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EMBEDDING METRIC ABSOLUTE BOREL SETS IN COMPLETELY REGULAR SPACES

BY

STEPHEN WILLARD (CLEVELAND, OHIO)

Introduction. All spaces we consider are assumed to be completely regular, and "completely regular" here implies "Hausdorff". We denote by ω_1 the first uncountable ordinal and reserve symbols like a, β and γ to mean ordinals less than ω_1.

It is a classical result that the following are equivalent for a metric space X (see [3] or [4] for notation and terminology):

(a) X is an absolute G_δ;

(b) X is a G_δ in βX;

(c) X is a G_δ in some compactification K of X;

(d) X is a G_δ in every compactification K of X;

(e) X is a G_δ in its closure in every completely regular space in which it is embedded.

In [4] we generalized the equivalence of (a), (b), and (c) to arbitrary G_a sets.

In this paper (1), in Section 1, we prove a lemma (1.1) which enables us to simplify considerably the treatment given in [5] and extend the generalization to parts (d) and (e). In fact, the lemma provides interesting information about the "absolute Borel" behavior of non-metric spaces as well; one of its consequences is: *Y is a G_a in βY if and only if Y is a G_a in its closure in every completely regular space in which it is embedded.*

Thus, predictably, βX assumes the role for completely regular spaces that the metric completion $\hat{X}$ plays for metric spaces, at least where "absolute Borel" properties involving the open sets are concerned. Hope for similar theorems involving the closed sets is considerably dimmed by a result of Choquet (namely, every K-analytic set is Lindelöf) from which it follows, for example, that whenever a metric space X is an F_σ in βX, it must be separable. Since non-separable metric absolute F_σ

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sets abound, the best one could hope for is an embedding theorem restricted to the separable or Lindelöf spaces. Here a result of Frolík [1] is of interest: a metric space X is a separable absolute Borel set if and only if X is Baire (generated from the compact G_δ sets by countable union and intersection) in βX.

In Section 2 we provide the obvious extension to Souslin $\mathcal{G}$ sets. The theorems of this section are stated, as nearly as possible, in the same order and form as the corresponding theorems of Section 1, and the proofs, being essentially the same, are either abbreviated or omitted altogether. Again, the principal result is the following: *Y is Souslin $\mathcal{G}$ in βY if and only if Y is Souslin $\mathcal{G}$ in its closure in every completely regular space in which it is embedded.*

1. Absolute Borel sets. We need some terminology. If $\mathcal{H}$ is any family of sets in a space Y, we define the collection $\mathcal{H}_\beta$ of sets for $\beta < \omega_1$, inductively, as follows: $\mathcal{H}_\beta = \mathcal{H}$, if $\beta = 0$; $\mathcal{H}_\beta$ consists of all countable unions and intersections of sets from $\bigcup_{\gamma<\beta} \mathcal{H}_\gamma$, if $\beta > 0$.

The G_a sets in Y, when Y is a topological space, are the sets in the collection g_a, where g is the collection of open sets. We will use the fact, easily proved by induction, that any G_a set belongs to $\mathcal{H}_a$ for some countable subcollection $\mathcal{H}$ of g.

Our proof of the main lemma (1.1) makes use of the following well-known fact:

1.0. Lemma. *If h is a continuous mapping of a space Z into a space K whose restriction to a dense set Y is a homeomorphism, then h carries $Z-Y$ into $K-h(Y)$.*

For a proof see [2], Lemma 6.11.

1.1. Lemma. *If Y is a dense subset of Z and h is a closed continuous map of Z onto K such that $h|Y$ is a homeomorphism, then to each G_a set B in Z we can assign a G_a set B^* in K such that $h^{-1}(B^*) \subset B$ and $Y \cap h^{-1}(B^*) = Y \cap B$.*

Proof. The proof proceeds by induction. Let B be open in Z and define $B^* = K-h(Z-B)$. Then B^* is open in K since h is a closed map. Moreover, if $x \in h^{-1}(B^*)$, we must have $h(x) \notin h(Z-B)$, so $x \notin Z-B$, from which we obtain $h^{-1}(B^*) \subset B$. Finally, we have $Y \cap h^{-1}(B^*) = Y \cap B$, for one inclusion is obvious since $h^{-1}(B^*) \subset B$, while on the other hand, let $x \in Y \cap B$ and suppose $h(x) = h(y)$. It follows that $y \in Y$ since otherwise $h(Y) \cap h(Y-Z) \neq \emptyset$, which violates the Lemma 1.0 stated above, and hence $y = x$ since otherwise h is not one-one on Y. Thus if $x \in Y \cap B$, $h(x) \notin h(Z-B)$, therefore $h(x) \in K-h(Z-B)$, whence $x \in h^{-1}(B^*)$. This completes the proof of the inductive assertion for the case $a = 0$.

Suppose now that all G_β sets for $\beta < \alpha$ have the property of the theorem, and let B be a G_α set in Z. If $B = \bigcap_{n=1}^{\infty} B_n$, where each B_n is a G_{β_n} set in Z for some $\beta_n < \alpha$, then we can set $B^* = \bigcap_{n=1}^{\infty} B_n^*$, while if $B = \bigcup_{n=1}^{\infty} B_n$, we set $B^* = \bigcup_{n=1}^{\infty} B_n^*$, where in either case the B_n^* are G_{β_n} sets in K each having the appropriate properties relative to the corresponding B_n by the inductive hypothesis. It then easily follows, in either case, that B^* is a G_α set in Z with the properties that $h^{-1}(B^*) \subset B$ and $h^{-1}(B^*) \cap Y = B \cap Y$.

This completes the proof of Lemma 1.1.

We are now prepared to prove the main theorem. Throughout, if $f: Y \to K$ is a map of Y into the compact space K, f^β will denote the well-known Stone extension map $f^\beta: \beta Y \to K$. We will use properties of f^β and βY without apology, referring the reader unfamiliar with these properties to [2].

1.2. Theorem. *If Y is a G_α in βY, then Y is a G_α in every compactification K of Y.*

Proof. Let $f: Y \to K$ be the embedding of Y into K. Since Y is a G_α in βY and dense in βY, while f^β is closed, continuous and a homeomorphism when restricted to Y, we can apply Lemma 1.1, which then asserts the existence of a G_α subset Y^* of K such that $(f^\beta)^{-1}(Y^*) \cap Y = Y$, while $(f^\beta)^{-1}(Y^*) \subset Y$. It follows that $Y^* = f^\beta(Y)$, which proves the theorem.

Note that the following corollary to Theorem 1.2 generalizes a well-known result, namely that a locally compact space X (i.e. a space X which is open in βX) is open in its closure in whatever Hausdorff space it is embedded, except that in the result below, we allow only completely regular embeddings of X.

1.3. Corollary. *If Y is a G_α in βY, then Y is a G_α in its closure in every space Z in which Y is embedded. Hence, Y is absolute Borel (among completely regular spaces).*

Proof. If Y is embedded in Z, then $\beta(\mathrm{Cl}_Z Y)$ is a compactification of Y, so if Y is a G_α in βY, it is a G_α in $\beta(\mathrm{Cl}_Z Y)$ and thus in $\mathrm{Cl}_Z Y$. This completes the proof of 1.3.

1.4. Corollary. *For a metric space X, the following are equivalent:*

(a) *X is absolute Borel (among metric spaces);*

(b) *X is an absolute G_α, for some α;*

(c) *X is a G_α in βX;*

(d) *X is a G_α in K, whenever K is a compactification of X;*

(e) *X is a G_α in every perfectly normal space Y in which it is embedded;*

(f) *X is a G_a in $\mathrm{Cl}_Z X$ whenever X is embedded in the completely regular space Z;*

(g) *X is absolute Borel (among completely regular spaces).*

The scheme of proof is (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a), while also (d) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (a).

Proof. (a) is equivalent to (b) by the classical result that a metric space is a G_a in its completion if and only if it is an absolute G_a.

If X is an absolute G_a, then X is a G_a in its completion $\hat{X}$ which, by a classical result due to Čech, is a G_δ in $\beta\hat{X}$. Thus, X is a G_a in $\beta\hat{X}$. Since $\beta\hat{X}$ is a continuous image of βX by a map whose restriction to X is the identity, by taking inverses under this map, X must be a G_a in βX. Thus, (b) implies (c).

That (c) implies (d) follows from Theorem 1.2.

The same argument used in Corollary 1.3 can easily be applied to yield the conclusion that (d) implies (e), and (e) obviously implies (a).

Again, the argument of Corollary 1.3 shows that (d) implies (f), while certainly (f) implies (g), and (g) implies (a) are clear.

This completes the proof that the seven properties listed are equivalent.

2. Analytic sets. Let $\mathscr{H}$ be a collection of subsets of Y. A set X in Y will be called *Souslin $\mathscr{H}$* if and only if X is a result of the operation $\mathscr{A}$ (see [4]) carried out on sets of $\mathscr{H}$. It is a classical result, easily proved with the aid of Lavrentieff's Theorem, that a subset of a complete metric space Y is Souslin $\mathscr{G}$, where $\mathscr{G}$ is the collection of open sets in Y, if and only if it is Souslin g in whatever metric space it is embedded. Of course, in a metric space, if $\mathscr{F}$ denotes the collection of closed sets, "Souslin $\mathscr{G}$" and "Souslin $\mathscr{F}$" refer to the same collection of sets. In more general spaces (in particular, in non-perfectly normal spaces) this need not be so.

Our purpose here is to show that βX plays the role of the complete metric spaces, in the passage to more general spaces, at least if one deals with the Souslin $\mathscr{G}$ sets. The key is the following restatement of Lemma 1.1 for Souslin sets:

LEMMA 2.1. *If Y is a dense subset of Z and h is a closed continuous map of Z onto K such that $h|Y$ is a homeomorphism, then to each Souslin $\mathscr{G}$ set B in Z we can assign a Souslin $\mathscr{G}$ set B^* in K such that $h^{-1}(B^*) \subset B$ and $Y \cap h^{-1}(B^*) = Y \cap B$.*

Proof. Let S denote all sequences of positive integers. Let B be a Souslin $\mathscr{G}$ set in Z, say,

$$B = \bigcup_{\{n_1, n_2, \dots\} \in S} \bigcap_{k=1}^{\infty} B_{n_1 \dots n_k},$$

where $B_{n_1 \ldots n_k}$ is an open set for each k-tuple $n_1, \ldots, n_k$. For each $B_{n_1 \ldots n_k}$, define

$$B^*_{n_1 \ldots n_k} = K - h(Z - B_{n_1 \ldots n_k}).$$

Finally, let

$$B^* = \bigcup_{\{n_1, n_2, \ldots\} \in S} \bigcap_{k=1}^{\infty} B^*_{n_1 \ldots n_k}.$$

Then B^* is Souslin $\mathscr{G}$ in K and routine checking verifies that B^* has the properties asked for in the lemma.

2.2. THEOREM. *If Y is Souslin $\mathscr{G}$ in βY, then Y is Souslin $\mathscr{G}$ in every compactification of Y.*

Proof. If $f: Y \to K$ is the embedding of Y into K, and f^β is the Stone extension of f, then Y is dense in βY and Souslin $\mathscr{G}$, and f^β is a closed and continuous map whose restriction to Y is a homeomorphism, so by Lemma 2.1, there is a Souslin g set Y^* in K such that $(f^\beta)^{-1}(Y^*) \subset Y$, and $Y \cap (f^\beta)^{-1}(Y^*) = Y$. It follows that $Y^* = f^\beta(Y)$, which proves the theorem.

2.3. COROLLARY. *If Y is Souslin $\mathscr{G}$ in βY, then Y is Souslin $\mathscr{G}$ in its closure in every space Z in which it is embedded.*

We remark that it is well known that for separable metric spaces the analytic sets, that is, the sets which are continuous images of Borel sets, are precisely the Souslin $\mathscr{G}$ sets. This fails in more generality, however, although in any metric space, every analytic set is Souslin $\mathscr{G}$.

The following result is now clear:

2.4. THEOREM. *For a metric space X the following propositions are equivalent:*

(a) *X is Souslin $\mathscr{G}$ in every metric embedding;*

(b) *X is Souslin $\mathscr{G}$ in its metric completion $\hat{X}$;*

(c) *X is Souslin $\mathscr{G}$ in βX;*

(d) *X is Souslin $\mathscr{G}$ in every compactification K of X;*

(e) *X is Souslin $\mathscr{G}$ in every perfectly normal space in which it is embedded;*

(f) *X is Souslin $\mathscr{G}$ in $\mathrm{Cl}_Z X$ whenever X is embedded in a completely regular space Z.*

Proof. The scheme of proof is (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a), while also (d) $\Rightarrow$ (f) $\Rightarrow$ (a). The details are in no essential way different from the details of the corresponding parts of the proof of corollary in 1.4.

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CASE WESTERN RESERVE UNIVERSITY

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