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MOST MARKOV OPERATORS ON $C(X)$ ARE QUASI-COMPACT AND UNIQUELY ERGODIC

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1. Uniquely ergodic Markov operators. We denote by $C(X)$ the Banach space of continuous real-valued functions on a compact Hausdorff space X. A linear operator T on $C(X)$ which is positive ($f \geqslant 0 \Rightarrow Tf \geqslant 0$) and takes 1 into 1 is said to be *Markov*. A probability (Radon) measure μ on X is said to be *T-invariant* if $T^* \mu = \mu$. By the Markov–Kakutani fixed point theorem the set of all invariant probability measures is non-empty. T is *uniquely ergodic* if there exists only one invariant probability measure. It is well known (see, e.g., [8]) that T is uniquely ergodic if and only if the Cesàro means

$$A_n = n^{-1}(I+T+ \ldots +T^{n-1})$$

converge in the strong operator topology to a one-dimensional projection which necessarily is of the form $\mathrm{E}_\mu f := (\mu, f)\,1$, where μ is a probability measure on X.

T is said to be *uniformly ergodic* if the norm closed convex hull of $\{T^n;\, n \geqslant 0\}$ contains an operator P such that $TP = P$ (see [7]). If T is a uniformly ergodic Markov operator on $C(X)$, then by, e.g., Proposition 1 in [7], the Cesàro means A_n converge in the norm topology.

A Markov operator T is said to be *quasi-compact* if $\|T^n - K\| < 1$ for some positive integer n and some compact operator K. It is well known (see [9]) that quasi-compactness implies uniform ergodicity.

We shall use the above results to prove Lemma 2.

LEMMA 1. *Let a Markov operator T, a probability measure μ, and $0 \leqslant \alpha < 1$ be given. Then the iterates of the operator $S = \alpha T + (1-\alpha)\mathrm{E}_\mu$ converge in the norm topology to a one-dimensional projection.*

Proof. It is easily seen that

$$S^n = \alpha^n T^n + \mathrm{E}_{\mu_n} \quad \text{for some } \mu_n \in C^*(X).$$

Let ν be an S-invariant probability measure. Then

$$\|S^n - \mathrm{E}_\nu\| \leqslant \alpha^n + \|\mathrm{E}_\nu - \mathrm{E}_{\mu_n}\| = \alpha^n + \|\mathrm{E}_\nu S^n - \mathrm{E}_{\mu_n}\|$$
$$= \alpha^n + \|\alpha^n \mathrm{E}_\nu T^n\| \leqslant 2\alpha^n.$$

By Lemma 1, the uniquely ergodic quasi-compact Markov operators are norm dense in the set of Markov operators on $C(X)$.

LEMMA 2. *If there exists a sequence* (μ_k) *of probability measures on* X *such that*

$$\|A_{n_k} - \mathrm{E}_{\mu_k}\| \to 0,$$

then T *is quasi-compact and uniquely ergodic.*

Proof. For sufficiently large k the Markov operators A_{n_k} are quasi-compact. Hence, for those k, the Cesàro means $m^{-1} \sum_{j=0}^{m-1} A_{n_k}^j$ converge in the norm operator topology to a projection P_k. Clearly, $P_k A_{n_k} = P_k$. Moreover, we have $P_k A_n = A_n P_k$ for all k and n. Hence

$$\|P_k - \mathrm{E}_{\mu_k}\| = \|P_k A_{n_k} - P_k \mathrm{E}_{\mu_k}\| \leqslant \|A_{n_k} - \mathrm{E}_{\mu_k}\| \to 0.$$

Since

$$\|P_j - P_k A_{n_j}\| \leqslant \|P_j - \mathrm{E}_{\mu_j}\| + \|P_k \mathrm{E}_{\mu_j} - P_k A_{n_j}\| \to 0$$

uniformly in k, and

$$\begin{aligned}\|P_k A_{n_j} - P_k\| &\leqslant \|P_k - \mathrm{E}_{\mu_k}\| + \|\mathrm{E}_{\mu_k} - P_k A_{n_j}\| \\ &= \|P_k - \mathrm{E}_{\mu_k}\| + \|A_{n_j} \mathrm{E}_{\mu_k} - A_{n_j} P_k\| \to 0\end{aligned}$$

uniformly in j, the sequence (P_k) is Cauchy. As $A_{n_k}^j$ belong to the convex hull $\mathrm{co}\{T^n;\ n \geqslant 0\}$ for all k, j, the operators P_k and $P := \lim P_k = \lim \mathrm{E}_{\mu_k}$ belong to the closure of $\mathrm{co}\{T^n;\ n \geqslant 0\}$. Furthermore, $PT = TP = P$, so T is uniformly ergodic. Hence T is quasi-compact and uniquely ergodic since $P = \lim \mathrm{E}_{\mu_k} = \lim A_k$ is one-dimensional (see, e.g., Proposition 1 in [7] and Theorem 1 in [6]).

THEOREM 1. *The uniquely ergodic quasi-compact operators form a dense* G_δ*-set for the norm topology in the set of Markov operators on* $C(X)$.

Proof. Let $\mathscr{U}$ be the set of all Markov operators T for which the Cesàro means A_n converge in the norm topology to a one-dimensional projection. By Theorem 1 in [6], $\mathscr{U}$ consists of all uniquely ergodic quasi-compact operators. Let

$$\mathscr{U}_1 = \bigcap_k \bigcap_n \bigcup_{m \geqslant n} \bigcup_\mu \{T;\ T \text{ is Markov and } \|A_m - \mathrm{E}_\mu\| < 1/k\},$$

where $\bigcup_\mu$ is the union over all probability measures on X. Clearly, $\mathscr{U} \subset \mathscr{U}_1$ and $\mathscr{U}_1$ is a G_δ-set. By Lemma 2, each T from $\mathscr{U}_1$ is uniquely ergodic and quasi-compact, so $\mathscr{U} = \mathscr{U}_1$. By the remark following Lemma 1, $\mathscr{U}$ forms a norm dense subset.

2. Stochastic operators on L^1. In this section we apply Theorem 1 to some stochastic operators. Let (Ω, Σ, m) be a probability space. A linear operator T on $L^1(m)$ is said to be *stochastic* if T is positive (i.e., $f \geqslant 0 \Rightarrow Tf \geqslant 0$) and satisfies the equality $T^*1 = 1$ (or, equivalently, $f \geqslant 0 \Rightarrow \|Tf\| = \|f\|$). The set of all stochastic operators will be denoted by $\mathscr{S}$.

A stochastic operator T is said to be *conservative* if $\sum_{n=1}^{\infty} T^n f = \infty$ a.e. for every (or, equivalently, for some) strictly positive function $f \in L^1(m)$ (see, e.g., [2], Chapter 2). The set of all conservative operators in $\mathscr{S}$ will be denoted by $\mathscr{C}$. The set of all stochastic operators T for which $T^*1_A = 1_A$ implies $m(A)(1-m(A)) = 0$ $(A \in \Sigma)$ will be denoted by $\mathscr{O}$.

In this section we discuss operators from $\mathscr{C} \cap \mathscr{O}$, the set of conservative and ergodic stochastic operators. In [5] and [1] topological properties of $\mathscr{C} \cap \mathscr{O}$ have been studied. It has been shown (for $\Omega = [0, 1]$) that conservative and ergodic operators form a dense G_δ-set for both strong operator and norm topologies in $\mathscr{S}$. The Harris operators form an important subset of $\mathscr{C} \cap \mathscr{O}$. Let $T \in \mathscr{C} \cap \mathscr{O}$ and $T^n = Q_n + R_n$, where Q_n is a positive integral operator with kernel $q_n(x, y)$, and R_n is such that there is no non-zero integral operator K with $0 \leqslant K \leqslant R_n$ (see [2], Chapter 5, and [3]). T is said to be a *Harris operator* if $Q_n \neq 0$ for some $n \geqslant 1$ (see [2] and [3]).

By the Gelfand–Naĭmark theorem there is a 0-dimensional compact Hausdorff space $\tilde{X}$ such that $L_\infty(m)$ and $C(\tilde{X})$ are isometrically isomorphic. For each $T \in \mathscr{S}$ let $\tilde{T}$ denote the corresponding operator on $C(\tilde{X})$. Clearly, $\tilde{T}$ is a Markov operator. If $T \in \mathscr{C}$, then $\tilde{T}$ is uniquely ergodic and quasi-compact if and only if $T \in \mathscr{C} \cap \mathscr{O}$ and T^* is quasi-compact on $L_\infty(m)$ (see Theorem 4.1 in [4]).

Now we can prove the following

THEOREM 2. *The conservative and ergodic quasi-compact operators form a dense G_δ-set for the norm topology in $\mathscr{S}$.*

Proof. From Theorem 1 and the remark above, the conservative and ergodic quasi-compact operators form a G_δ-subset of $\mathscr{C}$ for the norm topology. As in [5] (Lemma 2) we can show that the conservative operators form a G_δ-set for the norm (even for strong operator) topology in $\mathscr{S}$. Hence the conservative and quasi-compact operators, being an intersection of two G_δ's, form a G_δ-set for the norm topology in $\mathscr{S}$.

To complete the proof, it remains to show that this set is norm dense in $\mathscr{S}$. Let $T \in \mathscr{S}$. Then for each α $(0 \leqslant \alpha < 1)$ the operator $S = \alpha T + (1-\alpha) \mathrm{E}_m$ is conservative, ergodic and quasi-compact. Indeed, by Lemma 2 in [1], S has an equivalent invariant probability measure μ, and therefore S is conservative. Clearly, S is ergodic. Similarly as in the proof of Lemma 1, S^n converge to E_μ in the norm topology, so S is quasi-compact. Clearly, $S = S(\alpha) \to T$ in the norm topology.

Now, from Theorem 4.1 in [4] we obtain

COROLLARY. *The set of Harris operators is norm residual in the set of stochastic operators.*

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REFERENCES

[1] W. Bartoszek, *Norm residuality of ergodic operators*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 29 (1981), p. 165–167.

[2] S. R. Foguel, *The ergodic theory of Markov processes*, New York 1969.

[3] – *Harris operators*, Israel Journal of Mathematics 33 (1979), p. 281–309.

[4] S. Horowitz, *Transition probabilities and contraction of L^∞*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 24 (1972), p. 263–274.

[5] A. Iwanik, *Approximation theorems for stochastic operators*, Indiana University Mathematics Journal 29 (1980), p. 415–425.

[6] M. Lin, *Quasi-compactness and uniform ergodicity of positive operators*, Israel Journal of Mathematics 29 (1978), p. 309–311.

[7] H. P. Lotz, *Uniform ergodic theorems for Markov operators on $C(X)$*, Mathematische Zeitschrift 178 (1981), p. 145–156.

[8] R. Sine, *Geometric theory of a single Markov operator*, Pacific Journal of Mathematics 27 (1968), p. 155–166.

[9] K. Yosida and S. Kakutani, *Operator theoretical treatment of Markoff's process and mean ergodic theorem*, Annals of Mathematics (2) 42 (1941), p. 188–228.

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