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CYCLIC STRUCTURES OF GEOMETRIC OBJECTS INVOLVING A CONNECTION AND LIE DERIVATIVES

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1. In this paper certain cyclic sequences generated by geometric objects involving a connection are obtained and their certain properties are discussed in the first two sections. Such sequences are then applied to deduce Lie derivatives and some known formulae. In this connection certain Lie derivatives, called here *primary Lie derivatives*, have been introduced and made use of.

In a previous paper [2] the following algebraic system and its application in Riemannian geometry was considered.

Let S be an algebraic system generated by a single element having the following properties:

(1) Corresponding to every element a of S there exist two elements, denoted by a^* and a', which are governed by the involutory properties $a^{**} = a'' = a$.

(2) A commutative composition is defined in S, where every pair of elements a, b of S is composed to form an element $a\circ b$ of S with the properties:

$a\circ a = a$; if $a\circ b = a\circ c$, then $b = c$; $(a\circ b)^* = a^*\circ b^*$; $(a\circ b)' = a'\circ b'$.

Elements a^* and a' are called the *associate* and the *conjugate* of a, respectively. The symbols $*$ and $'$ appearing as upper indices may be regarded as operations by which the associates and the conjugates are formed. An element which is equal to its associate is called a *self-associate*, shortly, s.a. element. Similarly, for the conjugate — a *self-conjugate*, shortly, s.c. element. E.g., elements $a\circ a^*$ and $a\circ a'$ are such.

In the paper [2] we were looking into the problem of finding, if possible, an element of S which is both s.a. and s.c., and as a result we were led to the following property stated here in the form of a theorem:

THEOREM 1. *Let S be an algebraic system satisfying properties* (1) *and* (2). *Assume that S has an element, say u, which is both s.a. and s.c. If the sequence of elements generated by a,*

(3) $$t_1 = a,\ t_2 = a^*,\ t_3 = a^{*\prime},\ t_4 = a^{*\prime *},\ t = a^{*\prime *\prime}, \ldots,$$

happens to be a cyclic sequence having an even number p of terms, then under certain circumstances u is given by

$$u = t_r \circ t_{r+p/2}, \quad r = 1, 2, 3, \ldots \tag{4}$$

provided that $p/2$ is even.

Application of this theorem was made in Riemannian geometry in the following way. Let an n-dimensional Riemannian space with the fundamental tensor g_{ij} admit an arbitrary affine connection represented by the coefficients Γ_{ij}^h and let the covariant derivative with respect to Γ_{ij}^h be denoted by comma followed by indices. Further, let L_{ij}^h also represent a connection. Put

$$\begin{aligned} a &= \Gamma_{ij}^h, \quad a^* = \Gamma_{ij}^h + g^{ht} g_{it,j}, \quad a' = \Gamma_{ji}^h, \\ b &= L^h{}_{ij}, \quad a \circ b = \tfrac{1}{2}(\Gamma_{ij}^h + L^h{}_{ij}). \end{aligned} \tag{1.1}$$

This representation can be considered as giving a system of affine connections satisfying properties (1) and (2) of an algebraic system S as defined above. Evidently, Γ_{ij}^h is s.a. if the covariant derivative of g_{ij} with respect to it vanishes; it is s.c. if it is symmetric in the two lower indices; and it is both s.a. and s.c. if it is the Christoffel symbol $\{{}_{ij}^{h}\}$. In order to construct sequence (3) it will be advantageous to adopt, besides a and a', the following notations:

$$\Gamma_{ij}^h - \Gamma_{ji}^h = \Lambda_{ij}^h, \quad g^{ht} g_{it,j} = \alpha_{ij}^h = \alpha,$$
$$g^{ht} g_{ij,t} = \gamma_{ij}^h = \gamma, \quad g^{ht} g_{im} \Lambda_{ij}^m = \beta_{ij}^h = \beta.$$

Let us also write

$$\alpha_{ji}^h = \alpha_c, \quad \beta_{ji}^h = \beta_c \quad (\gamma_{ji}^h = \gamma_{ij}^h = \gamma).$$

Sequence (3) can, as in [2], be represented in the Riemannian geometry as follows:

$$\left\{\begin{aligned} & t_1 = a, \quad t_2 = a + \alpha, \quad t_3 = a' + \alpha_c, \quad t_4 = a + \alpha + \beta - \gamma, \\ & t_5 = a' + \alpha_c + \beta_c - \gamma, \quad t_6 = a + \alpha + \alpha_c + \beta + \beta_c - \gamma, \\ & t_7 = a' + \alpha + \alpha_c + \beta + \beta_c - \gamma, \quad t_8 = a' + \alpha_c + \beta + \beta_c - \gamma, \\ & t_9 = a + \alpha + \beta + \beta_c - \gamma, \quad t_{10} = a' + \alpha_c + \beta_c, \\ & t_{11} = a + \alpha + \beta, \quad t_{12} = a'. \end{aligned}\right. \tag{1.2}$$

Sequence (1.2) is a cyclic sequence of 12 terms which are not necessarily all distinct.

However, it should be mentioned that in the system described by (1.1), the representation of (3) by (1.2) is not unique. From (1.1) it follows that the terms of (3) are all connections and this fact had indeed been mainly taken into consideration in obtaining representation (1.2). For a different representation every term of (3) can be regarded, by virtue of (1.1), as a geometric object $f(x)$ involving a connection a and can, therefore, be denoted by $f = f(a)$. We can then write $f^* = f(a^*)$ and $f' = f(a')$. Of course, we can also write $f(a^*) = \varphi(a)$, $f(a') = \psi(b)$, so that $f^{*\prime} = \varphi(a')$ and $f'^{*} = \psi(b^*)$. With the help of this understanding we find, by a calculation,

$$a^* = -a, \quad \gamma^* = -\gamma, \quad \beta^* = a+\beta-\gamma,$$

$$a' = a - a' + a + \beta, \quad \gamma' = \gamma - \beta - \beta_c, \quad \beta' = -\beta.$$

So

$$a'^* = a' + a_c, \quad a^{*\prime} = a' + a' = a + a + \beta, \quad \text{etc.}$$

Using these results, we find that a representation of (3) under system (1.1) is given by

$$(1.3) \quad \begin{cases} t_1 = a, \quad t_2 = a + a, \quad t_3 = a + a + \beta, \quad t_4 = a + a + \beta - \gamma, \\ t_5 = a + a + \beta + \beta_c - \gamma, \quad t_6 = a + a + a_c + \beta + \beta_c - \gamma, \\ t_7 = a' + a + a_c + \beta + \beta_c - \gamma, \quad t_8 = a' + a_c + \beta + \beta_c - \gamma, \\ t_9 = a' + a_c + \beta_c - \gamma, \quad t_{10} = a' + a_c + \beta_c, \quad t_{11} = a' + a_c, \\ t_{12} = a'. \end{cases}$$

This is also a cyclic sequence of 12 terms, but different from (1.2), and the terms are, as before, not necessarily all distinct. Finally, it can be observed that both sequences (1.2) and (1.3) satisfy the condition imposed on (3) in theorem 1. Therefore, result (4) of the theorem holds, i.e.,

$$(1.4) \qquad \left\{ {h \atop ij} \right\} = \frac{1}{2}(t_r + t_{r+b}) = \frac{1}{2}(a + a' + a + a_c + \beta + \beta_c - \gamma)$$

$$r = 1, 2, 3, \ldots$$

Above-mentioned results can be stated in the form of the following

THEOREM 2. *Any connection which is not both s.a. and s.c. can be made to generate a cyclic sequence of connections from which the Christoffel symbol, which is the connection both s.a. and s.c., can be obtained as shown by formula* (1.4). *However, the generated sequence is not unique and its terms,* 12 *in number, are not necessarily all distinct. If the connection is s.a. or s.c., the number of distinct terms cannot exceed* 6.

2. Throughout the paper we suppose that Γ_{ij}^h is any connection ether than the Christoffel symbol, that the covariant derivative with rospect to it is denoted by a comma followed by indices, and that the covariant derivative with respect to the Christoffel symbol is denoted by a crooked bracket followed by indices. For future reference to the sequence generated by Γ_{ij}^h let us choose one of the two sequences (1.2) or (1.3), say (1.3), together with the method used to obtain it and stick to it always. It is convenient to denote the terms of this sequence by

$$(2.1)\qquad {}^1\Gamma^h_{ij},\ {}^2\Gamma^h_{ij},\ {}^3\Gamma^h_{ij},\ \ldots,\ {}^{11}\Gamma^h_{ij},\ {}^{12}\Gamma^h_{ij}.$$

Let $F(x)$ be any geometric object involving Γ^h_{ij}. Let us write, as before, $F = F(\Gamma^h_{ij})$. Then, evidently, F generates a cyclic sequence whose successive terms can be denoted by

$$(2.2)\qquad {}^1F = F({}^1\Gamma^h_{ij}),\ {}^2F = F({}^2\Gamma^h_{ij}), \ldots, {}^{12}F = F({}^{12}\Gamma^h_{ij}).$$

For example, let $F(\Gamma^h_{ij}) = T^i_{j,k}$ be the covariant derivative of a tensor T^i_j with respect to Γ^h_{ij}. Then, the sequence generated by $T^i_{j,k}$ consists of the covariant derivatives of T^i_j with respect to connections (2.1). That is to say, if, in accordance with (2.2), the successive terms of the sequence are written as ${}^1T^i_{j,k}, {}^2T^i_{j,k}, \ldots, {}^{12}T^i_{j,k}$, the successive terms of the cyclic sequence are

$$\begin{aligned}
{}^1T^i_{j,k} &= \partial_k T^i_j + T^s_j \Gamma^{\ i}_{sk} - T^i_s \Gamma^{\ s}_{jk}, \ldots,\\
{}^7T^i_{j,k} &= \partial_k T^i_j + T^s_j(T^i_{ks} + \alpha^i_{sk} + \alpha^i_{ks} + \beta^i_{sk} + \beta^i_{ks} - \gamma^i_{sk}) -\\
&\qquad - T^i_s(\Gamma^{\ s}_{kj} + \alpha^s_{jk} + \alpha^s_{kj} + \beta^s_{jk} + \beta^s_{kj} - \gamma^s_{jk}), \ldots,\\
{}^{12}T^i_{j,k} &= \partial_k T^i_j + T^s_j \Gamma^{\ i}_{ks} - T^i_s \Gamma^{\ s}_{kj}.
\end{aligned}$$

It would follow from this sequence that

$$(2.3)\qquad \frac{1}{2}({}^rT^i_{j,k} + {}^{r+6}T^i_{j,k}) = \{T^i_j\}_{,k} \qquad r = 1, 2, 3, \ldots$$

However, it must be clearly understood that this formula holds if Γ^h_{ij} occurs linearly in $T^i_{j,k}$. In fact, we have the following general formula, provided Γ^h_{ij} occurs linearly in F:

$$(2.4)\qquad \frac{1}{2}\big({}^rF(\Gamma^h_{ij}) + {}^{r+6}F(\Gamma^h_{ij})\big) = F(\{{}^{\,h}_{ij}\}), \qquad r = 1, 2, 3, \ldots$$

Consider, for example, the case of a repeated covariant derivative of a tensor, say $T^i_{j,kl}$, in which Γ^h_{ij} occurs quadratically. In this case, although $T^i_{j,kl}$ generates, as before, a cyclic sequence of 12 terms, we cannot infer from the sequence a property of type (2.3), i.e.,

$$\frac{1}{2}({}^rT^i_{j,kl} + {}^{r+6}T^i_{j,kl}) \neq \{T^i_j\}_{kl}.$$

We can, however, arrive to the result $\{T^i_j\}_{kl}$ in the following manner:

Let the terms of the cyclic sequence generated by $T^i_{j,k}$ be denoted by $p_1, p_2, \ldots, p_{12}$ and let us write for the moment

$$\frac{1}{2}(p_r + p_{r+6}) = K^i_{jk}, \qquad r = 1, 2, 3, \ldots$$

Of course, $K^i_{jk} = \{T^i_j\}_k$. Further, let the terms of the cyclic sequence generated by $K^i_{jk,l}$ be denoted by $q_1, q_2, \ldots, q_{12}$. Then, for $r = 1, 2, 3, \ldots,$

$$\frac{1}{2}(q_r + q_{r+6}) = \{K^i_{jk}\}_l = \{T^i_j\}_{kl}.$$

This procedure can be extended to $T^i_{j,klm}, \ldots$ It can, however, be observed that the above-mentioned method could be applied solely because Γ^h_{ij} was not allowed to appear explicitly in K^i_{jk}. However, even such methods cannot be applied in all cases where Γ^h_{ij} does not occur linearly. For example, from the curvature tensor $\Gamma_{ijk}^{\;h}$ defined by

$$\Gamma_{ijk}^{\;h} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^{\;h}_{sj}\Gamma^{\;s}_{ik} - \Gamma^{\;h}_{sk}\Gamma^{s}_{ij}, \tag{2.5}$$

which occurs in $T^h_{,jk} - T^h_{,kj} = -T^t\Gamma^{\;h}_{tjk} - T^h_{,s}\Lambda^s_{jk}$, the Riemannian curvature tensor defined, as in Eisenhart [1], by

$$R^h_{ijk} = \partial_j \begin{Bmatrix} h \\ ik \end{Bmatrix} - \partial_k \begin{Bmatrix} h \\ ij \end{Bmatrix} + \begin{Bmatrix} h \\ sj \end{Bmatrix}\begin{Bmatrix} s \\ ik \end{Bmatrix} - \begin{Bmatrix} h \\ sk \end{Bmatrix}\begin{Bmatrix} s \\ ij \end{Bmatrix} \tag{2.6}$$

cannot be obtained, as before, by means of sequences. However, the results obtained in this section can be stated in the form of the following

THEOREM 3. *Any geometric object involving a connection other than the Christoffel symbol can be made to generate a cyclic sequence of* 12 *such objects, not necessarily all distinct, by allowing the connection involved to follow the procedure described in theorem* 2. *If the connection occurs linearly, then from the sequence we can obtain a geometric object in which the involved connection has been replaced by the Christoffel symbol, as is shown by formula* (2.4).

We give two useful formulae of which use will be made later.

First, construct the cyclic sequence generated by $v^h_{,ij}$, where v^h is an arbitrary vector. According to the adopted notation, the terms are ${}^1v^h_{,ij}$, ${}^2v^h_{,ij}, \ldots, {}^{12}v^h_{,ij}$. For the sake of definiteness, we shall choose in the rest of the paper a fixed value of r in (1.4), say $r = 1$. It can be seen after straightforward calculation that

$$\frac{1}{2}({}^1v^h_{,ij} + {}^7v^h_{,ij}) = \{v^h\}_{ij} + v^t\Bigg[\left(\begin{Bmatrix} h \\ sk \end{Bmatrix}\begin{Bmatrix} s \\ tj \end{Bmatrix} - \begin{Bmatrix} h \\ ts \end{Bmatrix}\begin{Bmatrix} s \\ jk \end{Bmatrix}\right) + (\Gamma^{\;h}_{sk}\Gamma^{s}_{tj} - \Gamma^{h}_{ts}\Gamma^{s}_{jk}) +$$
$$+ \left(\begin{Bmatrix} h \\ ts \end{Bmatrix}\Gamma^{\;s}_{jk} + \Gamma^h_{ts}\begin{Bmatrix} s \\ jk \end{Bmatrix}\right) - \left(\begin{Bmatrix} h \\ sk \end{Bmatrix}\Gamma^s_{tj} + \Gamma^{\;h}_{sk}\begin{Bmatrix} s \\ tj \end{Bmatrix}\right).$$

Put

$$S^h_{ij} = {}^1\Gamma^{\;h}_{ij} - {}^7\Gamma^{\;h}_{ij}. \tag{2.7}$$

Then

$$\begin{Bmatrix} h \\ ij \end{Bmatrix} = \frac{1}{2}({}^1\Gamma^{\;h}_{ij} + {}^7\Gamma^{\;h}_{ij}) = \frac{1}{2}(2\Gamma^{\;h}_{ij} - S^h_{ij}).$$

It is now a matter of a simple algebra to see that

$$\text{(2.8)} \qquad \frac{1}{2}\,({}^{1}v_{,ij}^{\;h}+{}^{7}v_{,ij}^{h}) = \{v^h\}_{ij}+\tfrac{1}{4}v^t(S_{sj}^h S_{ti}^s - S_{ts}^h S_{ij}^s).$$

Secondly, let L_{ij}^h be a connection, let $\nabla_{ij}^h = \frac{1}{2}(\Gamma_{ij}^h + l_{ij}^h)$, and let L_{ikj}^h and ∇_{ikj}^h be curvature tensors formed from L_{ij}^n and ∇_{ij}^h in the same manner as Γ_{ijk}^h from Γ_{ij}^h, as defined in (2.5). If we now put $Q_{ij}^h = \Gamma_{ij}^h - L_{ij}^h$, it can be seen that

$$\text{(2.9)} \qquad \nabla_{ijk}^h - \frac{1}{2}\,(\Gamma_{ijk}^h + L_{ijk}^h) = \frac{1}{4}\,(Q_{sk}^h Q_{ij}^s - Q_{sj}^h Q_{ik}^s).$$

Now consider the cyclic sequence generated by Γ_{ijk}^h and, as usual, denote its terms by ${}^{1}\Gamma_{ijk}^h, {}^{2}\Gamma_{ijk}^h, \ldots, {}^{12}\Gamma_{ijk}^h$.

It then follows from (2.9) and (2.7) that

$$\text{(2.10)} \qquad \frac{1}{2}\,({}^{1}\Gamma_{ijk}^h + {}^{7}\Gamma_{ijk}^h) = R_{ijk}^h - \frac{1}{4}\,(S_{sk}^h S_{ij}^s - S_{sj}^h S_{ik}^s),$$

where R_{ijk}^h is given by (2.6).

3. It is known that the Lie derivative of a tensor with respect to a vector is generally expressed in terms of the covariant derivatives with respect to the Christoffel symbol, although the Lie derivative is independent with respect to the symbols appearing in the covariant derivatives (cf. [3], p. 14-20), e.g.,

$$\text{(3.1)} \qquad \underset{v}{\mathscr{L}} T_j^i = v^t\{T_j^i\}_t - T_j^t\{v^i\}_t + T_t^i\{v^t\}_j = v^t\partial_t T_j^i - T_j^t\partial_t v^i + T_t^i\partial_j v^t.$$

That the terms in (3.1) involving Christoffel symbol cancel out each other is due to the property that the symbol is s.c. and has nothing to do with the property that the symbol is s.a. Therefore, the Lie derivative remains unaltered if it is expressed in terms of the covariant derivatives with respect to any symmetric connection.

Let us discuss the situation in some detail. Consider the geometric object (its form can be compared with that of (3.1))

$$\text{(3.2)} \qquad \underset{v}{N} T_{j_1\ldots j_q}^{i_1\ldots i_p} = v^t T_{j_1\ldots j_q,t}^{i_1\ldots i_p} - T_{j_1\ldots j_q}^{s\ldots i_p} v_{,s}^{i_1} - \ldots - T_{j_1\ldots j_q}^{i_1\ldots s} v_{,s}^{i_p} + \\ + T_{s\ldots j_q}^{i_1\ldots i_p} v_{,j_1}^{s} + \ldots + T_{j_1\ldots s}^{i_1\ldots i_p} v_{,j_q}^{s},$$

where $T_{j_1\ldots j_q}^{i_1\ldots i_p}$ is a tensor and a comma followed by indices denotes covariant derivative with respect to an arbitrary connection Γ_{ij}^h. It can be seen that $p+q$ Γ's occur in the first term of the right-hand member of (3.2) and that the same number of Γ's appear in the same number of remaining terms, one in each, with the positive and negative terms equally distributed. It can be easily seen that these terms would cancel out provided that Γ_{ij}^h were symmetric, and a Lie derivative would follow. Now $\frac{1}{2}({}^{1}\Gamma_{ij}^h + {}^{7}\Gamma_{ij}^h)$ and $\frac{1}{2}({}^{1}\Gamma_{ij}^h + {}^{12}\Gamma_{ij}^h)$ are symmetric connections, although they are, in general, different. But from the sequence generated by the geometric object $\underset{v}{N}T^{i_1\cdots i_p}_{j_1\cdots j_q}$ defined in (3.2) it would follow that

$$\text{(3.2a)}\quad \underset{v}{\mathscr{L}}T^{i_1\cdots i_p}_{j_1\cdots j_q} = \tfrac{1}{2}({}^1\underset{v}{N}T^{i_1\cdots i_q}_{j_1\cdots j_q} + {}^7\underset{v}{N}T^{i_1\cdots i_p}_{j_1\cdots j_q}) = \tfrac{1}{2}({}^1\underset{v}{N}T^{i_1\cdots i_p}_{j_1\cdots j_q} + {}^{12}\underset{v}{N}T^{i_1\cdots i_p}_{j_1\cdots j_q}).$$

This shows that

$${}^7\underset{v}{N}{}^{i_1\cdots i_p}_{j_1\cdots j_q} = {}^{12}\underset{v}{N}{}^{i_1\cdots i_p}_{j_1\cdots j_q}.$$

This is a speciality of the form of the geometric object (3.2). As a matter of fact, if for a moment terms of the sequence generated by $\underset{v}{N}{}^{i_1\cdots i_p}_{j_1\cdots i_q}$ are denoted by $t_1, t_2, \ldots, t_{12}$, it will be seen that they are not all distinct, but appear as $t_1, t_2, t_3, t_4 = t_3, t_5 = t_2, t_6 = t_1, t_7, t_8, t_9, t_{10} = t_9, t_{11} = t_8, t_{12} = t_7$.

It is thus seen that the third and the ninth terms are s.a. and this makes the seventh and the twelfth terms equal, and conversely. These results may be stated in the form of the following

THEOREM 4. *The geometric object* $\underset{v}{N}T^{i_1\cdots i_p}_{j_1\cdots j_q}$ *defined by* (3.2), *where* $T^{i_1\cdots i_p}_{j_1\cdots j_q}$ *is a tensor and a comma followed by indices denotes covariant derivative with respect to an arbitrary connection, has special significance in the formation of the Lie derivative* $\mathscr{L}T^{i_1\cdots i_p}_{j_1\cdots j_q}$ *in which the sequence generated by it has its seventh and twelfth terms equal or, which comes to the same thing, its third and ninth terms self-associate.*

Definition. If a Lie derivative is capable of generating a cyclic sequence of 12 terms, it will be called a *primitive Lie derivative* and will be denoted by $\mathscr{P}\mathscr{L}$, otherwise it will be denoted, as usual, by $\mathscr{L}$. Thus we write a Lie derivative as $\mathscr{L}T^{\cdot\cdot}_{\cdot\cdot}$ if $T^{\cdot\cdot}_{\cdot\cdot}$ involves either no connection or involves only the Christoffel symbol, otherwise we write it as $\mathscr{P}\mathscr{L}T^{\cdot\cdot}_{\cdot\cdot}$

E.g., $\mathscr{L}R^h_{ijk}$, but $\mathscr{P}\mathscr{L}\Gamma_{ijk}^{\ \ \ h}$, where R^h_{ijk} and $\Gamma_{ijk}^{\ \ \ h}$ are defined by (2.6) and (2.5). In a simplified form we have

$$\text{(3.3)}\quad \underset{v}{\mathscr{P}\mathscr{L}}\Gamma_{ijk}^{\ \ \ h} = v^t\partial_t\Gamma_{ijk}^{\ \ \ h} - \Gamma_{ijk}^{\ \ \ s}\partial_s v^h + \Gamma_{sjk}^{\ \ \ h}\partial_i v^s + \Gamma_{isk}^{\ \ \ h}\partial_j v^s + \Gamma_{ijs}^{\ \ \ h}\partial_k v^s.$$

Throughout the rest of the paper we assume that the connection $\Gamma^h_{ij} \neq \{^{\,h}_{ij}\}$ is symmetric, the covariant derivative with respect to this connection is denoted, as said before, by a comma followed by indices, and that the $\mathscr{L}$ as well as the $\mathscr{P}\mathscr{L}$ derivatives are always taken with respect to a vector v^h which, for the sake of brevity and without any ambiguity, shall not be specifically indicated in writing below $\mathscr{L}$ and $\mathscr{P}\mathscr{L}$. It should, however, be mentioned that since Γ^h_{ij} is symmetric, the tensor $\Gamma_{ijk}^{\ \ \ h}$, like R^h_{ijk}, has the following properties:

It satisfies the cyclic law with respect to its three lower indices, it satisfies the law analogous to the Bianchi identities, and it satisfies the formulae analogous to ones known as Ricci identities.

We now formulate a number of formulae. We start by defining the following $\mathscr{PL}$ derivative:

$$\mathscr{PL}\Gamma^h_{ij} = v^h_{,ij} + v^t \Gamma^{\ h}_{itj}. \tag{3.4}$$

Let us justify the definition. It follows from (3.4) that

$$\frac{1}{2}\,({}^1\mathscr{PL}\Gamma^h_{ij} + {}^7\mathscr{PL}\Gamma^h_{ij}) = \frac{1}{2}\,({}^1v^h_{,ij} + {}^7v^h_{,ij}) + \frac{1}{2}\,v^t({}^1\Gamma^{\ h}_{itj} + {}^7\Gamma^{\ h}_{itj}).$$

The left-hand member of this equation is $\mathscr{L}\{^{\,h}_{ij}\}$. So, by (2.8) and (2.10), the equation reduces to

$$\mathscr{L}\{^{\,h}_{ij}\} = \{v^h\}_{ij} + v^t R^h_{itj} + \frac{1}{4}\,(S^h_{sj}S^s_{ti} - S^h_{ts}S^s_{ij}) - \frac{1}{4}\,(S^h_{sj}S^s_{it} - S^h_{st}S^s_{ij}).$$

Now, since Γ^h_{ij} is symmetric, so S^h_{ij} is also symmetric in i, j and, therefore, the terms in S's cancel. Hence, from formula (3.4) in $\mathscr{PL}$ we have deduced the following known formula in $\mathscr{L}$ and, therefore, the justification:

$$\mathscr{L}\{^{\,h}_{ij}\} = \{v^h\}_{ij} + v^t R^h_{itj}. \tag{3.4.L}$$

Again

$$\mathscr{PL}u^h_{,j} = v^t u^h_{,jt} - u^t_{,j} v^h_{,t} + u^h_{,t} v^t_{,j},$$

and

$$(\mathscr{L}u^h)_{,j} = v^t u^h_{,tj} + u^h_{,t} v^t_{,j} - u^t_{,j} v^h_{,t} - u^t v^h_{,tj}.$$

Therefore,

$$\mathscr{PL}u^h_{,j} - (\mathscr{L}u^h)_{,j} = v^t(u^h_{,jt} - u^h_{,tj}) + u^t v^h_{,tj} = u^s(v^t\Gamma^{\ h}_{stj} + v^h_{,sj}).$$

Hence, by (3.4), we establish the formula

$$\mathscr{PL}u^h_{,j} - (\mathscr{L}u^h)_{,j} = u^s \mathscr{PL}\Gamma^h_{sj}. \tag{3.5}$$

It follows immediately from (3.5) that

$$\frac{1}{2}\,({}^1\mathscr{PL}u^h_{,j} + {}^7\mathscr{PL}u^h_{,j}) - \frac{1}{2}\,[{}^1(\mathscr{L}u^h)_{,j} + {}^7(\mathscr{L}u^h)_{,j}] = \frac{1}{2}\,u^s({}^1\mathscr{PL}\Gamma^h_{sj} + {}^7\mathscr{PL}\Gamma^h_{sj})$$

or

$$\mathscr{L}\{u^h\}_j - \{\mathscr{L}u^h\}_j = u^s \mathscr{L}\{^{\,h}_{sj}\}. \tag{3.5L}$$

Thus we have deduced from formula (3.5) in $\mathscr{PL}$ the known formula (3.5L) fully in $\mathscr{L}$.

In a similar manner we get

$$\mathscr{PL}\omega_{i,j} - (\mathscr{L}\omega_i)_{,j} = v^t(\omega_{i,jt} - \omega_{i,tj}) - \omega_t v^t_{,ij}.$$

Hence, using (3.4), we establish the formula

$$(3.6)\qquad \mathscr{P}\mathscr{L}\omega_{i,j}-(\mathscr{L}\omega_i)_{,j} = -\omega_s\mathscr{P}\mathscr{L}\Gamma_{ij}^{s}.$$

It follows, as before, by taking half the sums of the first and the seventh terms of the sequences generated by members of both sides of equation (3.6), that

$$(3.6\mathrm{L})\qquad \mathscr{L}\{\omega_i\}_j-\{\mathscr{L}\omega_i\}_j = -\omega_s\mathscr{L}\{{}_{ij}^{s}\}.$$

Thus we have deduced from formula (3.6) in $\mathscr{P}\mathscr{L}$ the known formula (3.6L) fully in $\mathscr{L}$.

Again, in the same way as above, we get

$$\begin{aligned}\mathscr{P}\mathscr{L}T_{ij,k}^{h}-(\mathscr{L}T_{ij}^{h})_{,k} &= v^t(T_{ij,kt}^{h}-T_{ij,tk}^{h})+T_{ij}^{t}v_{,tk}^{h}-T_{tj}^{h}v_{,ik}^{t}-T_{it}^{h}v_{,jk}^{t}\\ &= T_{ij}^{s}(v_{,sk}^{h}+v^t\Gamma_{stk}^{h})-T_{sj}^{h}(v_{,ik}^{s}+v^t\Gamma_{itk}^{s})-T_{is}^{h}(v_{,jk}^{s}+v^t\Gamma_{jtk}^{s}).\end{aligned}$$

Hence, by (3.4), we establish the formula

$$(3.7)\qquad \mathscr{P}\mathscr{L}T_{ij,k}^{h}-(\mathscr{L}T_{ij}^{h})_{,k} = T_{ij}^{s}\mathscr{P}\mathscr{L}\Gamma_{sk}^{h}-T_{sj}^{h}\mathscr{P}\mathscr{L}\Gamma_{ik}^{s}-T_{is}^{h}\mathscr{P}\mathscr{L}\Gamma_{jk}^{s}.$$

Further, by taking half the sums of the first and seventh terms of the sequences generated by members of formula (3.7), we get, as before,

$$(3.7\mathrm{L})\qquad \mathscr{L}\{T_{ij}^{h}\}_k-\{\mathscr{L}T_{ij}^{h}\}_k = T_{ij}^{s}\mathscr{L}\{{}_{sk}^{h}\}-T_{sj}^{h}\mathscr{L}\{{}_{ik}^{s}\}-T_{is}^{h}\mathscr{L}\{{}_{jk}^{s}\}.$$

Thus we have deduced from formula (3.7) in $\mathscr{P}\mathscr{L}$ the known formula (3.7L) fully in $\mathscr{L}$.

Finally, we have

$$\begin{aligned}(\mathscr{P}\mathscr{L}\Gamma_{ij}^{h})_{,k}-(\mathscr{P}\mathscr{L}\Gamma_{ik}^{h})_{,j} &= (v_{,ij}^{h}+v^t\Gamma_{itj}^{h})_{,k}-(v_{,ik}^{h}+v^t\Gamma_{itk}^{h})_{,j}\\ &= -v_{,i}^{t}\Gamma_{tjk}^{h}+v_{,s}^{h}\Gamma_{ijk}^{s}+v^t\Gamma_{ijk,t}^{h}+\Gamma_{itj}^{h}v_{,k}^{t}-\Gamma_{itk}^{h}v_{,j}^{t}.\end{aligned}$$

Hence, we establish the formula

$$(3.8)\qquad (\mathscr{P}\mathscr{L}\Gamma_{ij}^{h})_{,k}-(\mathscr{P}\mathscr{L}\Gamma_{ik}^{h})_{,j} = \mathscr{P}\mathscr{L}\Gamma_{ikj}^{h}.$$

Further, from (3.4) we get

$$({}^7\mathscr{P}\mathscr{L}\Gamma_{ij}^{h})_{,k}-({}^7\mathscr{P}\mathscr{L}\Gamma_{ik}^{h})_{,j} = ({}^7v_{,ij}^{h}+v^t\,{}^7\Gamma_{itj}^{h})_{,k}-({}^7v_{,ik}^{h}+v^t\,{}^7\Gamma_{itk}^{h})_{,j}.$$

By (2.8),

$${}^7v_{,ij}^{h} = 2\{v^h\}_{ij}-v_{,ij}^{h}+\frac{1}{2}v^tS_{itj}^{h},$$

and, by (2·10),

$${}^7\Gamma_{itj}^{h} = 2R_{itj}^{h}-\Gamma_{itj}^{h}-\frac{1}{2}v^tS_{itj}^{h},$$

where $S^h_{itj} = S^h_{sj}S^s_{ti} - S^h_{ts}S^s_{ij}$, and $S^h_{ij} = {}^1\Gamma^h_{ij} - {}^7\Gamma^h_{ij}$.

Therefore, we get

$$(3.8a)\quad ({}^7\mathscr{P}\mathscr{L}\Gamma^h_{ij})_{,k} - ({}^7\mathscr{P}\mathscr{L}\Gamma^h_{ik})_{,j}$$
$$= 2(\{v^h\}_{ij,k} - \{v^h\}_{ik,j}) + 2v^t(R^h_{itj,k} - R^h_{itk,j}) + 2(v^t_{,k}R^h_{itj} - v^t_{,j}R^h_{itk}) - \mathscr{P}\mathscr{L}\Gamma^h_{ikj}.$$

Taking half the sums of (3.8) and (3.8a), we get

$$(3.8b)\quad (\mathscr{L}\{^{\,h}_{ij}\})_{,k} - (\mathscr{L}\{^{\,h}_{ik}\})_{,j}$$
$$= (\{v^h\}_{ij,k} - \{v^h\}_{ik.j}) + v^t(R^h_{itj,k} - R^h_{itk,j}) + (v^t_{,k}R^h_{itj} - v^t_{,j}R^h_{itk}).$$

Finally, taking half the sums of the first and seventh terms of the sequences generated by both the sides of equation (3.8b), we get after simplification

$$(3.8L)\qquad \{\mathscr{L}\{^{\,h}_{ij}\}\}_k - \{\mathscr{L}\{^{\,h}_{ik}\}\}_j = \mathscr{L}R^h_{ikj}.$$

Thus we have deduced from formula (3.8) in $\mathscr{P}\mathscr{L}$ the known formula (3.8L) in $\mathscr{L}$.

Again we have the known formula

$$\mathscr{L}g_{ij} = v^t\partial_t g_{ij} - g_{im}\partial_j v^m - g_{jm}\partial_i v^m$$
$$= g_{im}\{v^m\}_j + g_{jm}\{v^m\}_i = \{v_i\}_j + \{v_j\}_i.$$

If $\mathscr{L}g_{ij} = 0$, the vector v_i is known as a Killing vector.

Now it can be seen by calculation, although heavy, that

$$\frac{1}{2}g^{hs}[(\mathscr{L}g_{is})_{,j} + (\mathscr{L}g_{js})_{,i} - (\mathscr{L}g_{ij})_{,s}]$$
$$= \mathscr{P}\mathscr{L}(\{^{\,h}_{ij}\} - \Gamma^h_{ij}) + \mathscr{P}\mathscr{L}\Gamma^h_{ij} + [\partial_m v^h + g^{hs}(v^t\partial_t g_{sm} + g_{pm}\partial_s v^p)](\{^{\,m}_{ij}\} - \Gamma^m_{ij})$$
$$= \mathscr{L}\{^{\,h}_{ij}\} + [2\{v^h\}_m - g^{hs}(\{v_s\}_m - \{v_m\}_s)](\{^{\,m}_{ij}\} - \Gamma^m_{ij}).$$

Therefore, we establish the formula

$$(3.9)\quad \frac{1}{2}g^{hs}[(\mathscr{L}g_{is})_{,j} + (\mathscr{L}g_{js})_{,i} - (\mathscr{L}g_{ij})_{,s}] = \mathscr{L}\{^{\,h}_{ij}\} + H^h_m(\{^{\,m}_{ij}\} - \Gamma^m_{ij}),$$

where $H^h_m = \{v^h\}_m + g^{hs}\{v_m\}_s$.

Taking half the sums of the first and the seventh terms of the sequences generated by the two sides of (3.9), we have

$$(3.9L)\qquad \frac{1}{2}g^{hs}[\{\mathscr{L}g_{is}\}_j + \{\mathscr{L}g_{js}\}_i - \{\mathscr{L}g_{ij}\}_{,s}] = \mathscr{L}\{^{\,h}_{ij}\}.$$

Thus from formula (3.9), which involves $\mathscr{L}$ but nevertheless is not a usual formula in $\mathscr{L}$, we have deduced the known formula (3.9L) wholly in $\mathscr{L}$.

Regarding the term H^h_m in formula (3.9), it can be noted that if v_i is a Killing vector, then

$$H^h_m = \{v^h\}_m - g^{hs}\{v_s\}_m = 0.$$

This is consistent with the formula, because if v_i is a Killing vector, there is $\mathscr{L}g_{ij} = 0 = \mathscr{L}\{{}^{\,h}_{ij}\}$, and this makes both sides zero.

We may state above-mentioned results in the form of the following

THEOREM 5. *Introducing the notion of a primitive Lie derivative* $\mathscr{PL}$, *it has been possible to establish formulae* (3.4) *to* (3.9), *all but the last one, in* $\mathscr{PL}$ *and to deduce from them the corresponding known formulae* (3.4L) *to* (3.9L) *in* $\mathscr{L}$.

4. It is known that if a point transformation $x'^i = f^i(x)$ leaves geodesics of the space unaltered, it is a projective motion. If the point transformation reduces to the infinitesimal transformation $x'^i = x^i + v^i\,\delta t$, where v^i is a vector, the condition for the projective motion is

$$\mathscr{L}\{{}^{\,h}_{ij}\}[= \tfrac{1}{2}({}^1\mathscr{PL}\Gamma^h_{ij} + {}^7\mathscr{PL}\Gamma^h_{ij})] = \delta^h_i p_j + \delta^h_j p_i, \tag{4.1}$$

where δ^i_j is a Kronecker delta and p_i is a gradient vector.

Let us write equation (3.8b) as

$$(\mathscr{L}\{{}^{\,h}_{ij}\})_{,k} - (\mathscr{L}\{{}^{\,h}_{ik}\})_{,j} = B^h_{ikj}. \tag{4.2}$$

Obviously, B^h_{ikj} satisfies

$$B^h_{ikj} + B^h_{ijk} = 0 \quad \text{and} \quad B^h_{ikj} + B^h_{kji} + B^h_{jik} = 0.$$

From (4.1) we find

$$B^h_{ikj} = \delta^h_j p_{i,k} - \delta^h_k p_{i,j}. \tag{4.2a}$$

Writing $B^h_{inj} = -B_{ij}$, we get

$$B_{ij} = (n-1)p_{i,j}. \tag{4.2b}$$

So (4.2a) can be written as

$$(n-1)B^h_{ikj} = \delta^h_j B_{ik} - \delta^h_k B_{ij}. \tag{4.2c}$$

Now, taking half the sums of the first and seventh terms of sequences generated by both sides of equations (4.2a), (4.2b) and (4.2c), we get the following known equations in $\mathscr{L}$ under infinitesimal projective motion:

$$\mathscr{L}R^h_{ikj} = \delta^h_j\{p_i\}_k - \delta^h_k\{p_i\}_j, \tag{4.2aL}$$

$$\mathscr{L}R_{ij} = (n-1)\{p_i\}_j, \tag{4.2bL}$$

$$(n-1)\mathscr{L}R^h_{ikj} = \delta^h_j\mathscr{L}R_{ik} - \delta^h_k\mathscr{L}R_{ij}. \tag{4.2cL}$$

Besides, the projective motion, it is also known that if a point transformation $x'^i = f^i(x)$ leaves angle between two directions in the space invariant, it is a conformal motion. If the point transformation reduces to the infinitesimal transformation $x'^i \doteq x^i + v^i \delta t$, where v^i is a vector, the condition for a conformal motion is

$$(4.3)\quad \begin{aligned} &\mathscr{L} g_{ij} = 2\Phi g_{ij}, && \text{where } \Phi \text{ is a scalar,} \\ &\mathscr{L}\left\{ {h \atop ij} \right\} [= \tfrac{1}{2}({}^1\mathscr{P}\mathscr{L}\Gamma^h_{ij} + {}^7\mathscr{P}\mathscr{L}\Gamma^h_{ij})] = \delta^h_i \Phi_j + \delta^h_j \Phi_i, && \text{where } \Phi_i = \partial_i \Phi. \end{aligned}$$

Referring to (3.9), it follows from (4.3) that in the case of an infinitesimal conformal motion there is

$$(4.3\text{a})\qquad H^h_m = 2\Phi \delta^h_m, \quad \text{where } H (= H^m_m) = 2\Phi = 2\{v^m\}_m.$$

Referring further to (4.2), it follows from the second of equations (4.3) that

$$(4.4)\qquad B^h_{ikj} = \delta^h_j \Phi_{i,k} - \delta^h_k \Phi_{i,j} - \Phi^h (g_{ij,k} - g_{ik,j}) - (g_{ij}\Phi^h_{,k} - g_{ik}\Phi^h_{,j}).$$

Writing $B_{ij} = B^h_{ijh}$, $B^h_j = g^{hi}B_{ij}$ and $B = g^{ij}B_{ij}$, it can be seen from (4.4) that $B_{ij} = (n-2)\Phi_{i,j} + (\Phi^h g_{ij})_{,h}$; similarly, for B^h_j and B.

Put

$$E_{ij} = -\frac{B_{ij}}{n-2} + \frac{B g_{ij}}{2(n-1)(n-2)}, \qquad E^h_j = g^{hi}E_{ij}.$$

It can then be seen that

$$(4.4\text{a})\ E_{ij} = -\Phi_{i,j} - \frac{\Phi^t}{n-2} g_{ij,t} - \frac{g_{ij}\Phi_s}{2(n-1)} g^{st}_{\ ,t} - \frac{g_{ij}\Phi^t g_{pq}}{2(n-1)(n-2)} g^{pq}_{\ ,t}.$$

Write

$$D^h_{ijk} = B^h_{ijk} + \delta^h_k E_{ij} - \delta^h_j E_{ik} + E^h_k g_{ij} - E^h_j g_{ik}.$$

Substituting from (4.4a), we find after simplification that

$$(4.4\text{b})\quad D^h_{ijk} = (\delta^h_j g_{ik} - \delta^h_k g_{ij}) \left\{ \frac{\Phi_s}{n-1} g^{st}_{\ ,t} + \frac{\Phi^t g_{pq}}{(n-1)(n-2)} g^{pq}_{\ ,t} \right\} +$$
$$+ \Phi_1 \{(g_{ij}g^{hl})_{,k} - (g_{ik}g^{hl})_{,j}\} \frac{\Phi^t g^{hl}}{n-2} (g_{ij}g_{kl} - g_{ik}g_{jl})_{,t}.$$

Now write

$$A_{ij} = -\frac{R_{ij}}{n-2} + \frac{R g_{ij}}{2(n-1)(n-2)},$$

and take half the sums of the first and seventh terms of the sequences generated by both sides of equations (4.4), (4.4a) and (4.4b). Then, remembering that the covariant derivatives of the g_{ij} with respect to the Christoffel symbol vanish, we get the known equations in $\mathscr{L}$ under infinitesimal conformal motion,

$$\mathscr{L}R^h_{ijk} = \delta^h_j\{\Phi_i\}_k - \delta^h_k\{\Phi_i\}_j - g_{ij}\{\Phi^h\}_k + g_{ik}\{\Phi^h\}_j, \tag{4.4L}$$

$$\mathscr{L}A_{ij} = -\{\Phi_i\}_j, \tag{4.4aL}$$

$$\mathscr{L}C^h_{ijk} = 0, \tag{4.4bL}$$

where C^h_{ijk} is the conformal tensor. We can state the above-mentioned results in the form of the following

THEOREM 6. *Under infinitesimal projective and conformal motions, equations* (4.2a), (4.2b), (4.2c) *and equations* (4.4), (4.4a), (4.4b) *are obtained, and from them the corresponding known equations* (4.2aL), (4.2bL), (4.2cL) *and* (4.4L), (4.4aL), (4.4bL), *all in* $\mathscr{L}$, *are deduced.*

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