1/2}\int_{\mathbf{R}^n}|Tb_P(x)|(1\wedge 2^\nu|x-x_P|)\,dx\,.$$

Thus

$$\sum_{Q\in\mathcal{Q}_3} |\langle Tb_P, a_Q\rangle|\,|P|^{1/2}|Q|^{1/2} \leq c\sum_{\nu<\mu_0}|P|^{1/2}\int_{\mathbf{R}^n}|Tb_P(x)|(1\wedge 2^\nu|x-x_P|)\,dx\,.$$

We will split the integral into two terms and use Lemma 2.5:

$$\sum_{\nu<\mu_0}|P|^{1/2}\int_{19\sqrt{n}P}|Tb_P(x)|(1\wedge 2^\nu|x-x_P|)\,dx$$

$$\leq c\sum_{\nu<\mu_0}|P|^{1/2}|19\sqrt{n}P|\,|P|^{-1/2}2^\nu\ell(P)$$

$$\leq c|P|\ell(P)\sum_{\nu<\mu_0}2^\nu = c|P|\ell(P)2^{\mu_0}\,.$$

Sum over $P\subset P_0$ of fixed length $\ell(P)=2^{-\mu}$ and then over $\mu\geq\mu_0$ to get the bound

$$c|P_0|\sum_{\mu\geq\mu_0}2^{\mu_0-\mu} = 2c|P_0|\,. \tag{50}$$

Now we use (45):

$$\sum_{\nu<\mu_0}|P|^{1/2}\int_{\mathbf{R}^n\setminus 19\sqrt{n}P}|Tb_P(x)|(1\wedge 2^\nu|x-x_P|)\,dx$$

$$\leq \sum_{\nu<\mu_0}|P|^{1/2}\int_{\mathbf{R}^n\setminus 19\sqrt{n}P}\int_{y\in 3P}\int_{|u|\leq 3\sqrt{n}\ell(P)}|K(x,y)-K(x,y+u)|$$

$$\times\,(1\wedge 2^k\ell(P)2^\nu)\,du\,dy\,dx$$

$$\leq c|3P|^{-1}\sum_{\nu<\mu_0}\int_{y\in 3P}\int_{|u|\leq 3\sqrt{n}\ell(P)}\sum_{k=1}^{\infty}\int_{2^k 3\sqrt{n}\ell(P)\leq|x-x_P-y|\leq 2^{k+1}3\sqrt{n}\ell(P)}$$

$$|K(x,y)-K(x,y+u)|(1\wedge 2^k\ell(P)2^\nu)\,du\,dy\,dx$$

$$\leq c|P|\sum_{k=1}^{\infty}\sum_{\nu<\mu_0}(1\wedge 2^k\ell(P)2^\nu)\epsilon(k)\,.$$

Sum over $P \subset P_0$ with $\ell(P) = 2^{-\mu}$ fixed and then over $\mu \geq \mu_0$ to get the bound

$$
\begin{aligned}
(51)\qquad c|P_0|\sum_{k=1}^{\infty}\Big(\sum_{\mu\geq\mu_0}\sum_{\nu<\mu_0}(1\wedge 2^k\ell(P)2^\nu)\Big)\epsilon(k)& \\
&= c|P_0|\sum_{k=1}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{\infty}(1\wedge 2^{k-(s+t)})\epsilon(k) \\
&= c|P_0|\sum_{k=1}^{\infty}\left(\binom{k+1}{2}+(k+2)\right)\epsilon(k).
\end{aligned}
$$

Combine (46), (47), (50), and (51). We obtain

$$\sum_{P\subset P_0}\sum_{Q\not\subset 25\sqrt{n}P_0}|\langle Tb_P, a_Q\rangle|\,|P|^{1/2}|Q|^{1/2} \leq c\Big(1+\sum_{k=1}^{\infty}k^2\epsilon(k)\Big)|P_0|.$$

This completes the proof of Theorem 3.3. ∎

Remark. The conditions in the last theorem can be somewhat relaxed. For example, instead of (34) we may assume

$$\int_{2^k r\leq|x-y|}|K(x+u,y+v)-K(x,y)|\,dy \leq \tilde{\epsilon}(k), \qquad x\in\mathbf{R}^n,\ k\geq 1,\ r>0,$$

provided $|u|+|v|\leq r$, and make a similar change to (35). It is now enough to require that $\sum_k k\tilde{\epsilon}(k) < \infty$.This can be relaxed even further by carefully checking the proof.

3.3. *Another version.* We turn next to a version of a slightly different nature. In this subsection we will be considering operators T that have a kernel that satisfies (32) and (33) but in place of (34) and (35), with conditions on the sequence $\{\epsilon(k)\}$, satisfies:

$$(52)\qquad \int_{|x-y|\geq 2|v|}|K(x,y)-K(x,y+v)|\,dx \leq c, \qquad v,y\in\mathbf{R}^n,$$

$$(53)\qquad \int_{|x-y|\geq 2|u|}|K(x,y)-K(x+u,y)|\,dx \leq c, \qquad u,x\in\mathbf{R}^n.$$

These conditions are satisfied if the kernel satisfies (34) and (35) with $\{\epsilon(k)\}$ summable, which is certainly true if (36) or (37) holds. These conditions are the analogues for a general kernel of the conditions proposed by Hörmander [14] for convolution kernels.

To motivate our next theorem we note that a convolution operator is bounded from $\dot{B}_1^{0,1}$ to $\dot{B}_1^{0,1}$ if and only if it is bounded from $\dot{B}_1^{0,1}$ to $\dot{B}_1^{0,\infty}$. (We return to this circle of ideas at the end of proof of our theorem.) Of course, for a general kernel this equivalence does not hold and we need to study each of these cases separately. Meyer's result, Proposition 3.2, deals with the first case. The second case is the subject of the following theorem.

PROPOSITION 3.4. *Suppose T is an operator with a kernel that is continuous off the diagonal in $\mathbf{R}^n \times \mathbf{R}^n$ and satisfies* (32), (33), (52), *and* (53), *and that $T \in$ WBP, $T1 = 0$. Then T is bounded from $\dot{B}_1^{0,1}$ to $\dot{B}_1^{0,\infty}$.*

Proof. The proof is easy. It will suffice to show that the matrix $\{\langle Tb_P, a_Q\rangle\}_{QP}$ is bounded from $\dot{b}_1^{0,1}$ to $\dot{b}_1^{0,\infty}$, where $\{a_Q\}$ and $\{b_Q\}$ are families of smooth atoms. It is easy to check that this will follow if

$$(54)\qquad \sup_P \sup_\nu \sum_{Q:\ell(Q)=2^{-\nu}} |\langle Tb_P, a_Q\rangle| (|Q|/|P|)^{1/2} \leq \infty\,.$$

That is, given a dyadic cube P and $\nu \in \mathbf{Z}$,

$$(55)\qquad \sum_{Q:\ell(Q)=2^{-\nu}} |\langle Tb_P, a_Q\rangle| \leq c|P|^{1/2} 2^{n\nu/2}\,,$$

where c is independent of P and ν. The crucial fact for this proof is that by an extension of the remarks at the beginning of the previous subsection the conclusion of Proposition 2.5 and its corollary hold for operators satisfying the conditions of this theorem. In particular,

$$(56)\qquad |Tb_P(x)| \leq c|P|^{-1/2} \qquad \text{for all } x \in \mathbf{R}^n,$$

where $\{b_Q\}$ is a family of smooth atoms, and c is independent of the choice of the family. Use the size and support condition for a_Q to get

$$|\langle Tb_P, a_Q\rangle| \leq |Q|^{-1/2} \int_{3Q} |Tb_P(x)|\,dx = 2^{n\nu/2} \int_{3Q} |Tb_P(x)|\,dx\,.$$

Consequently,

$$\sum_{Q:\ell(Q)=2^{-\nu}} |\langle Tb_P, a_Q\rangle| \leq 3^n 2^{n\nu/2} \int_{\mathbf{R}^n} |Tb_P(x)|\,dx\,.$$

Thus, to establish (55) it will suffice to show that

$$(57)\qquad \int_{\mathbf{R}^n} |Tb_P(x)|\,dx \leq c|P|^{1/2}$$

We may assume that $x_P = 0$. Use (56):

$$(58)\qquad \int_{9\sqrt{n}\ell(P)} |Tb_P(x)|\,dx \leq c|P|\,|P|^{-1/2} = c|P|^{1/2}\,.$$

Now use $\int b_P(x)\,dx = 0$ and the size condition on b_P. If $|x| > 9\sqrt{n}\ell(P)$

then, as in the estimate (45), we have

$$|Tb_P(x)| \le c|P|^{-3/2} \int\limits_{3P} \int\limits_{|u|\le 3\sqrt{n}\ell(P)} |K(x,y) - K(x,y+u)|\, du\, dy\,.$$

Use this estimate, Fubini's theorem, and (52) to obtain

$$\text{(59)} \qquad \int\limits_{|x|\ge 9\sqrt{n}\ell(P)} |Tb_P(x)|\, dx$$

$$\le c|P|^{-3/2} \int\limits_{|x|\ge 9\sqrt{n}\ell(P)} \int\limits_{3P} \int\limits_{|u|\le 3\sqrt{n}\ell(P)} |K(x,y) - K(x,y+u)|\, du\, dy\, dx$$

$$\le c|P|^{-3/2} \int\limits_{3P} \int\limits_{|u|\le 3\sqrt{n}\ell(P)} \int\limits_{|x|\ge 9\sqrt{n}\ell(P)} |K(x,y) - K(x,y+u)|\, dx\, du\, dy$$

$$\le c|P|^{-3/2}|P|^2 = c|P|^{1/2}\,.$$

From (58) and (59) we get (57), and this completes the proof. ■

COROLLARY 3.5. *Suppose T is an operator with a kernel that is continuous off the diagonal in $\mathbf{R}^n \times \mathbf{R}^n$ and satisfies* (32), (33), (52), *and* (53), *and that $T \in$ WBP, $T1 = 0$, $T^*1 = 0$. Then T is bounded from $\dot{B}_2^{0,1}$ to $\dot{B}_2^{0,\infty}$.*

Proof. From the symmetry of the conditions on T and T^* we see that T^* also satisfies (54). This can be reformulated as

$$\text{(60)} \qquad \sup_Q \sup_\mu \sum_{P:\ell(P)=2^{-\mu}} |\langle Tb_P, a_Q\rangle| (|P|/|Q|)^{1/2} \le \infty\,.$$

It is easy to see that this implies that the matrix $\{\langle Tb_P, a_Q\rangle\}$ is bounded from $\dot{b}_\infty^{0,1}$ to $\dot{b}_\infty^{0,\infty}$, which implies that T is bounded from $\dot{B}_\infty^{0,1}$ to $\dot{B}_\infty^{0,\infty}$. By interpolation (see, for example, [19, Chap. 5, Thm. 6(i)]) it follows that T is bounded from $\dot{B}_2^{0,1}$ to $\dot{B}_2^{0,\infty}$. ■

Remark. In the particular case of a convolution operator the corollary guarantees the L^2-boundedness of the operator. From the results of Hörmander [14] for the classical singular integral operator theory we then get the boundedness of the operator on all L^p-spaces, $1 < p < \infty$. In this way the classical result of Hörmander for convolution operators can be incorporated into the theory for more general kernels.

For the sake of completeness we finish this section with the proof of our claim.

PROPOSITION 3.6. *Suppose that T is a convolution operator that maps $\dot{B}_2^{0,1}$ continuously into $\dot{B}_2^{0,\infty}$. Then T is bounded on L^2.*

Proof. Let φ and ψ be functions that satisfy (1)–(4) where $\widehat{\psi} = 1$ on $\operatorname{supp}\widehat{\varphi}$. For $g \in \mathcal{S}'$

$$\|\varphi_\nu * g\|_{\dot{B}_2^{0,1}} \sim \sum_{\mu=\nu-1}^{\nu+1} \|\varphi_\mu * \varphi_\nu * g\|_{L^2} \le \sum_{\mu=\nu-1}^{\nu+1} \|\varphi\|_{L^1}\|\varphi_\nu * g\|_{L^2}$$
$$= 3\|\varphi\|_{L^1}\|\varphi_\nu * g\|_{L^2} = 3\|\varphi\|_{L^1}\|\psi_\nu * \varphi * g\|_{L^2}$$
$$\le 3\|\varphi\|_{L^1}\sup_\nu \|\psi_\nu * \varphi * g\|_{L^2} \sim 3\|\varphi\|_{L^1}\|\varphi_\nu * g\|_{\dot{B}_2^{0,\infty}} .$$

But we also have the continuous embedding $\dot{B}_2^{0,1} \hookrightarrow \dot{B}_2^{0,\infty}$ and so $\|\cdot\|_{\dot{B}_2^{0,q_1}} \sim \|\cdot\|_{\dot{B}_2^{0,q_2}}$, $1 \le q_1, q_2 \le \infty$. Since $\dot{B}_2^{0,2} = L^2$,

$$\|Tf\|_{L^2} \sim \Big(\sum_\nu (\|\varphi_\nu * Tf\|_{L^2})^2\Big)^{1/2} \le c\Big(\sum_\nu (\|\varphi_\nu * Tf\|_{\dot{B}_2^{0,\infty}})^2\Big)^{1/2}$$
$$= c\Big(\sum_\nu (\|T(\varphi_\nu * f)\|_{\dot{B}_2^{0,\infty}})^2\Big)^{1/2} \le c\Big(\sum_\nu (\|\varphi_\nu * f\|_{\dot{B}_2^{0,\infty}})^2\Big)^{1/2}$$
$$\le c\Big(\sum_\nu (\|\varphi_\nu * f\|_{L^2})^2\Big)^{1/2} \sim c\|f\|_{L^2} . \ \blacksquare$$

This result is a simple consequence of the Littlewood–Paley characterization of the Triebel–Lizorkin and Besov–Lipschitz spaces by means of equations (5)–(7), together with the fact that the characterization does not depend on the choice of the function φ.

4. ϵ-Families of operators. Recently, Christ and Journé [2] have studied families of operators which capture enough of the structure of Calderón–Zygmund operators to make it possible to study them with techniques used for the study of Calderón–Zygmund operators. We will consider such families in this section.

Throughout this section we fix parameters p, q, α, and ϵ with $1 \le p, q \le \infty$ and $|\alpha| < \epsilon \le 1$. We say that $D = \{D_\nu\}_{\nu\in\mathbf{Z}}$ is an *ϵ-family of operators* and write $D \in \mathrm{F}_\epsilon$ if each operator D_ν is given by a continuous kernel $D_\nu(x,y)$ that satisfies the following size and smoothness conditions:

$$(61) \qquad |D_\nu(x,y)| \le c2^{\nu n}\omega_\epsilon(2^\nu(x-y)), \qquad x, y \in \mathbf{R}^n,$$

$$(62) \qquad |D_\nu(x,y) - D_\nu(x,y')| \le c2^{\nu n}(2^\nu|y-y'|^\epsilon\omega_{2\epsilon}(2^\nu(x-y)), \qquad x, y, y' \in \mathbf{R}^n,$$

whenever $2^\nu|y-y'| \le \frac{1}{2}(1 + 2^\nu|x-y|)$.

We are interested in conditions on a family, $D = \{D_\nu\}$, that will imply that

$$(63)\qquad \|f\|_{\dot F_p^{\alpha,q}} \sim \Big\|\Big(\sum_{\nu\in\mathbb{Z}}(2^{\nu\alpha}|D_\nu f|)^q\Big)^{1/q}\Big\|_{L^p} \qquad \text{if } p\neq\infty,$$

$$(64)\qquad \|f\|_{\dot F_\infty^{\alpha,q}} \sim \sup_P\Big(|P|^{-1}\int_P \sum_{\nu=-\log_2 \ell(P)}^{\infty} (2^{\nu\alpha}|D_\nu f(x)|)^q\,dx\Big)^{1/q},$$

$$(65)\qquad \|f\|_{\dot B_p^{\alpha,q}} \sim \Big(\sum_{\nu\in\mathbb{Z}}(2^{\nu\alpha}\|D_\nu f\|_{L^p})^q\Big)^{1/q},$$

for an appropriate range of α, p, and q, where we interpret this in the usual way if $q=\infty$.

Remark. We are using the convenient, but sometimes confusing, convention of denoting an operator and its kernel with the same symbol.

4.1. *Boundedness results.* Suppose $D=\{D_\nu\}_{\nu\in\mathbb{Z}}$ is an ϵ-family of operators. We say that D is *bounded* on the Triebel–Lizorkin space $\dot F_p^{\alpha,q}$ if

$$(66)\qquad \Big\|\Big(\sum_{\nu\in\mathbb{Z}}(2^{\nu\alpha}|D_\nu f|)^q\Big)^{1/q}\Big\|_{L^p} \le c\,\|f\|_{\dot F_p^{\alpha,q}}$$

for all $f\in\dot F_p^{\alpha,q}$, for some constant c, modifying this as in (64) when $p=\infty$. Similarly, we say that D is bounded on the Besov space $\dot B_p^{\alpha,q}$ if

$$(67)\qquad \Big(\sum_{\nu\in\mathbb{Z}}(2^{\nu\alpha}\|D_\nu f\|_{L^p})^q\Big)^{1/q} \le c\,\|f\|_{\dot B_p^{\alpha,q}}$$

for all $f\in\dot B_p^{\alpha,q}$, for some constant c. For the case $p=q=2$ and $\alpha=0$, where $\dot F_2^{0,2}=\dot B_2^{0,2}=L^2$, this definition of boundedness for an ϵ-family was introduced in [2].

Our next lemma makes a useful connection between ϵ-families and almost diagonal matrices.

LEMMA 4.1. *Suppose $1\le p,q\le\infty$ and $|\alpha|<\epsilon\le 1$. Suppose further that $D=\{D_\nu\}_{\nu\in\mathbb{Z}}\in \mathrm{F}_\epsilon$ and $D_\nu 1=0$ for all $\nu\in\mathbb{Z}$. Let $\{a_Q\}_Q$ be a family of smooth atoms. For each dyadic cube Q, $\ell(Q)=2^{-\nu}$, let*

$$(68)\qquad A_{QP}=|Q|^{1/2}\sup_{x\in Q}|D_\nu a_P(x)|\,.$$

Then $A=\{A_{QP}\}_{QP}$ is almost diagonal for α, p, and q.

Proof. Fix a cube Q and an $x\in Q$, and set $m_Q(y)=|Q|^{1/2}D_\nu(x,y)$ where $\ell(Q)=2^{-\nu}$. It is routine to check that the function m_Q satisfies the following three conditions:

$$\int m_Q(y)\,dy=0\,,$$

$$|m_Q(y)| \le c|Q|^{-1/2}\omega_\epsilon(|y - x_Q|/\ell(Q)),$$
$$|m_Q(y) - m_Q(y')| \le c|Q|^{-1/2-\epsilon/n}|y - y'|^\epsilon \sup_{|z|\le|y-y'|} \omega_{2\epsilon}(2^\nu(|y - z - x_Q|)),$$

where c is a positive constant that does not depend on Q or the point x in Q. That is, $\{m_Q/c\}$ satisfies the three conditions (20)–(22). Since $D_\nu a_P(x) = \langle m_Q, \overline{a_P}\rangle$ the lemma follows from Proposition 2.1. ∎

Remark. The factor $|Q|^{1/2}$ in the definition of A_{QP} above is due to the fact that the definitions of smooth atoms, almost diagonal matrices, and related concepts all involve an L^2-normalization, while the definition of an ϵ-family has an L^1-normalization.

The lemma gives us the boundedness result.

THEOREM 4.2. *Suppose* $1 \le p, q \le \infty$ *and* $|\alpha| < \epsilon \le 1$. *Suppose further that* $D = \{D_\nu\}_{\nu\in\mathbf{Z}} \in \mathrm{F}_\epsilon$ *and* $D_\nu 1 = 0$ *for all* $\nu \in \mathbf{Z}$. *Then* D *is bounded on* $\dot F_p^{\alpha,q}$ *and* $\dot B_p^{\alpha,q}$.

Proof. Let A be the matrix with entries A_{QP} defined as in (68). According to the lemma, A is almost diagonal. Suppose $f = \sum_P s_P a_P$ is a smooth atomic decomposition of f. Let s denote the sequence $\{s_Q\}_Q$. Since

$$\sum_{Q:\ell(Q)=2^{-\nu}} \chi_Q(x) \equiv 1$$

we get

$$|D_\nu f(x)| \le \sum_{Q:\ell(Q)=2^{-\nu}} (As)_Q\tilde\chi_Q(x).$$

Thus,

$$\Big(\sum_{\nu\in\mathbf{Z}}(2^{\nu\alpha}\|D_\nu f\|_{L^p})^q\Big)^{1/q} \le \Big\{\sum_\nu\Big(|Q|^{-\alpha/n}\Big\|\sum_{Q:\ell(Q)=2^{-\nu}}|s_Q|\tilde\chi_Q\Big\|_{L^p}\Big)^q\Big\}^{1/q}$$
$$\le \|As\|_{\dot b_p^{\alpha,q}} \le c\|s\|_{\dot b_p^{\alpha,q}}.$$

Taking the infimum over such representations the boundedness on the Besov spaces follows from (16).

The result for the Triebel–Lizorkin formula follows from a similar argument, and is left to the reader. ∎

4.2. *Converse estimates.* Inherent in the Littlewood–Paley characterizations of the Triebel–Lizorkin and Besov–Lipschitz spaces (which is to say: in equations (5)–(7)) is a non-degeneracy condition such as equation (4), which implies that $f \hookrightarrow \sum_\nu \varphi_\nu * \varphi_\nu * f$ is the identity. In the case of an ϵ-family of operators, $D = \{D_\nu\}$, we shall require a similar condition. We shall assume that

$$\sum_{\nu\in\mathbf{Z}} D_\nu = I. \tag{69}$$

The adjoint of D_ν is the operator D_ν^* that is represented by the kernel

$$D_\nu^*(x,y) = D_\nu(y,x). \tag{70}$$

We set $D^* = \{D_\nu^*\}_{\nu\in\mathbf{Z}}$. In order to get our converse estimates we shall assume that

$$D,\ D^* \in \mathrm{F}_\epsilon \tag{71}$$

for some ϵ, $0 < \epsilon \le 1$, and that

$$D_\nu 1 = 0, \quad D_\nu^* 1 = 0 \tag{72}$$

for all ν.

EXAMPLE. Consider a system of kernels $\{S_\nu\}_{\nu\in\mathbf{Z}}$ with $S_\nu(x,y) = S_\nu(y,x)$ for all $x,y\in\mathbf{R}^n$ and assume that each S_ν satisfies conditions (61) and (62) for an ϵ-family. Suppose further that $\int_{\mathbf{R}^n} S_\nu(x,y)\,dy = 1$ for all $x\in\mathbf{R}^n$. Such systems are easily constructed (even in the more general setting of a space of homogeneous type). Let $D_\nu(x,y) = S_{\nu+1}(x,y) - S_\nu(x,y)$ for all $\nu\in\mathbf{Z}$, $x,y\in\mathbf{R}^n$. Let $D = \{D_\nu\}_{\nu\in\mathbf{Z}}$ be the family of operators associated with these kernels. Then D is an example of type we are considering.

We continue with our general assumption that $|\alpha| < \epsilon \le 1$ and $1 \le p, q \le \infty$.

Let us return to consideration of condition (69) in order to motivate our next two lemmas. Operating at a purely formal level we have

$$I = \sum_\nu D_\nu = \sum_\nu D_\nu \sum_\mu D_\mu = \sum_\mu \sum_\nu D_{\mu+\nu} D_\nu$$

$$= \sum_{\mu=-\infty}^{-M-1} \sum_{\nu=-\infty}^{\infty} D_{\mu+\nu} D_\nu + \sum_{\nu=-\infty}^{\infty} \Big(\sum_{\nu=-M}^{M} D_{\mu+\nu} \Big) D_\nu + \sum_{\mu=M+1}^{\infty} \sum_{\nu=-\infty}^{\infty} D_{\mu+\nu} D_\nu .$$

If we set

$$E_\mu = \sum_{\nu=-\infty}^{\infty} D_{\mu+\nu} D_\nu, \tag{73}$$

$$D_\nu^M = \sum_{\mu=-M}^{M} D_{\mu+\nu}, \tag{74}$$

$$\Phi_M = \sum_{\nu=-\infty}^{\infty} D_\nu^M D_\nu = \sum_{\mu=-M}^{M} E_\mu, \tag{75}$$

we see that

$$(76) \qquad I - \Phi_M = \sum_{\mu=-\infty}^{-M-1} E_\mu + \sum_{\mu=M+1}^{\infty} E_\mu .$$

In the two lemmas that follow we will show that on the basis of (70)–(72), $\sum_\nu D_\nu \in \mathrm{CZO}(\epsilon) \cap \mathrm{WBP}$ and that each E_μ is bounded on the Besov and Triebel–Lizorkin spaces with a norm bounded by $c2^{-|\mu|\sigma}$ for some $\sigma > 0$. This justifies the formal manipulations above. Furthermore, it follows that if M is large enough then the norm of $I - \Phi_M$ is less than 1 and this implies that Φ_M is invertible. From this we will obtain the equivalences (63)–(65).

As our first step we establish the properties of $\sum_\nu D_\nu$.

LEMMA 4.3. *Suppose $D = \{D_\nu\}_{\nu\in\mathbf{Z}}$ is a family of operators such that $D \in \mathrm{F}_\epsilon$ and $D_\nu 1 = 0$ for all $\nu \in \mathbf{Z}$; and set $E = \sum_\nu D_\nu$. Then $E1 = 0$ and $E \in \mathrm{CZO}_x(\epsilon') \cap \mathrm{WBP}$ when $0 < \epsilon' < \epsilon \leq 1$.*

Proof. It is trivial that $E1 = 0$. Sum the estimate (61) over ν to get the estimate $|E(x,y)| \leq c|x-y|^{-n}$. Sum the estimate (62) over ν to get the estimate $|E(x,y) - E(x,y')| \leq c|y-y'|^{\epsilon'}|x-y|^{-n-\epsilon'}$ when $2|y-y'| \leq |x-y|$. To complete the proof we show that $E \in \mathrm{WBP}$.

Suppose θ and η are functions in $\mathcal{D}$ and there are points x_0 and y_0 such that $\operatorname{supp}\theta \subset \{x : |x - y_0| \leq t\}$ and $\operatorname{supp}\eta \subset \{x : |x - x_0| \leq t\}$ for some $t > 0$. In the Appendix we will show that

$$(77) \qquad |\langle D_\nu\theta, \eta\rangle| = \left| \int D_\nu(x,y)\theta(y)\overline{\eta(x)}\,dx\,dy \right| \leq \begin{cases} ct^{2n}2^{\nu n}\|\theta\|_{L^\infty}\|\eta\|_{L^\infty} \\ ct^{n-\epsilon'}2^{-\nu\epsilon'}(\|\theta\|_{L^\infty} + t\|\nabla\theta\|_{L^\infty})\|\eta\|_{L^\infty} \end{cases}$$

for all $0 < \epsilon' < \epsilon \leq 1$ where c depends on ϵ' but is independent of $t > 0$. Sum these estimates over ν:

$$|\langle D_\nu\theta, \eta\rangle| \leq ct^n(\|\theta\|_{L^\infty} + t\|\nabla\theta\|_{L^\infty})(\|\eta\|_{L^\infty} + t\|\nabla\eta\|_{L^\infty}).$$

We see that E satisfies equation (28) and so satisfies the Weak Boundedness Property. This completes the proof. ∎

Our next step is to establish the properties of E_μ.

LEMMA 4.4. *Suppose that $D = \{D_\nu\}$ is a family of operators such that $D, D^* \in \mathrm{F}_\epsilon$, and $D_\nu 1 = D_\nu^* 1 = 0$ for all $\nu \in \mathbf{Z}$, and that $|\alpha| < \epsilon \leq 1$, $1 \leq p, q \leq \infty$. Then $E_\mu = \sum_\nu D_{\mu+\nu}D_\nu$ is bounded on $\dot{F}_p^{\alpha,q}$ and $\dot{B}_p^{\alpha,q}$ with a norm bounded by $c2^{-|\mu|\sigma}$ for positive constants c and σ.*

Proof. We claim that $E_\mu \in \mathrm{CZO}(\epsilon')$ for each $0 < \epsilon' < \epsilon$ and that its kernel satisfies

$$|E_\mu(x,y| \le c2^{-|\mu|\epsilon'}|x-y|^{-n}, \tag{78}$$

$$\begin{aligned}|E_\mu(x,y)-E_\mu(x,y')| &+ |E_\mu(y,x)-E_\mu(y',x)| \\ &\le c2^{-|\mu|\sigma}|y-y'|^{\epsilon'}|x-y|^{n+\epsilon'}\end{aligned} \tag{79}$$

whenever $|y-y'| \le \frac{1}{2}|x-y|$ for positive constants c and σ. Furthermore, $E_\mu \in$ WBP with constant $c2^{-|\mu|\epsilon'}$. It is clear that $E_\mu 1 = E_\mu^* 1 = 0$. Since operators with these properties are bounded on $\dot{F}_p^{\alpha,q}$ and $\dot{B}_p^{\alpha,q}$ (see Theorem 3.1 and the remark immediately after the proof of that theorem) with norms proportional to the constants in (78), (79), and in the Weak Boundedness Property, this will prove the lemma.

To prove (78) we note that

$$E_\mu(x,y) = \sum_\mu \int_{\mathbf{R}^n} D_{\mu+\nu}(x,z)D_\nu(z,y)\,dz.$$

In the Appendix we show that

$$\left|\int_{\mathbf{R}^n} D_{\mu+\nu}(x,z)D_\nu(z,y)\,dz\right| \le c2^{-|\mu|\epsilon'}(k_{\nu\mu})^n\omega_{\epsilon'}(k_{\nu\mu}(x-y)) \tag{80}$$

where $k_{\nu\mu} = 2^\nu \wedge 2^{\nu+\mu}$ and c is a constant that depends on ϵ' and ϵ. Consider the cases $\mu \ge 0$ and $\mu < 0$ separately, and sum over ν to obtain (78). In the Appendix we will show that

$$\begin{aligned}\left|\int_{\mathbf{R}^n} D_{\mu+\nu}(x,z)D_\nu(z,y)\,dz - \int_{\mathbf{R}^n} D_{\mu+\nu}(x,z)D_\nu(z,y')\,dz\right| \\ \le c2^{-|\mu|\sigma}2^{\nu n}(2^\nu|y-y'|)^{\epsilon'}\omega_{\epsilon'+\tau}(2^\nu(x-y))\end{aligned} \tag{81}$$

whenever $|y-y'| \le \frac{1}{4}|x-y|$, where c, σ, and τ are constants that depend on ϵ' and ϵ. Sum this estimate over ν to obtain one half of (79). From the symmetry of the conditions we get the other half. To extend the result to $\frac{1}{4}|x-y| < |y-y'| \le \frac{1}{2}|x-y|$ use (78). This shows that $E_\mu \in \mathrm{CZO}(\epsilon')$ with constant bounded by $c2^{-|\mu|\sigma}$.

To show that $E_\mu \in$ WBP with constant bounded by $c2^{-|\mu|\sigma}$ we will show that it satisfies (28). That is, we take two functions $\theta, \eta \in \mathcal{D}$, $\operatorname{supp}\theta \subset \{x : |x-y_0| \le t\}$ and $\operatorname{supp}\eta \subset \{x : |x-x_0| \le t\}$, where $t > 0$ and $x_0, y_0 \in \mathbf{R}^n$. We will show that

$$|\langle E_\mu\theta, \eta\rangle| \le c2^{-|\mu|\epsilon'}t^n B(\theta,\eta) \tag{82}$$

with

$$B(\theta,\eta) = (\|\theta\|_{L^\infty} + t\|\nabla\theta\|_{L^\infty})(\|\eta\|_{L^\infty} + t\|\nabla\eta\|_{L^\infty})$$

where c depends on ϵ' and ϵ but is independent of θ, η, t, and μ. Let

$$I_{\mu\nu} = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} D_{\nu+\mu}(x,z) D_\nu(z,y) \theta(y) \overline{\eta(x)}\, dx\, dy\, dz .$$

Then

$$\langle E_\mu \theta, \eta \rangle = \sum_{\nu \in \mathbf{Z}} I_{\mu\nu} .$$

In the Appendix we show that

$$|I_{\mu\nu}| \le c 2^{-|\mu|\epsilon'} (k_{\nu\mu})^{-\epsilon'} t^{n-\epsilon'} B(\theta, \eta), \tag{83}$$

$$|I_{\mu\nu}| \le c 2^{-|\mu|\epsilon'} (k_{\nu\mu})^{n} t^{2n} B(\theta, \eta). \tag{84}$$

Consider the cases $\mu \ge 0$ and $\mu < 0$ separately, sum over ν and (82) follows. This completes the proof of the lemma. ■

We can now prove the following converse of Theorem 4.2:

THEOREM 4.5. *Suppose that $D = \{D_\nu\}$ is a family of operators such that $D, D^* \in \mathrm{F}_\epsilon$, $D_\nu 1 = D_\nu^* 1 = 0$ for all $\nu \in \mathbf{Z}$, and $\sum_\nu D_\nu = I$. Then*

$$\|f\|_{\dot F_p^{\alpha,q}} \le c \Big\| \Big(\sum_{\nu \in \mathbf{Z}} (2^{\nu\alpha} |D_\nu f|)^q \Big)^{1/q} \Big\|_{L^p} \qquad \text{if } p \ne \infty, \tag{85}$$

$$\|f\|_{\dot F_\infty^{\alpha,q}} \le c \sup_P \Big(|P|^{-1} \int_P \sum_{\nu = \log_2 \ell(P)}^{\infty} (2^{\nu\alpha} |D_\nu f(x)|)^q \, dx \Big)^{1/q}, \tag{86}$$

$$\|f\|_{\dot B_p^{\alpha,q}} \le c \Big(\sum_{\nu \in \mathbf{Z}} (2^{\nu\alpha} \|D_\nu f\|_{L^p})^q \Big)^{1/q} . \tag{87}$$

Proof. We will prove (85) for $1 < p < \infty$, and offer a hint for $p = 1$ and for (86). We leave (87) as an easy exercise for the reader.

As we saw at the beginning of this subsection

$$I - \Phi_M = \sum_{\mu=-\infty}^{-M-1} E_\mu + \sum_{\mu=M+1}^{\infty} E_\mu .$$

By Lemma 4.4

$$\begin{aligned} \|(I - \Phi_M) f\|_{\dot F_p^{\alpha,q}} &\le c \sum_{|\mu| > M} \|E_\mu f\|_{\dot F_p^{\alpha,q}} \\ &\le c \sum_{|\mu| > M} 2^{-|\mu|\sigma} \|f\|_{\dot F_p^{\alpha,q}} \le c 2^{-M\sigma} \|f\|_{\dot F_p^{\alpha,q}} . \end{aligned}$$

It follows that if we pick M large enough then Φ_M^{-1} exists and is bounded on $\dot F_p^{\alpha,q}$. This implies that

$$\|f\|_{\dot F_p^{\alpha,q}} = \|\Phi_M^{-1} \Phi_M f\|_{\dot F_p^{\alpha,q}} \le c_M \|\Phi_M f\|_{\dot F_p^{\alpha,q}} .$$

To complete the proof it is enough to show that

$$\|\Phi_M f\|_{\dot F_p^{\alpha,q}} \le c\Big\|\Big(\sum_{\nu\in\mathbf{Z}}(2^{\nu\alpha}|D_\nu f|)^q\Big)^{1/q}\Big\|_{L^p}. \tag{88}$$

To see this we use the usual "converse to Hölder's inequality argument". Suppose $g \in \dot F_{p'}^{-\alpha,q'}$ with $\|g\|_{\dot F_{p'}^{-\alpha,q'}} = 1$. Suppose that $p \neq 1,\infty$. From the definition of Φ_M we have

$$\begin{aligned} |\langle \Phi_M f, g\rangle| &= \Big|\sum_\nu \int_{\mathbf{R}^n} D_\nu^M(D_\nu f)(x)g(x)\,dx\Big| \\ &\le c\Big\|\Big(\sum_\nu(2^{\nu\alpha}|D_\nu f|)^q\Big)^{1/q}\Big\|_{L^p}\Big\|\Big(\sum_\nu(2^{-\nu\alpha}|D_\nu^* g|)^{q'}\Big)^{1/q'}\Big\|_{L^{p'}}. \end{aligned} \tag{89}$$

By Theorem 4.2

$$\Big\|\Big(\sum_\nu(2^{-\nu\alpha}|D_\nu^* g|)^{q'}\Big)^{1/q'}\Big\|_{L^{p'}} \le c\|g\|_{\dot F_{p'}^{-\alpha,q'}} \le c.$$

Insert this into (89) and take the supremum over all such functions g to get (88) provided $p \neq 1$ and $p \neq \infty$. For the limiting cases, $p = 1$ or $p = \infty$, one needs a modified version of the "converse to Hölder's inequality". See [12, §5] for details. ∎

5. A concluding remark. It is generally understood that much of classical Calderón–Zygmund theory does not depend on the special structure of $\mathbf{R}^n$. Indeed, a natural setting for the theory is provided by the spaces of homogeneous type (see [4] and [5]). It has recently been shown by David, Journé, and Semmes [7] that the $T1$-theorem (and, more generally, the Tb-theorem) also has an analog on these more general spaces. Similarly, it is now possible to develop a theory of Triebel–Lizorkin and Besov spaces on spaces of homogeneous type. For this we need to avoid the convolution structure in the definitions of the spaces (as in Theorems 4.2 and 4.4) and we need a substitute for the identity $I = \sum_\nu \varphi_\nu * \varphi_\nu$ (as in equation (69)). See our discussion in the Example of the preceding subsection. Details will appear elsewhere.

A. Some details for Section 4. We begin with an elementary lemma.

LEMMA A.1. *Suppose θ is a C^1 function and* $\operatorname{supp}\theta \subset \{x : |x - x_0| \le t\}$ *for some $x_0 \in \mathbf{R}^n$ and a $t > 0$. Then if $0 < \epsilon' \le 1$,*

$$|\theta(y) - \theta(z)| \le 4|y - z|^{\epsilon'} t^{-\epsilon'}(\|\theta\|_{L^\infty} + t\|\nabla\theta\|_{L^\infty})$$

for all $y, z \in \mathbf{R}^n$.

Proof. Let $G = \{ x : |x - x_0| \le t\}$ and let G^* be the double of G. There are three cases to consider.

Case I: $y, z \notin G$. Then $\theta(y) = \theta(z) = 0$.

Case II: $y, z \in G^*$. Then $|y - z| \le 4t$ and so

$$\begin{aligned}|\theta(y) - \theta(z)| &\le |y - z| \|\nabla\theta\|_{L^\infty} = |y - z|^{\epsilon'} t^{-\epsilon'} t\|\nabla\theta\|_{L^\infty} (|y - z|/t)^{1-\epsilon'}\\ &\le 4^{1-\epsilon'} |y - z|^{\epsilon'} t^{-\epsilon'} t\|\nabla\theta\|_{L^\infty} .\end{aligned}$$

Case III: One of y, z is in G, the other is not in G^*, say $y \in G$, $z \notin G^*$. In this case $|y - z| \ge t$ and therefore

$$\begin{aligned}|\theta(y) - \theta(z)| = |\theta(y)| \le \|\theta\|_{L^\infty} &= |y - z|^{\epsilon'} t^{-\epsilon'} \|\theta\|_{L^\infty} (t/|y - z|)^{\epsilon'}\\ &\le |y - z|^{\epsilon'} t^{-\epsilon'} \|\theta\|_{L^\infty} .\end{aligned}$$

This completes the proof. ∎

Proof of (77). One half of this is trivial. Since $|D_\nu(x, y| \le 2^{\nu n}$,

$$\begin{aligned}\left| \int\int D_\nu(x, y)\theta(y)\overline{\eta(x)}\, dx\, dy\right| &\le 2^{\nu n} \int\limits_{|y-y_0|\le t} \int\limits_{|x-x_0|\le t} 1\, dx\, dy\, \|\theta\|_{L^\infty} \|\eta\|_{L^\infty}\\ &\le c2^{\nu n} t^{2n} \|\theta\|_{L^\infty} \|\eta\|_{L^\infty} .\end{aligned}$$

Use $D_\nu 1 = 0$, (61), and Lemma A.1 to get

$$\begin{aligned}|\langle D_\nu\theta, \eta\rangle| &= \left| \int\int D_\nu(x, y)\theta(y)\overline{\eta(x)}\, dx\, dy\right|\\ &= \left| \int\int D_\nu(x, y)(\theta(y) - \theta(x))\overline{\eta(x)}\, dx\, dy\right|\\ &\le \int\limits_{|x-x_0|\le t} \int\limits_{\mathbf{R}^n} \frac{2^{\nu n}}{(1 + 2^\nu|x - y|)^{n+\epsilon}} |y - x|^{\epsilon'} t^{-\epsilon'}\, dy\, dx\\ &\quad \times (\|\theta\|_{L^\infty} + t\|\nabla\theta\|_{L^\infty})\|\eta\|_{L^\infty}\\ &= 2^{\nu\epsilon'} t^{-\epsilon'} \int\limits_{\mathbf{R}^n} \int\limits_{|x-x_0|\le t} \frac{2^{\nu n}(2^\nu|x - y|)^{\epsilon'}}{(1 + 2^\nu|x - y|)^{n+\epsilon}}\, dx\, dy\, C(\theta, \eta)\\ &= 2^{\nu\epsilon'} t^{-\epsilon'} \int\limits_{\mathbf{R}^n} F(y)\, dy\, C(\theta, \eta),\end{aligned}$$

where

$$C(\theta, \eta) = (\|\theta\|_{L^\infty} + t\|\nabla\theta\|_{L^\infty})\|\eta\|_{L^\infty}$$

and F is the convolution of the integrable function

$$2^{\nu n}(2^\nu|x - y|)^{\epsilon'} (1 + 2^\nu|x - y|)^{-n-\epsilon}$$

and the characteristic function of $\{ x : |x - x_0| \le t\}$. Therefore $\int_{\mathbf{R}^n} F\, dy \le ct^n$ where c depends on ϵ' and ϵ but is independent of t. Thus,

$$|\langle D_\nu\theta, \eta\rangle| \le 2^{-\nu\epsilon'} t^{n-\epsilon'} C(\theta, \eta) .$$

This completes the proof of (77). ■

Proof of (78). An obvious change of variables and the symmetry of the conditions reduces (78) to the following lemma.

LEMMA A.2. *Suppose $\epsilon > 0$ and*

$$|g(x,y)| \le \frac{1}{(1+|x-y|)^{n+\epsilon_1}} \qquad \text{for all } x,y \in \mathbf{R}^n,$$

$$|g(x,z)-g(x,y)| \le \frac{|z-y|^{\epsilon_2}}{(1+|x-y)^{n+\epsilon_1+\epsilon_2}} \qquad \text{for all } x,y,z \in \mathbf{R}^n$$

$$\text{with } |y-z| \le \tfrac{1}{2}\max\{1+|x-y|, 1+|x-z|\},$$

$$\int h(z,y)\,dy = 0\,,$$

$$|h(z,y)| \le \frac{2^{kn}}{(1+2^k|z-y|)^{n+\delta}} \qquad \text{for all } \epsilon_1,\ \epsilon_2,\ \delta \text{ with } 0 \le \epsilon_1, \epsilon_2, \delta \le \epsilon.$$

If k is a non-negative integer and $0 < \epsilon' < \epsilon$ then

$$\Big| \int_{\mathbf{R}^n} g(x,z)h(z,y)\,dz \Big| \le c\frac{2^{-k\epsilon'}}{(1+2^k|x-y|)^{n+\epsilon'}}\,,$$

where c depends on ϵ' and ϵ but does not depend on k.

Proof. It is enough to estimate

$$\int_{\mathbf{R}^n} |g(x,z)-g(x,y)|\,|h(z,y)|\,dz\,.$$

Dissect $\mathbf{R}^n$: $\mathbf{R}^n = I \cup II \cup III$ where

$$I = \{\, z : 2|y-z| \le \max\{1+|x-y|, 1+|x-z|\}\}\,,$$
$$II = \{\, z : 2|y-z| > 1+|x-z| \ge 1+|x-y|\}\,,$$
$$III = \{\, z : 2|y-z| > 1+|x-y| > 1+|x-z|\}\,.$$

We have

$$\int_I \dots\,dz \le \frac{2^{-k\epsilon_2}}{(1+|x-y|)^{n+\epsilon_1+\epsilon_2}} \int_{\mathbf{R}^n} \frac{2^{kn}(2^k|y-z|)^{\epsilon_2}}{(1+2^k|y-z|)^{n+\delta}}\,dz$$
$$= c\frac{2^{-k\epsilon_2}}{(1+|x-y|)^{n+\epsilon_1+\epsilon_2}}\,,$$

provided $\delta > \epsilon_2$. Take $\epsilon_2 = \epsilon'$, $\epsilon_1 = 0$, and $\delta = \epsilon$. Then

$$\int_{II} \dots\,dz \le \frac{2}{(1+|x-y|)^{n+\epsilon_1}} \int_{II} \frac{2^{kn}}{(1+2^k|x-z|)^{n+\delta}}\,dz$$
$$\le 2^{1+n+\delta}\frac{2^{-k\delta}}{(1+|x-y|)^{n+\epsilon_1}} \int_{|z-y|\ge(1+|x-y|)/2} \frac{1}{|z-y|^{n+\delta}}\,dz$$

$$= c\frac{2^{-k\delta}}{(1+|x-y|)^{n+\epsilon_1+\delta}},$$

provided $\delta > 0$. Take $\delta = \epsilon'$ and $\epsilon_1 = 0$. Then

$$\int_{III} \dots dz \le 2 \int_{III} \frac{1}{(1+|x-z|)^{n+\epsilon_1}} \frac{2^{kn}}{(1+2^k|x-z|)^{n+\delta}} dz$$

$$\le 2^{1+n+\delta}\frac{2^{-k\delta}}{(1+|x-y|)^{n+\delta}} \int_{\mathbf{R}^n} \frac{1}{(1+|x-z|)^{n+\epsilon_1}} dz$$

$$= c\frac{2^{-k\delta}}{(1+|x-y|)^{n+\delta}},$$

provided $\epsilon_1 > 0$. Take $\delta = \epsilon'$ and $\epsilon_1 = \epsilon$. This completes the proof. ∎

P r o o f o f (83) a n d (84). (84) follows easily from (80):

$$|I_{\nu\mu}| \le \int_{|x-x_0|\le t} \int_{|y-y_0|\le t} \left| \int_{\mathbf{R}^n} D_{\nu+\mu}(x,z)D_\nu(z,y)\, dz \right| dx\, dy\, \|\theta\|_{L^\infty} \|\eta\|_{L^\infty}$$

$$\le c2^{-|\mu|\epsilon'}(k_{\nu\mu})^n t^{2n} \|\theta\|_{L^\infty} \|\eta\|_{L^\infty}.$$

Consider now (83). Use Lemma A.1, $D_\nu 1 = 0$, and the size estimates for D_ν to get

$$|I_{\nu\mu}| = \left| \int\int\int D_{\nu+\mu}(x,z)D_\nu(z,y)(\theta(y)-\theta(z))\overline{\eta(x)}\, dx\, dy\, dz \right|$$

$$\le c \int_{z\in\mathbf{R}^n} \int_{y\in\mathbf{R}^n} \int_{|x-x_0|\le t} \frac{2^{(\nu+\mu)n}}{(1+2^{\nu+\mu}|x-z|)^{n+\epsilon}}$$

$$\times \frac{2^{\nu n}}{(1+2^\nu|y-z|)^{n+\epsilon}} |y-z|^{\epsilon'} t^{-\epsilon'}\, dx\, dy\, dz$$

$$\times (\|\theta\|_{L^\infty} + t\|\nabla\theta\|_{L^\infty})\|\eta\|_{L^\infty}$$

$$= ct^{-\epsilon'}2^{-\epsilon'} \int_{z\in\mathbf{R}^n} \int_{y\in\mathbf{R}^n} \frac{2^\nu(2^\nu n|y-z|)^{\epsilon'}}{(1+2^\nu|y-z|)^{n+\epsilon'}}$$

$$\times \int_{|x-x_0|\le t} \frac{2^{(\nu+\mu)n}}{(1+2^{\nu+\mu}|y-z|)^{n+\epsilon}} dx\, dy\, dz$$

$$\times (\|\theta\|_{L^\infty} + t\|\nabla\theta\|_{L^\infty})\|\eta\|_{L^\infty}$$

$$= ct^{-\epsilon'}2^{-\nu\epsilon'} \int\int \frac{2^{\nu n}(2^\nu n|y-z|)^{\epsilon'}}{(1+2^\nu|y-z|)^{n+\epsilon'}} dy F(z)\, dz$$

$$\times (\|\theta\|_{L^\infty} + t\|\nabla\theta\|_{L^\infty})\|\eta\|_{L^\infty},$$

where $F(z)$ is the convolution of an integrable function with the characteristic function of a ball of radius t centered at x_0. Thus, $\int F(z)\, dz \le ct^n$. If

$\mu \leq 0$ then $k_{\nu\mu} = 2^{\nu+\mu}$ and so

$$\begin{aligned}|I_{\nu\mu}| &\leq ct^{n-\epsilon'}2^{-\nu\epsilon'}(\|\theta\|_{L^\infty} + t\|\nabla\theta\|_{L^\infty})\|\eta\|_{L^\infty} \\ &= ct^{n-\epsilon'}(k_{\nu\mu})^{-\epsilon'}2^{-|\mu|\epsilon'}(\|\theta\|_{L^\infty} + t\|\nabla\theta\|_{L^\infty})\|\eta\|_{L^\infty}.\end{aligned}$$

If $\mu > 0$ then $k_{\nu\mu} = 2^\nu$, and by an analogous calculation

$$\begin{aligned}|I_{\nu\mu}| &\leq ct^{n-\epsilon'}2^{-(\nu+\mu)\epsilon'}(\|\eta\|_{L^\infty} + t\|\nabla\eta\|_{L^\infty})\|\theta\|_{L^\infty} \\ &= ct^{n-\epsilon'}(k_{\nu\mu})^{-\epsilon'}2^{-|\mu|\epsilon'}(\|\eta\|_{L^\infty} + t\|\nabla\eta\|_{L^\infty})\|\theta\|_{L^\infty}.\end{aligned}$$

This completes the proof of (83) and (84). ∎

Proof of (81). Unlike the situation with the proof of (78) we can not reduce this to the case where k is a non-negative integer, but by the usual change of variables we can reduce it to the following lemma:

LEMMA A.3. *Suppose k is an integer, $\epsilon > 0$, while g and h are functions on $\mathbf{R}^n \times \mathbf{R}^n$ that satisfy the following conditions:*

$$|g(x,y)| \leq \frac{1}{(1+|x-z|)^{n+\epsilon}},$$

$$|g(x,z) - g(x,y)| \leq \frac{|y-z|^\epsilon}{(1+|x-y|)^{n+2\epsilon}}$$

$$\text{if } 2|z-y| \leq \max\{1+|x-y|, 1+|x-z|\}.$$

$$|h(z,y)| \leq \frac{2^{kn}}{(1+2^k|y-z|)^{n+\epsilon}},$$

$$|h(z,y) - h(z,y')| \leq \frac{2^{kn}(2^k|y-y'|)^\epsilon}{(1+2^k|y-z|)^{n+2\epsilon}}$$

$$\text{if } 2\cdot 2^k|y-y'| \leq \max\{1+2^k|z-y|, 1+|z-y'|\}.$$

Set

$$F(x,y,y') = \int (g(x,z) - g(x,y))(h(z,y) - h(z,y'))\,dz.$$

If $0 < \epsilon' < \epsilon$ and $4|y-y'| \leq |x-y|$ then

$$|F(x,y,y')| \leq c2^{-|k|\sigma}\frac{|y-y'|^{\epsilon'}}{(1+|x-y|)^{n+\epsilon'+\tau}}, \tag{90}$$

where c, σ, and τ are positive constants that depend on ϵ and ϵ' but are independent of k, x, y, and y'.

Proof. A consequence of (80) is that

$$|F(x,y,y')| \leq c(\epsilon_1)2^{-k\epsilon_1} \tag{91}$$

whenever $0 < \epsilon_1 < \epsilon$ and $4|y-y'| \le |x-y|$. If we can show that, under the conditions of the lemma,

$$|F(x,y,y')| \le c(\epsilon_2,\epsilon_3)\frac{|y-y'|^{\epsilon_2}}{(1+|x-y|)^{n+\epsilon_2+\epsilon_3}} \tag{92}$$

whenever $0 < \epsilon_2, \epsilon_3 < \epsilon$ then (90) will follow. To see this take a θ, $0 < \theta < 1$. From (91) and (92) we have

$$|F(x,y,y')| \le c(\epsilon_1)^\theta c(\epsilon_2,\epsilon_3)^{1-\theta}\frac{2^{-|k|\epsilon_1\theta}|y-y'|^{\epsilon_2(1-\theta)}}{(1+|x-y|)^{(n+\epsilon_3)(1-\theta)+\epsilon_2(1-\theta)}}.$$

Fix an ϵ_3, $\epsilon' < \epsilon_3 < \epsilon$. Then take a θ such that

$$0 < \theta < \min\{\epsilon_3/(n+\epsilon_3), (\epsilon-\epsilon')/\epsilon\}.$$

It follows that $(n+\epsilon_3)(1-\theta) > n$ and that there is an ϵ_2, $0 < \epsilon_2 < \epsilon$, such that $(1-\theta)\epsilon_2 = \epsilon'$. For ϵ_1 take any admissible value. Let $\sigma = \theta\epsilon_1$ and $\tau = (n+\epsilon_3)(1-\theta) - n$. This yields (90).

Our basic plan is to split $\mathbf{R}^n$ into five pieces and establish the estimate (92) for the integral over each piece. This works except in one case. We will deal with that when we come to it. We proceed with the estimate:

$$\begin{aligned}|F(x,y,y')| &\le \int |g(x,z)-g(x,y)|\,|h(z,y)-h(z,y')|\,dz \\ &= \int \Delta_0(x,y,z)\Delta_k(z,y,y')\,dz.\end{aligned}$$

For each of Δ_0 and Δ_k we split $\mathbf{R}^n$ into three pieces, much as we did in the proof of Lemma A.3. On each piece we get the indicated estimate.

$$\begin{aligned}
I_0 &= \{z : 2|y-z| \le \max\{1+|x-z|, 1+|x-y|\}\}, \\
&\qquad \Delta_0 \le \frac{|y-z|^\epsilon}{(1+|x-y|)^{n+2\epsilon}}; \\
II_0 &= \{z : 2|y-z| > 1+|x-z| \ge 1+|x-y|\}, \\
&\qquad \Delta_0 \le 2\frac{1}{(1+|x-y|)^{n+\epsilon}}; \\
III_0 &= \{z : 2|y-z| > 1+|x-y| > 1+|x-z|\}, \\
&\qquad \Delta_0 \le 2\frac{1}{(1+|x-z|)^{n+\epsilon}}; \\
I_k &= \{z : 2\cdot 2^k|y-y'| \le \max\{1+2^k|y-z|, 1+2^k|y'-z|\}\}, \\
&\qquad \Delta_k \le \frac{2^{kn}(2^k|y-y'|)^\epsilon}{(1+2^k|y-z|)^{n+2\epsilon}}; \\
II_k &= \{z : 2\cdot 2^k|y-y'| > 1+2^k|y'-z| \ge 1+2^k|y-z|\}, \\
&\qquad \Delta_k \le 2\frac{2^{kn}}{(1+2^k|y-z|)^{n+\epsilon}};
\end{aligned}$$

$$III_k = \{\, z : 2 \cdot 2^k|y - y'| > 1 + 2^k|y - z| > 1 + 2^k|y' - z| \,\},$$

$$\Delta_k \le 2\frac{2^{kn}}{(1 + 2^k|y' - z|)^{n+\epsilon}}.$$

This presents us with nine (logically possible) cases. Fortunately, four of these are void.

Notice that $(II_0 \cup III_0) \cap (II_k \cup III_k) = \emptyset$. To see this note that if $z \in II_0 \cup III_0$ then $|y - z| \ge \frac{1}{2}(1 + |x - y|)$ and that if $z \in II_k \cup III_k$ then $2^k|y - y'| \ge \frac{1}{2}(1 + 2^k|y - z|)$. Thus

$$2^k|y - y'| \ge \frac{1}{2}\left(1 + 2^k \cdot \frac{1}{2}(1 + |x - y|)\right) = \frac{1}{2} + \frac{1}{4}2^k + \frac{1}{4}2^k|x - y|.$$

This implies that $4|y - y'| > |x - y|$, but $4|y - y'| \le |x - y|$, a contradiction.

Five cases remain: (I_0, I_k), (I_0, II_k), (I_0, III_k), (II_0, I_k), and (III_0, I_k). Four of these verifications are routine. The fifth requires special consideration when k is negative.

In what follows, $0 < \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 < \epsilon$, and they will be selected in each individual case as needed.

(I_0, I_k):

$$\begin{aligned}
\int\limits_{I_0 \cap I_k} \ldots\, dz &\le \int \frac{|y - z|^{\epsilon_1}}{(1 + |x - y|)^{n+\epsilon_1+\epsilon_2}} \frac{2^{kn}(2^k|y - y'|)^{\epsilon_3}}{(1 + 2^k|y - z|)^{n+\epsilon_3+\epsilon_4}}\, dz \\
&\le \frac{|y - y'|^{\epsilon_3} 2^{k(\epsilon_3 - \epsilon_1)}}{(1 + |x - y|)^{n+\epsilon_1+\epsilon_2}} \int\limits_{\mathbf{R}^n} \frac{2^{kn}(2^k|y - z|)^{\epsilon_1}}{(1 + 2^k|y - z|)^{n+\epsilon_3+\epsilon_4}}\, dz \\
&= c(\epsilon_2, \epsilon_3) \frac{|y - y'|^{\epsilon_3}}{(1 + |x - y|)^{\epsilon_1+\epsilon_2}},
\end{aligned}$$

where ϵ_1 and ϵ_2 are free and $\epsilon_3 = \epsilon_4 = \epsilon_1$.

(I_0, II_k):

$$\begin{aligned}
\int\limits_{I_0 \cap II_k} \ldots\, dz &\le 2 \int \frac{|y - z|^{\epsilon_1}}{(1 + |x - y|)^{n+\epsilon_1+\epsilon_2}} \frac{2^{kn}}{(1 + 2^k|y - z|)^{n+\epsilon_3}}\, dz \\
&\le 2^{1+\epsilon_1} \frac{|y - y'|^{\epsilon_1}}{(1 + |x - y|)^{n+\epsilon_1+\epsilon_2}} \int\limits_{\mathbf{R}^n} \frac{2^{kn}}{(1 + 2^k|y - z|)^{n+\epsilon_3}}\, dz \\
&\le c(\epsilon_1) \frac{|y - y'|^{\epsilon_1}}{(1 + |x - y|)^{n+\epsilon_1+\epsilon_2}},
\end{aligned}$$

where ϵ_1 and ϵ_2 are free and $\epsilon_3 = \epsilon_1$.

(I_0, III_k): The argument is identical to that immediately above except that $|y' - z|$ replaces $|y - z|$ in the denominator of the integrands.

(II_0, I_k):

(93) $$\int_{II_0\cap I_k} \dots dz$$
$$\le 2\int \frac{1}{(1+|x-y|)^{n+\epsilon_1}} \frac{2^{kn}(2^k|y-y'|)^{\epsilon_3}}{(1+2^k|y-z|)^{n+\epsilon_3+\epsilon_4}}\, dz$$
$$\le 2^{1+\epsilon_3}\frac{|y-y'|^{\epsilon_3}}{(1+|x-y|)^{n+\epsilon_3+\epsilon_1}} \int_{|y-z|\ge(1+|x-y|)/2} \frac{2^{kn}}{(1+2^k|y-z|)^{n+\epsilon_4}}\, dz$$
$$\le c(\epsilon_1,\epsilon_3)\frac{|y-y'|^{\epsilon_3}}{(1+|x-y|)^{n+\epsilon_1+\epsilon_3}},$$

where ϵ_1 and ϵ_3 are free and $\epsilon_4 = \epsilon_1$.

(III_0, I_k):

(94) $$\int_{III_0\cap I_k} \dots dz$$
$$\le 2\int \frac{1}{(1+|x-z|)^{n+\epsilon_1}} \frac{2^{kn}(2^k|y-y'|)^{\epsilon_3}}{(1+2^k|y-z|)^{n+\epsilon_3+\epsilon_4}}\, dz$$
$$\le 2^{1+n+\epsilon_3+\epsilon_4}\frac{|y-y'|^{\epsilon_3}}{(1+|x-y|)^{n+\epsilon_3+\epsilon_4}} 2^{-k\epsilon_4}\int_{\mathbf{R}^n} \frac{1}{(1+|x-z|)^{n+\epsilon_1}}\, dz$$
$$\le c(\epsilon_3,\epsilon_4)2^{-k\epsilon_4}\frac{|y-y'|^{\epsilon_3}}{(1+|x-y|)^{n+\epsilon_3+\epsilon_4}},$$

where ϵ_3 and ϵ_4 are free and $\epsilon_1 = \epsilon_3$. If k is non-negative this is just fine. In fact, it gives us (90) directly with $\sigma = \tau = \epsilon_4$.

To complete the proof we suppose that $k < 0$. From (80) we have the bound

(95) $$c(\delta)2^{k(n+\delta)}$$

provided $0 < \delta < \epsilon$. If $0 < \theta < 1$ then from (94) and (95) we have the bound

(96) $$c(\epsilon_3,\epsilon_4)^{1-\theta}c(\delta)^{\theta}2^{k(\theta(n+\delta)-(1-\theta)\epsilon_4)}\frac{|y-y'|^{\epsilon_3(1-\theta)}}{(1+|x-y|)^{(n+\epsilon_4)(1-\theta)+\epsilon_3(1-\theta)}}.$$

We are done if we can show that given $0 < \epsilon' < \epsilon \le 1$ we can choose ϵ_3, ϵ_4, δ, and θ such that

(97) $$(1-\theta)\epsilon_3 = \epsilon',$$

(98) $$(n+\epsilon_4)(1-\theta) > n, \qquad \theta(n+\delta) - (1-\theta)\epsilon_4 > 0.$$

We can rewrite (98) as

(99) $$n < \frac{1-\theta}{\theta}\epsilon_4 < n+\delta.$$

The first thing to do is to take a δ, $0 < \delta < \epsilon$, and fix it. From (97) we see that the admissible range for $(1-\theta)/\theta$ is $(\epsilon'/(\epsilon-\epsilon'),\infty)$. If ϵ_4 is small enough we see that $((\epsilon-\epsilon')/\epsilon')\epsilon_4 < n$. Take such an ϵ_4 and fix it. For this value of ϵ_4 there is an admissible value of θ such that $((1-\theta)/\theta)\epsilon_4 > n+\delta$. By continuity there is an admissible value of θ such that (99) is satisfied. That value of θ determines an ϵ_3 for which (97) is satisfied. For δ, ϵ_3, ϵ_4, and θ so chosen we have the bound

$$c(\epsilon')2^{-|k|\sigma}\frac{|y-y'|^{\epsilon'}}{(1+|x-y|)^{n+\epsilon'+\tau}}$$

where $\sigma = \theta(n+\delta) - (1-\theta)\epsilon_4 > 0$ and $\tau = (1-\theta)(n+\epsilon_4) - n > 0$.

This completes the proof of Lemma A.3. ∎

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