COLLOQUIUM MATHEMATICUM

VOL. XLVI 1982 FASC. 1

DUALITIES FOR STONE ALGEBRAS, DOUBLE STONE ALGEBRAS, AND RELATIVE STONE ALGEBRAS

BY

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In [3]-[5] we developed duality theories for Stone algebras and relative Stone algebras. In both cases the dual of an algebra was a Hom-set endowed with a Boolean topology and the continuous action of the endomorphism monoid of an appropriate algebra; the dual of a Stone algebra was also endowed with the pointwise partial order. Here we prove a general duality theorem for a class of bounded-distributive-lattice-ordered algebras from which the above-mentioned dualities for Stone algebras and relative Stone algebras follow; the theorem is also applied to derive a new duality theorem for the equational class of double Stone algebras.

The lattice theory necessary for our purposes may be found in [1] and [9].

1. THE CATEGORY $\boldsymbol{A}$

In this section we introduce the algebras to be studied.

For objects B, C of a category $\boldsymbol{K}$, let $\boldsymbol{K}(B, C)$ denote the corresponding Hom-set; and denote the monoid $\boldsymbol{K}(A, A)$ of endomorphisms of A by $\mathrm{End}(A)$. Denote the category of bounded distributive lattices with bound-preserving homomorphisms by $\boldsymbol{D}$, and denote the two-element chain $\{0, 1\}$ with $0 < 1$ by $\mathit{2}$.

In this paper we consider only those classes $\boldsymbol{A}$ of algebras which enjoy the following properties:

(P_0) There is a finite bounded-distributive-lattice-ordered algebra A such that $\boldsymbol{A} = \boldsymbol{ISP}(A)$, and hence $\boldsymbol{A}$ is a subcategory of $\boldsymbol{D}$.

(P_1) There exists $a \in \boldsymbol{D}(A, \mathit{2})$ such that for all $B \in \boldsymbol{A}$ the map

$$\Phi : \boldsymbol{A}(B, A) \to \boldsymbol{D}(B, \mathit{2}),$$

given by $g\Phi = g\alpha$, has an order-preserving left inverse

$$\Theta : \boldsymbol{D}(B, \mathit{2}) \to \boldsymbol{A}(B, A), \quad \text{i.e.} \quad \Theta\Phi = \mathrm{id}_{\boldsymbol{D}(B,\mathit{2})}.$$

(P_2) Let $\hat{} = \Phi\Theta : \boldsymbol{A}(B, A) \to \boldsymbol{A}(B, A)$; then for all $g \in \boldsymbol{A}(B, A)$ there exists $e \in \mathrm{End}(A)$ such that $g = \hat{g}e$, and $\mathrm{Im}(g) = \bigcap(\mathrm{Im}(e) \mid g = \hat{g}e)$.

(P_3) For all $g \in \boldsymbol{A}(B, A)$ and all $\beta \in \boldsymbol{D}(A, \mathit{2})$ there exists $e \in \mathrm{End}(A)$ such that $(\hat{g}\beta)\Theta = \hat{g}e$.

Remark. It would be sufficient for our purposes to replace (P_0) by the assumption that A has a bounded-distributive-lattice reduct.

Before proceeding with the theorems, let us show that Stone algebras, double Stone algebras, and relative Stone algebras satisfy these conditions.

1.1. Stone algebras. Let $\mathit{3}$ denote the three-element *Stone algebra*, $0 < a < 1$. Then $\mathit{3}$ is an algebra of type $\langle 2, 2, 1, 0, 0\rangle$ with operations $\langle \vee, \wedge, {}^*, 0, 1\rangle$, where * is given by $0^* = 1$, $a^* = 1^* = 0$. Since $\mathit{3}$ and its subalgebra $\mathit{2}$ are (up to isomorphism) the only subdirectly irreducible Stone algebras (see [13] and [14]), $\boldsymbol{S} = \boldsymbol{ISP}(\mathit{3})$ is the equational class of Stone algebras.

It is easily seen that $\mathrm{End}(\mathit{3}) = \{e_0, e_1\}$, where e_0 is the identity map and e_1 is given by $ce_1 = c^{**}$ for all $c \in \mathit{3}$.

(P_1) Let $\alpha \in \boldsymbol{D}(\mathit{3}, \mathit{2})$ be given by $0\alpha = a\alpha = 0$, $1\alpha = 1$. For each prime filter F of a Stone algebra B there is a unique maximal filter $M(F)$ with $F \subseteq M(F)$ (see [9] and [10]). Define $\Theta : \boldsymbol{D}(B, \mathit{2}) \to \boldsymbol{S}(B, \mathit{2})$ by

$$x(\beta\Theta) = \begin{cases} 1 & \text{if } x \in 1\beta^{-1}, \\ a & \text{if } x \in M(1\beta^{-1}) - 1\beta^{-1}, \\ 0 & \text{if } x \in B - M(1\beta)^{-1}. \end{cases}$$

A lattice homomorphism $g : B \to \mathit{3}$ belongs to $\boldsymbol{S}(B, \mathit{3})$ if and only if $\{a, 1\}g^{-1} = M(1g^{-1})$ (see [2] and [9]); hence $\beta\Theta \in \boldsymbol{S}(B, \mathit{3})$ and, moreover, Θ is an order-preserving two-sided inverse of Φ.

(P_2) This condition holds trivially since $\Phi\Theta = \mathrm{id}_{\boldsymbol{S}(B,\mathit{3})}$, and hence $\hat{g} = g$ for all $g \in \boldsymbol{S}(B, \mathit{3})$.

(P_3) It is easily seen that $\boldsymbol{D}(\mathit{3}, \mathit{2}) = \{\beta_0, \beta_1\}$, where $\beta_0 = e_0\alpha = \alpha$ and $\beta_1 = e_1\alpha$. For all $g \in \boldsymbol{S}(B, \mathit{3})$ and each $k < 2$ we have

$$(g\beta_k)\Theta = (ge_k\alpha)\Theta = (ge_k)\Phi\Theta = ge_k,$$

and hence (P_3) holds since $\hat{g} = g$.

1.2. Double Stone algebras. Let $\mathit{4}$ denote the four-element *double Stone algebra*, $0 < a < b < 1$. Then $\mathit{4}$ is an algebra of type $\langle 2, 2, 1, 1, 0, 0\rangle$ with operations $\langle \wedge, \vee, {}^*, {}^+, 0, 1\rangle$, where * and $^+$ are given by

$$0^* = 1, a^* = b^* = 1^* = 0, \quad \text{and} \quad 0^+ = a^+ = b^+ = 1, 1^+ = 0.$$

Since $\boldsymbol{4}$ and its subalgebras $\boldsymbol{2}$ and $\boldsymbol{3}$ are (up to isomorphism) the only subdirectly irreducible double Stone algebras (see [12]), $\boldsymbol{T} = \boldsymbol{ISP}(\boldsymbol{4})$ is the equational class of double Stone algebras.

The algebra $\boldsymbol{4}$ has only three endomorphisms:

$$\mathrm{End}(\boldsymbol{4}) = \{e_0, e_1, e_2\},$$

where e_0 is the identity map, $0e_1 = 0$, $ae_1 = be_1 = b$, $1e_1 = 1$, and $0e_2 = 0$, $ae_2 = be_2 = a$, $1e_2 = 1$.

(P_1) Define $\alpha \in \boldsymbol{D}(\boldsymbol{4}, \boldsymbol{2})$ by $0\alpha = a\alpha = 0$, $b\alpha = 1\alpha = 1$. Since a double Stone algebra B is both a Stone algebra and a dual Stone algebra, for each prime filter F of B there are a unique maximal filter $M(F)$ and a unique minimal prime filter $m(F)$ such that $m(F) \subseteq F \subseteq M(F)$. Define $\Theta : \boldsymbol{D}(B, \boldsymbol{2}) \to \boldsymbol{T}(B, \boldsymbol{4})$ by

$$x(\beta\Theta) = \begin{cases} 1 & \text{if } x \in m(1\beta^{-1}), \\ b & \text{if } x \in 1\beta^{-1} - m(1\beta^{-1}), \\ a & \text{if } x \in M(1\beta^{-1}) - 1\beta^{-1}, \\ 0 & \text{if } x \in B - M(1\beta^{-1}). \end{cases}$$

By the corresponding result for Stone algebras, and its dual, a lattice homomorphism $g : B \to \boldsymbol{4}$ belongs to $\boldsymbol{T}(B, \boldsymbol{4})$ if and only if $\{a, b, 1\}g^{-1} = M(\{b, 1\}g^{-1})$ and $1g^{-1} = m(\{b, 1\}g^{-1})$. Hence $\beta\Theta \in \boldsymbol{T}(B, \boldsymbol{4})$; and again Θ is an order-preserving two-sided inverse of Φ.

(P_2) Using $\bar{g} = g$ and $\boldsymbol{2} = \bigcap(\mathrm{Im}(e) \mid e \in \mathrm{End}(\boldsymbol{4}))$, this is easily checked.

(P_3) It is easily seen that $\boldsymbol{D}(\boldsymbol{4}, \boldsymbol{2}) = \{\beta_0, \beta_1, \beta_2\}$, where $\beta_0 = e_0\alpha = \alpha$, $\beta_1 = e_1\alpha$, and $\beta_2 = e_2\alpha$. That (P_3) holds follows exactly as in the case of Stone algebras.

1.3. Relative Stone algebras. An *L-algebra* is a Heyting algebra satisfying the identity $(x*y) \vee (y*x) = 1$; the equational class of all L-algebras is denoted by $\boldsymbol{L}_\omega$. An L-algebra B is a *relative Stone algebra* in the sense that every interval $[a, b]$ of B is a Stone algebra; in $[a, b]$ the unary operation * is given by $x^* = (x*a) \wedge b$. For $2 \leqslant n < \omega$, let $\boldsymbol{n}$ denote the n-element chain as a Heyting algebra, $0 = c_0 < c_1 < \ldots < c_{n-1} = 1$. Then $\boldsymbol{n}$ is an algebra of type $\langle 2, 2, 2, 0, 0\rangle$ with operations $\langle \wedge, \vee, *, 0, 1\rangle$, where $*$ is given by

$$a*b = \begin{cases} 1 & \text{if } a \leqslant b, \\ b & \text{if } a > b. \end{cases}$$

Since $\boldsymbol{n}$ is an L-algebra, the equational class $\boldsymbol{L}_n$ generated by $\boldsymbol{n}$ is a subclass of $\boldsymbol{L}_\omega$; T. Hecht and T. Katriňák have shown that every nontrivial proper equational subclass of $\boldsymbol{L}_\omega$ is of the form $\boldsymbol{L}_n$ for some n. Since $\boldsymbol{n}$ and its subalgebras $\boldsymbol{2}, \boldsymbol{3}, \ldots, \boldsymbol{n-1}$ are (up to isomorphism) the only subdirectly irreducible algebras in $\boldsymbol{L}_n$ (see [11]), it follows that $\boldsymbol{L}_n = \boldsymbol{ISP}(\boldsymbol{n})$.

If $e \in \mathrm{End}(\boldsymbol{n})$, then

(i) $0e = 0$;

(ii) e is order-preserving;

(iii) there exists k with $1 \leqslant k < n$ such that $[c_k) = 1e^{-1}$ and, for $i, j < k$, $c_i e = c_j e$ implies $i = j$.

Conversely, any map $e : \boldsymbol{n} \to \boldsymbol{n}$ satisfying (i)-(iii) is an endomorphism of $\boldsymbol{n}$. It follows that $|\mathrm{End}(\boldsymbol{n})| = 2^{n-2}$.

(P_1) Let $\alpha \in \boldsymbol{D}(\boldsymbol{n}, \boldsymbol{2})$ be given by $c_k \alpha = 0$ for all $k < n-1$ and $1\alpha = 1$. It is shown in [11] that an L-algebra B is in $\boldsymbol{L}_n$ if and only if for each prime filter F of A the set of prime filters containing F forms a chain with at most $n-1$ elements. Let $B \in \boldsymbol{L}_n$ and let $F = F_k \subset F_{k-1} \subset \ldots \subset F_1$ be the chain of all prime filters containing the prime filter F. Define a map $g_F : B \to \boldsymbol{n}$ by

$$xg_F = \begin{cases} 1 & \text{if } x \in F = F_k, \\ c_i & \text{if } b \in F_i - F_{i+1} \ (1 \leqslant i < k), \\ 0 & \text{if } b \in B - F_1. \end{cases}$$

It is proved in [4], Proposition 1.1, that $g_F \in \boldsymbol{L}_n(B, \boldsymbol{n})$; g_F is called the *homomorphism determined by* F. Now define $\Theta : \boldsymbol{D}(B, \boldsymbol{2}) \to \boldsymbol{L}_n(B, \boldsymbol{n})$ by $\beta\Theta = g_F$, where $F = 1\beta^{-1}$. Then Θ is an order-preserving left inverse of Φ; and Θ is a two-sided inverse of Φ if and only if n equals 2 or 3.

(P_2) This property is proved in [4], Lemma 1.3; in that paper the map $\hat{g}$ is denoted by $g_{\downarrow}$.

(P_3) Let e_k be the endomorphism of $\boldsymbol{n}$ determined by the prime filter $[c_k)$. Then $\boldsymbol{D}(\boldsymbol{n}, \boldsymbol{2})$ is the set $\{\beta_1, \ldots, \beta_{n-1}\}$, where $\beta_k = e_k \alpha$ for all k; note that $\beta_{n-1} = \alpha$. If a map $g \in \boldsymbol{L}_n(B, \boldsymbol{n})$ satisfies $\mathrm{Im}(g) = (c_l] \cup \{1\}$ for some $l < n-1$, then $g\Phi\Theta = g$. Thus for each k with $1 \leqslant k < n$ we have

$$(\hat{g}\beta_k)\Theta = (\hat{g}e_k\alpha)\Theta = (\hat{g}e_k)\Phi\Theta = \hat{g}e_k,$$

since $\mathrm{Im}(\hat{g}e_k) = (c_l] \cup \{1\}$ for some $l \leqslant k$. Hence (P_3) holds.

2. THE REPRESENTATION THEOREM

We now describe the dual B^* of an algebra B in $\boldsymbol{A}$ and show that B is isomorphic to an algebra B^{**} of continuous functions defined on B^*. Throughout this section it is assumed that $\boldsymbol{A}$ is a class of algebras satisfying (P_0), (P_1), (P_2), and (P_3).

A *Boolean space* is a totally disconnected compact space or, equivalently, a compact space with a basis of clopen subsets. Any closed subspace of a product of finite discrete spaces is a Boolean space, and hence for all $B \in \boldsymbol{A}$ the Hom-set $\boldsymbol{A}(B, A)$ is a Boolean space (regarded as a subspace of A^B). The sets of the form

$$\langle b; a \rangle = \{g \in \boldsymbol{A}(B, A) \mid bg = a\},$$

where $b \in B$ and $a \in A$, form a subbasis for the topology on $\boldsymbol{A}(B, A)$; and hence $\langle b ; U\rangle = \bigcup(\langle b ; a\rangle \mid a \in U)$ is open in $\boldsymbol{A}(B, A)$ for every subset U of A. Order $\boldsymbol{A}(B, A)$ pointwise and define a continuous action of the monoid $\mathrm{End}(A)$ on the space $\boldsymbol{A}(B, A)$ by the map $e \to \tilde{e}$, where $\tilde{e}: \boldsymbol{A}(B, A) \to \boldsymbol{A}(B, A)$ is given by $g\tilde{e} = ge$.

Thus $D(B) = B^* = \boldsymbol{A}(B, A)$ is an object of the category $\boldsymbol{Z}$ of partially ordered Boolean spaces on which $\mathrm{End}(A)$ acts continuously; the morphisms of $\boldsymbol{Z}$ are continuous order-preserving maps which preserve the action of $\mathrm{End}(A)$. A functor $D : \boldsymbol{A} \to \boldsymbol{Z}$ is obtained when $D(h) = h^*$ is defined in the usual way: if $h \in \boldsymbol{A}(B, C)$, then $h^* : C^* \to B^*$ is defined by $gh^* = hg$ for all $g \in C^*$.

Define a $\boldsymbol{Z}$-structure on A as follows: the topology on A is discrete; the partial order on A, which we denote by $\leqslant^*$ to avoid confusion with the usual partial order $\leqslant$, is defined by $a \leqslant^* b$ if and only if $p(a) \leqslant p(b)$ for every unary polynomial p; the action of $\mathrm{End}(A)$ on A is simply $\tilde{e} = e$. Note that if $a \leqslant^* b$, then $a \leqslant b$, and hence if $\beta \in \boldsymbol{D}(A, 2)$, then β preserves the order $\leqslant^*$ on A. Denote the free algebra in $\boldsymbol{A}$ with free generator x_0 by $\boldsymbol{FA}(1)$. The following lemma is obvious:

2.1. LEMMA. *Define* $\varrho : \boldsymbol{FA}(1)^* \to A$ *by* $g\varrho = x_0 g$; *then* ϱ *is an isomorphism in* $\boldsymbol{Z}$.

It is, in fact, natural to define a $\boldsymbol{Z}$-structure on A so that it is isomorphic in $\boldsymbol{Z}$ to $\boldsymbol{FA}(1)^*$. For a discussion of this and of duality in general, we refer to [5].

For each object B of $\boldsymbol{A}$ let B^{**} denote $\boldsymbol{Z}(B^*, A)$. We may define a map $\eta_B : B \to B^{**}$ by $b\eta_B = \Gamma_b$, where $g\Gamma_b = bg$ for all $g \in B^*$. Observe that

(i) Γ_b is continuous;

(ii) if $g \leqslant h$ in B^*, then for every unary polynomial p we have

$$p(g\Gamma_b) = p(bg) = p(b)g \leqslant p(b)h = p(bh) = p(h\Gamma_b),$$

and hence $g\Gamma_b \leqslant^* h\Gamma_b$ in A;

(iii) Γ_b preserves the action of $\mathrm{End}(A)$ since

$$(g\tilde{e})\Gamma_b = b(ge) = (bg)\tilde{e} = (g\Gamma_b)\tilde{e}.$$

Thus Γ_b is an element of B^{**}, and so η_B is well defined. Before we can show that η_B is an isomorphism we require the following lemma:

2.2. LEMMA. (i) $g\Phi = h\Phi$ *if and only if* $\hat{g} = \hat{h}$.

(ii) $g\Phi = \hat{g}\Phi$, *i.e.* $ga = \hat{g}a$.

(iii) $\hat{\hat{g}} = \hat{g}$.

(iv) Φ *is order-preserving.*

(v) $g\Phi \leqslant h\Phi$ *if and only if* $\hat{g} \leqslant \hat{h}$.

(vi) *Let* $\varphi \in \boldsymbol{Z}(B^*, A)$; *then, for all* $g \in B^*$, $g\varphi \in \mathrm{Im}(g)$.

Proof. (i) This is trivial.

(ii) $\hat{g}\Phi = g\Phi\Theta\Phi = g\Phi$.

(iii) From (ii) we have $g\Phi = \hat{g}\Phi$, and hence $\hat{\hat{g}} = \hat{g}$ by (i).

(iv) This follows from the fact that a is order-preserving.

(v) Since Θ is order-preserving, $g\Phi \leqslant h\Phi$ implies $\hat{g} = g\Phi\Theta \leqslant h\Phi\Theta = \hat{h}$. Conversely, assume that $\hat{g} \leqslant \hat{h}$; then we have $g\Phi = \hat{g}\Phi \leqslant \hat{h}\Phi = h\Phi$ by (ii) and (iv).

(vi) Assume e satisfies $g = \hat{g}e$; then $g\varphi = (\hat{g}e)\varphi = (\hat{g}\varphi)e \in \mathrm{Im}(e)$. Hence

$$g\varphi \in \bigcap \big(\mathrm{Im}(e) \mid g = \hat{g}e\big) = \mathrm{Im}(g).$$

The proof of the representation theorem below follows the proof of the corresponding result, Theorem 2.4, in [4]; it is interesting to note that in the present more general setting the proof is slightly easier. The main tool is Priestley's duality for bounded distributive lattices (see [15] and [16]): for every bounded distributive lattice B, the set $\boldsymbol{D}(B, 2)$ is a partially ordered Boolean space with its order and topology inherited from 2^B; for each $b \in B$, the set $\langle b; 1\rangle = \{g \in \boldsymbol{D}(B, 2) \mid bg = 1\}$ is a clopen increasing subset of $\boldsymbol{D}(B, 2)$, Priestley's duality states, in part, that the map $b \to \langle b; 1\rangle$ is an isomorphism of B onto the lattice of clopen increasing subsets of $\boldsymbol{D}(B, 2)$. Alternatively, let $\mathfrak{X}(B)$ be the set of prime filters of B, ordered by set inclusion, with

$$\{\mathfrak{X}_b \mid b \in B\} \cup \{\mathfrak{X}(B) - \mathfrak{X}_b \mid b \in B\}, \quad \text{where } \mathfrak{X}_b = \{x \in \mathfrak{X}(B) \mid b \in x\},$$

as a subbasis for the topology; then the map $g \to 1g^{-1}$ is a homeomorphism and an order-isomorphism of $\boldsymbol{D}(B, 2)$ onto $\mathfrak{X}(B)$, and the map $b \to \mathfrak{X}_b$ is an isomorphism of B onto the lattice of clopen increasing subsets of $\mathfrak{X}(B)$.

2.3. THEOREM. *For all $B \in \boldsymbol{A}$ the map $\eta_B : B \to B^{**} = \boldsymbol{Z}\big(\boldsymbol{A}(B, A), A\big)$, is an isomorphism.*

Proof. Since $\boldsymbol{A} = \boldsymbol{ISP}(A)$, it follows that η_B is one-to-one. The operations in $B^{**} = \boldsymbol{Z}(B^*, A)$ are of course defined pointwise from the operations on A. Let f be an n-ary operation and let $g \in B^*$; then

$$\begin{aligned} g\big(f(b_0, \ldots, b_{n-1})\eta_B\big) &= f(b_0, \ldots, b_{n-1})g = f(b_0 g, \ldots, b_{n-1} g) \\ &= f\big(g(b_0\eta_B), \ldots, g(b_{n-1}\eta_B)\big), \end{aligned}$$

and so

$$f(b_0, \ldots, b_{n-1})\eta_B = f(b_0\eta_B, \ldots, b_{n-1}\eta_B).$$

Thus it remains to show that η_B is onto.

Let $R = \ker(\Phi)$, that is, by Lemma 2.2 (i),

$$g \equiv h(R) \Leftrightarrow g\Phi = h\Phi \Leftrightarrow ga = ha \Leftrightarrow \hat{g} = \hat{h}.$$

Note that, by Lemma 2.2 (ii), $[g]R = [\hat{g}]R$. The map Φ is continuous since, for all $b \in B$,

$$\langle b\,;1\rangle\Phi^{-1} = \langle b\,;1a^{-1}\rangle \quad \text{and} \quad \langle b\,;0\rangle\Phi^{-1} = \langle b\,;0a^{-1}\rangle.$$

Thus Φ induces a homeomorphism $\Phi' : B^*/R \to \boldsymbol{D}(B, 2)$. Define a partial order on B^*/R by

$$[g]R \leqslant [h]R \Leftrightarrow g\Phi \leqslant h\Phi$$

and note that, by Lemma 2.2 (v),

$$[g]R \leqslant [h]\Phi \Leftrightarrow \hat{g} \leqslant \hat{h}.$$

Thus Φ' is a homeomorphism and an order-isomorphism.

Let $\varphi \in \boldsymbol{Z}(B^*, A)$; the remainder of this proof is devoted to showing that there exists $b \in B$ such that $b\eta_B = \varphi$, i.e. $g\varphi = bg$ for all $g \in B^*$. Now $U = 1(\varphi a)^{-1}$ is clopen in B^*, and, by definition, $g \in U \Leftrightarrow g\varphi a = 1$. We claim that $g \in U \Leftrightarrow \hat{g} \in U$. Since $\hat{g}\varphi \in \operatorname{Im}(\hat{g})$ by Lemma 2.2 (vi), there exists $b \in B$ such that $\hat{g}\varphi = b\hat{g}$. Now assume that $g \in U$, i.e. $g\varphi a = 1$. Let $g = \hat{g}e$; then

$$\hat{g}\varphi a = b\hat{g}a = bga = b\hat{g}ea = \hat{g}\varphi ea = \hat{g}e\varphi a = g\varphi a = 1;$$

the second equality follows from Lemma 2.2 (ii). Thus $\hat{g} \in U$. Conversely, assume that $\hat{g} \in U$, i.e. $\hat{g}\varphi a = 1$. Observe that if $\hat{g}\varphi = b\hat{g}$, then $g\varphi = bg$; indeed,

$$g\varphi = \hat{g}e\varphi = \hat{g}\varphi e = b\hat{g}e = bg.$$

Thus $g\varphi a = bga = b\hat{g}a = \hat{g}\varphi a = 1$, and so $g \in U$.

It follows at once that U is a union of R-equivalence classes; for if $g \in U$ and $g \equiv h(R)$, then $\hat{g} = \hat{h}$, and hence $\hat{h} = \hat{g} \in U$, whence $h \in U$. Consequently, U/R is clopen in B^*/R. Moreover, U/R is increasing since U is, and U is increasing because $U = 1(\varphi a)^{-1}$ is the preimage of the increasing set $\{1\}$ under the order-preserving map φa. Hence $(U/R)\Phi'$ is a clopen increasing subset of $\boldsymbol{D}(B, 2)$, and so by Priestley's duality there exists $b \in B$ such that $(U/R)\Phi' = \langle b\,;1\rangle$. Since U is a union of R-equivalence classes, this implies that, for all $g \in B^*$, $g\varphi a = 1 \Leftrightarrow bga = 1$, and hence

(*) for all $g \in B^*$, $g\varphi a = bga$.

We claim that $b\eta_B = \varphi$, i.e. $g\varphi = bg$ for all $g \in B^*$. It is sufficient to prove that, for all $g \in B^*$, $\hat{g}\varphi = b\hat{g}$. Let $g \in B^*$; since $\boldsymbol{D}(A, 2)$ separates the points of A, we need only to show that $\hat{g}\varphi\beta = b\hat{g}\beta$ for all $\beta \in \boldsymbol{D}(A, 2)$. By Lemma 2.2 (vi), $\hat{g}\varphi \in \operatorname{Im}(\hat{g})$, and thus there exists $c \in B$ with $\hat{g}\varphi = c\hat{g}$. Let $\beta \in \boldsymbol{D}(A, 2)$ and set $h = (\hat{g}\beta)\Theta$; then $\hat{g}\beta = \hat{g}\beta\Theta\Phi = h\Phi = ha$. But by ($P_3$) there exists $e \in \operatorname{End}(A)$ such that $h = \hat{g}e$. Thus

$$c\hat{g}\beta = cha = c\hat{g}ea = \hat{g}\varphi ea = \hat{g}e\varphi a = h\varphi a.$$

Finally, using (*), we obtain

$$\hat{g}\varphi\beta = c\hat{g}\beta = h\varphi a = bha = b\hat{g}\beta,$$

as required.

In [4] the dual B^* of an algebra $B \in \boldsymbol{L}_n$ was endowed only with the action of $\mathrm{End}(A)$; no mention of a partial order on B^* was made. The proof of Theorem 2.3 is easily modified to cover this situation provided $\boldsymbol{A}$ satisfies the following condition:

(P_4) Let $B \in \boldsymbol{A}$; then for all $g, h \in B^*$ with $\hat{g} \leqslant \hat{h}$ there exists $e \in \mathrm{End}(A)$ such that $\hat{h} = \hat{g}e$.

It is easily seen that $\boldsymbol{L}_n$ satisfies (P_4); in fact, if $\mathrm{Im}(\hat{h}) = (c_k]\cup\{1\}$ with $k < n-1$, then for all $g \in B^*$ with $\hat{g} \leqslant \hat{h}$ we have $\hat{h} = \hat{g}e$, where e is the endomorphism of A determined by the prime filter $[c_{k+1})$.

Let $\boldsymbol{Y}$ be the category whose objects are Boolean spaces endowed with a continuous action of $\mathrm{End}(A)$ and whose morphisms are continuous maps which preserve the action of $\mathrm{End}(A)$. As an object of $\boldsymbol{Y}$, A is topologized discretely and $\mathrm{End}(A)$ acts on A via $\tilde{e} = e$.

2.4. Theorem. *Assume that* $\boldsymbol{A}$ *satisfies* (P_4). *Then for all* $B \in \boldsymbol{A}$ *the map* $\eta_B : B \to B^{**} = \boldsymbol{Y}(\boldsymbol{A}(B, A), A)$ *is an isomorphism.*

Proof. Firstly, we claim that if $\varphi : B^* \to A$ preserves the action of $\mathrm{End}(A)$, then φ preserves the pointwise partial order on the set $\{\hat{g} \mid g \in B^*\}$. Assume that $\hat{g} \leqslant \hat{h}$. Then by ($P_4$) there exists $e \in \mathrm{End}(A)$ such that $\hat{h} = \hat{g}e$. Now $a \in \mathrm{Im}(\hat{g})$ implies that $a = b\hat{g}$ for some $b \in B$, and so for each unary polynomial p we have

$$p(a) = p(b\hat{g}) = p(b)\hat{g} \leqslant p(b)\hat{h} = p(b\hat{h}) = p(b\hat{g}e) = p(ae);$$

thus $a \leqslant^* ae$ in A for all $a \in \mathrm{Im}(\hat{g})$. But, by Lemma 2.2 (vi), $\hat{g}\varphi \in \mathrm{Im}(\hat{g})$, and thus

$$\hat{g}\varphi \leqslant^* \hat{g}\varphi e = \hat{g}e\varphi = \hat{h}\varphi.$$

The fact that φ is order-preserving is used only once in the proof of Theorem 2.3; namely, to show that U/R is increasing. We claim that it is still true that U/R is increasing.

If $[g]R \in U/R$ and $[h]R \geqslant [g]R$, then $\hat{g} \in U$ and $\hat{g} \leqslant \hat{h}$. But $U = 1a^{-1}\varphi^{-1}$, and so $\hat{g}\varphi \in 1a^{-1}$. Since $\hat{g}\varphi \leqslant \hat{h}\varphi$ and $1a^{-1}$ is an increasing subset of A, it follows that $\hat{h}\varphi \in 1a^{-1}$, whence $\hat{h} \in U = 1a^{-1}\varphi^{-1}$. Consequently, $[h]R = [\hat{h}]R \in U/R$, and so U/R is increasing. Thus the proof of Theorem 2.3 carries over.

3. *THE DUALITIES*

Let $\boldsymbol{K}$ and $\boldsymbol{X}$ be categories and assume that $D : \boldsymbol{K} \to \boldsymbol{X}^{op}$ is left adjoint to $E : \boldsymbol{X}^{op} \to \boldsymbol{K}$; here $\boldsymbol{K}$ is to be thought of as a category whose objects are algebraic and the objects of $\boldsymbol{X}$ are to be thought of as topological.

Following [5], the pair $\langle D, E\rangle$ is a *duality* between $\boldsymbol{K}$ and $\boldsymbol{X}$ if the unit $\eta : \mathrm{id}_{\boldsymbol{K}} \to ED$ of the adjunction is a natural isomorphism, and $\langle D, E\rangle$ is a *full duality* if the counit $\varepsilon : \mathrm{id}_{\boldsymbol{X}} \to DE$ is also a natural isomorphism. It is natural, from an algebraic point of view, to concentrate on dualities rather than full dualities, although the latter yield more information of course.

In the previous section we defined the functor $D = \boldsymbol{A}(-, A)\colon \boldsymbol{A} \to \boldsymbol{Y}$. We define a functor $E = \boldsymbol{Y}(-, A)\colon \boldsymbol{Y} \to \boldsymbol{A}$ analogously: if f is an n-ary operation, then for all $\varphi_0, \ldots, \varphi_{n-1} \in \boldsymbol{Y}(X, A)$ the map $f(\varphi_0, \ldots, \varphi_{n-1})$ is continuous and preserves the action of $\mathrm{End}(A)$ (i.e. $\boldsymbol{Y}(X, A)$ is a subalgebra of A^X), whence E is well defined.

3.1. THEOREM. *Let A satisfy* (P_0)-(P_4). *Then* $\langle \boldsymbol{A}(-, A), \boldsymbol{Y}(-, A)\rangle$ *is a duality between* $\boldsymbol{A}$ *and* $\boldsymbol{Y}$ *with* $\eta : \mathrm{id}_{\boldsymbol{A}} \to ED$ *as the unit of the adjunction.*

Proof. By Theorem 2.3 it is sufficient to show that η is a natural transformation and that, for all $B \in \boldsymbol{A}$, $\eta_B : B \to ED(B)$ is universal to D from B. That η is a natural transformation is easily seen. For each homomorphism $a : B \to E(X)$ the unique fill-in map $\beta : X \to D(B)$ is given by $b(x\beta) = x(ba)$. It is easily checked that, for all $x \in X$, $x\beta : B \to A$ is a homomorphism, and thus β is well defined. For all $a \in A$ we have $\langle b\,;\, a\rangle\beta^{-1} = a(ba)^{-1}$, which is open in X since ba is continuous, and hence β is continuous since the sets of the form $\langle b\,;\, a\rangle$ form a subbasis for the topology on $D(B) = \boldsymbol{A}(B, A)$. Let $e \in \mathrm{End}(A)$; then for all $b \in B$

$$b\big((x\tilde{e})\beta\big) = x\tilde{e}(ba) = \big(x(ba)\big)e = \big(b(x\beta)\big)e = b(x\beta\tilde{e}),$$

since ba preserves the action of $\mathrm{End}(A)$, and thus β preserves the action of $\mathrm{End}(A)$. Hence $\beta \in \boldsymbol{Y}\big(X, D(B)\big)$ as required.

The category $\boldsymbol{Z}$ is not quite as well behaved as $\boldsymbol{Y}$; for though $\boldsymbol{Z}(B^*, A)$ is closed under the pointwise operations for all $B \in \boldsymbol{A}$, it does not necessarily follow that $\boldsymbol{Z}(X, A)$ is closed under the pointwise operations for all $X \in \boldsymbol{Z}$.

For the remainder of this section assume that $\boldsymbol{X}$ is a full subcategory of $\boldsymbol{Z}$ satisfying:

(i) $\boldsymbol{X}$ is closed under $\boldsymbol{Z}$-isomorphisms;

(ii) $\boldsymbol{X}$ contains the image of the functor $D = \boldsymbol{A}(-, A)$;

(iii) for all $X \in \boldsymbol{X}$, $E(X) = X^* = \boldsymbol{X}(X, A)$ is a subalgebra of A^X.

It follows easily that $E = \boldsymbol{X}(-, A)\colon \boldsymbol{X} \to \boldsymbol{A}$ is a well-defined functor.

3.2. THEOREM. *Let A satisfy* (P_0)-(P_3). *Then* $\langle \boldsymbol{A}(-, A), \boldsymbol{X}(-, A)\rangle$ *is a duality between* $\boldsymbol{A}$ *and* $\boldsymbol{X}$ *with* $\eta : \mathrm{id}_{\boldsymbol{A}} \to DE$ *as the unit of the adjunction.*

Proof. This follows immediately from the proof of Theorem 3.1 once we observe that if $\alpha : B \to E(X) = \boldsymbol{X}(X, A)$, then the map β is order-preserving, and so $\beta \in \boldsymbol{X}(X, D(B))$.

Under what conditions will these dualities be full? One approach is provided by the results of Davey [5]. Since Boolean spaces are precisely the zero-dimensional compact spaces, the category of Boolean spaces is denoted by ***ZComp***. For each Boolean space X, the set $\mathscr{C}(X, A)$ of continuous functions from X into A is a subalgebra of A^X, and hence $\mathscr{C}(-, A) : \boldsymbol{ZComp} \to \boldsymbol{A}$ is a well-defined functor. If $\langle D, E\rangle$ is a full duality between $\boldsymbol{A}$ and $\boldsymbol{X}$, say, then it is easily seen that $D\mathscr{C}(-, A)$: $\boldsymbol{ZComp} \to \boldsymbol{X}$ is left adjoint to the forgetful functor; and similarly for the categories $\boldsymbol{A}$ and $\boldsymbol{Y}$. The following is an immediate corollary to Theorem 2.25 of [5].

3.3. THEOREM. *Assume that A satisfies* (P_0)-(P_3) *and assume that*

(i) *$\boldsymbol{X}(X, A)$ separates the points of X for each $X \in \boldsymbol{X}$;*

(ii) *$D\mathscr{C}(-, A) : \boldsymbol{ZComp} \to \boldsymbol{X}$ is left adjoint to the forgetful functor;*

(iii) *A is injective in $\boldsymbol{A}$.*

Then $\langle \boldsymbol{A}(-, A), \boldsymbol{X}(-, A)\rangle$ is a full duality between $\boldsymbol{A}$ and $\boldsymbol{X}$. If A satisfies (P_0)-(P_4), *then an analogous result holds for the categories $\boldsymbol{A}$ and $\boldsymbol{Y}$.*

We can describe a reasonably general situation in which condition (ii) of Theorem 3.3 holds. For each Boolean space X let $\mathscr{F}(X) = X \times \mathrm{End}(A)$ and define the action of $\mathrm{End}(A)$ on $\mathscr{F}(X)$ by $\langle x, f\rangle\tilde{e} = \langle x, fe\rangle$; if $\psi \in \mathscr{C}(X, Y)$, then define $\mathscr{F}(\psi) \in \boldsymbol{Y}(\mathscr{F}(X), \mathscr{F}(Y))$ by $\langle x, e\rangle\mathscr{F}(\psi) = \langle x\psi, e\rangle$. Clearly, $\mathscr{F}$: $\boldsymbol{ZComp} \to \boldsymbol{Y}$ is a well-defined functor. In an attempt to lift $\mathscr{F}$ to a functor $\mathscr{F}$: $\boldsymbol{ZComp} \to \boldsymbol{X}$ we define a partial order on $\mathscr{F}(X)$ by

$$\langle x, e\rangle \leqslant \langle y, f\rangle \quad \text{if and only if} \quad x = y \text{ and } e \leqslant f.$$

Define μ_X: $\mathscr{F}(X) \to D\mathscr{C}(X, A) = \boldsymbol{A}(\mathscr{C}(X, A), A)$ by $\langle x, e\rangle\mu_X = \Gamma_x e$.

The following is an almost trivial consequence of Corollary 3.3 of [4].

3.4. THEOREM. *Let A be a finite nontrivial lattice-ordered algebra whose bounds,* 0 *and* 1, *are nullary operations, and assume that every subalgebra of A is subdirectly irreducible.*

(i) *$D\mathscr{C}(-, A)$: $\boldsymbol{ZComp} \to \boldsymbol{Y}$ is naturally isomorphic to $\mathscr{F}$: $\boldsymbol{ZComp} \to \boldsymbol{Y}$ and is left adjoint to the forgetful functor.*

(ii) *$D\mathscr{C}(-, A)$: $\boldsymbol{ZComp} \to \boldsymbol{X}$ is naturally isomorphic to $\mathscr{F}$: $\boldsymbol{ZComp} \to \boldsymbol{X}$ and is left adjoint to the forgetful functor.*

Proof. The proof is almost identical to the proof of Theorem 3.5 of [4] except that for part (ii) we must check that μ_X is an order-isomorphism. It is clear that μ_X is order-preserving. Now assume that $\langle x, e\rangle\mu_X \leqslant \langle y, f\rangle\mu_X$, i.e. $\Gamma_x e \leqslant \Gamma_y f$. Since $\mathscr{C}(X, A)$ contains the constant maps, it follows that $e \leqslant f$; it remains to prove that $x = y$. Suppose $x \neq y$; then let U be a clopen subset of X with $x \in U$ and $y \notin U$, and let $\varphi \in \mathscr{C}(X, A)$ be the characteristic function of U. Since 0 and 1 are nullary operations, we obtain the following contradiction:

$$0 = 0e = y\varphi e = \varphi(\Gamma_y e) \geqslant \varphi(\Gamma_x f) = x\varphi f = 1f = 1.$$

Note that $\mathscr{F}$: $\boldsymbol{ZComp} \to \boldsymbol{X}$ is well defined since, for every Boolean space X, $\mathscr{F}(X)$ is isomorphic in $\boldsymbol{Z}$ to $D\mathscr{C}(X, A)$ and we have assumed that $\boldsymbol{X}$ is closed under $\boldsymbol{Z}$-isomorphisms.

Under the assumptions of Theorem 3.4 it is easy to check condition (iii) of Theorem 3.3. The algebra A is called *self-injective* if every homomorphism from a subalgebra of A extends to an endomorphism of A. Theorem 4.1 of Day [8] yields the following result:

3.5. THEOREM. *Let A be a finite nontrivial lattice-ordered algebra and assume that every subalgebra of A is subdirectly irreducible. Then A is injective in $\boldsymbol{A}$ if and only if A is self-injective.*

We close by deriving the dualities which motivated this paper. Following [4], let $\boldsymbol{X}_n$ denote the category of Boolean spaces endowed with a continuous action of the endomorphism monoid, End$(\boldsymbol{n})$, of the Heyting algebra $\boldsymbol{n}$; that is, $\boldsymbol{X}_n$ denotes the category $\boldsymbol{Y}$ in the case $A = \boldsymbol{n}$.

3.6. Duality for relative Stone algebras. *$\langle \boldsymbol{L}_n(-, \boldsymbol{n}), \boldsymbol{X}_n(-, \boldsymbol{n})\rangle$ is a duality between $\boldsymbol{L}_n$ and $\boldsymbol{X}_n$. The duality is full if and only if n equals 2 or 3* (Davey [4]).

Proof. Note that $\boldsymbol{n}$ satisfies (P_0)-(P_4), 0 and 1 are nullary operations of $\boldsymbol{n}$, and every subalgebra of $\boldsymbol{n}$ is subdirectly irreducible; thus Theorems 3.1, 3.4, and 3.5 apply. It is easily seen that $\boldsymbol{n}$ is self-injective if and only if n equals 2 or 3. A very simple example is given in [4], which shows that the duality is not full for $n \geqslant 4$. Thus, by Theorem 3.3, it remains to prove that, for n equal to 2 or 3, $\boldsymbol{X}_n(X, \boldsymbol{n})$ separates the points of X for each $X \in \boldsymbol{X}_n$. For $n = 2$ this is a trivial consequence of the definition of Boolean space. For $n = 3$ we require a little sleight of hand. Note that End$(\boldsymbol{3})$ $= \{e_1, e_2\}$, where e_2 is the identity map and $e_1 e_1 = e_1$. Let $X \in \boldsymbol{X}_3$ and let $x, y \in X$ with $x \neq y$. Without loss of generality we may assume $x\tilde{e}_1 \neq y$. Let U be a clopen subset of X with $x, x\tilde{e}_1 \in U$ and $y \notin U$, and define φ_U: $X \to \boldsymbol{3}$ by

$$z\varphi_U = \begin{cases} 1 & \text{if } z \in U \cap U\tilde{e}_1^{-1}, \\ c_1 & \text{if } z \in (U\tilde{e}_1^{-1}) - U, \\ 0 & \text{if } z \notin U\tilde{e}_1^{-1}. \end{cases}$$

It is readily verified that $\varphi_U \in \boldsymbol{X}_3(X, \boldsymbol{3})$ and that $x\varphi_U = 1$ and $y\varphi_U \in \{c_1, 0\}$.

We turn now to Stone algebras. A partially ordered topological space X is *totally order-disconnected* if for all $x, y \in X$ with $x \not\leqslant y$ there is a clopen increasing subset U of X with $x \in U$ and $y \notin U$. Note that the underlying space of a totally order-disconnected compact space is a Boolean space. Recall that the endomorphism monoid End$(\boldsymbol{3})$ of the three-element Stone algebra consists of the identity map e_0 and the map e_1 given by $ce_1 = c^{**}$. For each Stone algebra B the space $S(B, \boldsymbol{3})$ is a totally order-disconnected compact space. For all $x \in S(B, \boldsymbol{3})$, $x\tilde{e}_1$ is the unique maximal element of $S(B, \boldsymbol{3})$ which dominates x, and of course $\tilde{e}_0$ is the identity map on $S(B, \boldsymbol{3})$. Let $\boldsymbol{V}$ be the full subcategory of $\boldsymbol{Z}$ whose objects are totally order-disconnected compact spaces X such that, for each $x \in X$, $x\tilde{e}_1$ is the maximal element of X which dominates x. Clearly, $\boldsymbol{V}$ is closed under $\boldsymbol{Z}$-isomorphisms and, by the remarks above, $\boldsymbol{V}$ contains the image of the functor $\boldsymbol{S}(-, \boldsymbol{3})$. The partial order on $\boldsymbol{3}$ as an object of $\boldsymbol{V}$ is determined by $x <^* y \Leftrightarrow (x = a$ and $y = 1)$. It is easily seen that $\boldsymbol{V}(X, \boldsymbol{3})$ is a subalgebra of $\boldsymbol{3}^X$ for all $X \in \boldsymbol{V}$. Hence Theorem 3.2 applies.

3.7. Duality for Stone algebras. (Davey [5]) $\langle \boldsymbol{S}(-, \boldsymbol{3}), \boldsymbol{V}(-, \boldsymbol{3})\rangle$ *is a full duality between* **S** *and* **V**.

Proof. By Theorem 3.2 we have a duality. Since 0 and 1 are nullary operations of $\boldsymbol{3}$ and every subalgebra of $\boldsymbol{3}$ is subdirectly irreducible, Theorems 3.4 and 3.5 apply. Clearly, $\boldsymbol{3}$ is self-injective, and so by Theorem 3.3 we must show that $\boldsymbol{V}(X, \boldsymbol{3})$ separates the points of X for each $X \in \boldsymbol{V}$. The proof, which is similar to the corresponding result for $\boldsymbol{X}_3$, is given in Proposition 4.5.3 of [5].

Since $\tilde{e}_0$ is the identity map, it may be ignored, and so the objects of $\boldsymbol{V}$ can be viewed as partially ordered spaces endowed with a single continuous retraction, namely $\tilde{e}_1$; this is the approach used in [5]. The proof of 3.7 given here is quite different to the proof in [5].

Finally, we present a new duality for the class $\boldsymbol{T} = \boldsymbol{ISP}(\boldsymbol{4})$ of double Stone algebras. Recall that $\boldsymbol{4}$ has two nontrivial endomorphisms denoted by e_1 and e_2.

Let $\boldsymbol{W}$ be the full subcategory of $\boldsymbol{Z}$ whose objects are totally order-disconnected compact spaces X such that, for each $x \in X$, $x\tilde{e}_1$ is the unique maximal element of X which dominates x, and $x\tilde{e}_2$ is the unique minimal element of X which is dominated by x. It is easily verified that $\boldsymbol{W}$ is closed under $\boldsymbol{Z}$-isomorphisms, $\boldsymbol{W}$ contains the image of the functor $\boldsymbol{T}(-, \boldsymbol{4})$, and $\boldsymbol{W}(X, \boldsymbol{4})$ is a subalgebra of $\boldsymbol{4}^X$ for all $X \in \boldsymbol{W}$. The partial order on $\boldsymbol{4}$, as an object of $\boldsymbol{W}$, is determined by $x <^* y \Leftrightarrow (x = a$ and $y = b)$. Again Theorem 3.2 applies.

3.8. Duality for double Stone algebras. $\langle \boldsymbol{T}(-, \boldsymbol{4}), \boldsymbol{W}(-, \boldsymbol{4})\rangle$ *is a full duality between* **T** *and* **W**.

Proof. Exactly as in the case of Stone algebras, by Theorems 3.2-3.5, it remains to prove that $W(X, 4)$ separates the points of X for all $X \in W$. If x and y are distinct points of X, then either $x \not\leqslant^* y$ or $y \not\leqslant^* x$, say $x \not\leqslant^* y$. Thus there exists a clopen increasing subset U of X with $x \in U$ and $y \notin U$. Put

$$U_0 = X - U\tilde{e}_1^{-1}, \quad U_a = U\tilde{e}_1^{-1} \cap (X - U), \quad U_b = (X - U)\tilde{e}_2^{-1} \cap U,$$
$$U_1 = X - (X - U)\tilde{e}_2^{-1}.$$

Then U_0 and U_1 are increasing and decreasing, U_a is decreasing, U_b is increasing, and $U_a \cup U_b = U\tilde{e}_1^{-1} \cap (X - U)\tilde{e}_2^{-1}$ is increasing and decreasing. Furthermore,

$$U_0\tilde{e}_1 \subseteq U_0, \quad U_a\tilde{e}_1 \subseteq U_a \cup U_b, \quad U_b\tilde{e}_1 \subseteq U_b, \quad \text{and} \quad U_1\tilde{e}_1 \subseteq U_1,$$

and similarly for $\tilde{e}_2$. Thus we may define a map $\varphi_U \in W(X, 4)$ by

$$z\varphi_U = \begin{cases} 1 & \text{if } z \in U_1, \\ b & \text{if } z \in U_b, \\ a & \text{if } z \in U_a, \\ 0 & \text{if } z \in U_0. \end{cases}$$

Since $x\varphi_U \in \{b, 1\}$ and $y\varphi_U \in \{0, a\}$, we are through.

These dualities have various applications. We noted in Section 1 that for Stone algebras, double Stone algebras, and for L_2 and L_3 the map Θ is a two-sided inverse of the map Φ. Thus, for such an algebra, say B, its dual, as defined here, is homeomorphic and order-isomorphic to its Priestley's dual $D(B, 2)$. It follows, e.g., that coproducts in S, T, L_2, and L_3 agree with coproducts in D. For further applications of the dualities, for example to describe free algebras and injectives, we refer to [4]-[7].

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Reçu par la Rédaction le 12. 6. 1978