COLLOQUIUM MATHEMATICUM

VOL. XLVII 1982 FASC. 2

Z_n-ACTIONS ON 3-MANIFOLDS

BY

JÓZEF H. PRZYTYCKI (WARSZAWA)

1. Introduction. The aim of this paper is to present a reduction of a given Z_n-action on a 3-manifold to simpler actions. The technique introduced here enables us to classify Z_n-actions on some manifolds, e.g., free Z_n-actions on handlebodies and compact surfaces. Tollefson [22] has constructed such a reduction for an involution of a 3-manifold M, provided no summand of M is a 2-sphere-bundle over S^1. Kim and Tollefson [8] have extended this result for involutions of an arbitrary closed 3-manifold.

We work in the PL-category. From now on, unless otherwise specified, manifolds are connected. We assume that if a manifold M is orientable, then it is oriented, and that if $\partial_i M$ is an orientable component of ∂M, then it is oriented (the orientation of ∂M is induced from M if M is oriented).

In Section 2 we introduce some necessary definitions and lemmas concerning 3-manifolds. In Section 3 we define a connected sum and a multiple of Z_n-manifolds. In Section 4 we prove the main Theorem 4.1 describing the decomposition of a Z_n-action along a compact surface and apply it for $F = S^2$, D^2, and P^2. In Section 5 we prove some facts which allow us to apply the theorems from Section 4 to concrete calculations. In Section 6 we prove some reduction theorems. In Section 7 we discuss our results and add some special corollaries on Z_n-actions.

The author wishes to thank very sincerely Agnieszka Bojanowska for her invaluable assistance.

The author is grateful to Krzysiek Nowiński and Krzysiek Jaczewski for their help in the preparation of this paper.

2. Preliminaries. The terminology of our paper is based on that of Hempel's book [5]. Moreover, we introduce some other definitions and notation and formulate some "folklore" results on 3-manifolds.

2.1. Definition. A 3-manifold M is called *decomposable* if it can be expressed as a connected sum of a finite number of prime factors. We put

$i(M) = \infty$ if M is not decomposable, and for M decomposable we define $i(M)$ to be the number of nontrivial ($\neq S^2$) prime factors of the prime factorization of M. Moreover, $i(S^3) = 0$.

2.2. Definition. (a) If a manifold M is orientable, then we put $\tilde{M} = M$, and if M is not orientable, then we define $\tilde{M}$ to be the orientable double cover of M.

(b) $\hat{M}$ is the manifold obtained from the n-manifold M by capping off each $(n-1)$-sphere component of ∂M with an n-cell.

Let M be a 3-manifold with compact boundary; we define:

(c) $k_{\partial M}$ to be the number of the boundary components of M.

(d) $\mathrm{gen}(M) = \frac{1}{2}\big(2k_{\partial M} - \chi(\partial M)\big)$.

2.3. Lemma. *Let M be a compact 3-manifold. Then $\pi_1(M) = 0$ iff M is a homotopy sphere with holes.*

2.4. Proposition. *If a 3-manifold M is a connected sum $M_1 \# M_2$, where neither M_1 nor M_2 is a homotopy sphere, then a 2-sphere S^2 separating two summands of $M_1 \# M_2$ generates an element of $\pi_2(M)$ which does not belong to the $\pi_1(M)$-invariant subgroup of $\pi_2(M)$ generated by classes of maps $P^2 \hookrightarrow \partial M$, where P^2 is a projective plane (if $\partial M = \emptyset$, then $[S^2] \neq 0$).*

Moreover: (a) *if $S^2 \hookrightarrow \mathrm{int}(M)$ does not separate M, then it generates a nontrivial element of $\pi_2(M)$;*

(b) *each P^2 embedded as a 2-sided submanifold in* $\mathrm{int} M$ *generates a nontrivial element of $\pi_2(M)$.*

The proposition seems to be a well-known "folk" theorem (cf. [13]).

2.5. Definition. (a) A closed surface $F \hookrightarrow \mathrm{int}(M^3)$ is *parallel to the boundary* if F is parallel to some component of the boundary, i.e., if there is an embedding $H\colon F \times I \hookrightarrow M^3$ such that

$$H(F \times \{1\}) = F \quad \text{and} \quad H(F \times I) \cap \partial M^3 = H(F \times \{0\}).$$

(b) A projective plane $P_0^2 \hookrightarrow \mathrm{int}(M^3)$ is *homotopy parallel to the boundary* if there is a 3-manifold $(W, \partial_1 W, \partial_2 W)$, embedded in M^3 with two boundary components $\partial_1 W = P_0^2$ and $\partial_2 W = P_1^2 \hookrightarrow \partial M$, such that W is obtained from a homotopy sphere Σ^3 by factorization of $\Sigma^3 - \big(\mathrm{int}(D_1^3) \cup \mathrm{int}(D_2^3)\big)$ by an orientation-reversing involution with two fixed points x_1 and x_2 with invariant, regular neighborhoods D_1^3 and D_2^3, respectively.

2.6. Remark. It follows from a theorem of Livesay [12] that if the Poincaré Conjecture is true, then P^2 is homotopy parallel to the boundary iff P^2 is parallel to the boundary.

2.7. Definition. Two 3-manifolds M_1 and M_2 are *congruent* if there exist two homotopy spheres Σ_1^3 and Σ_2^3 such that $M_1 \# \Sigma_1^3 = M_2 \# \Sigma_2^3$.

2.8. Definition (see [21]). A manifold M^3 with $\partial M^3 \neq \emptyset$ is called *∂-irreducible* if for any proper embedding $(D^2, \partial D^2) \hookrightarrow (M^3, \partial M^3)$ there exists a disk D^3 embedded in M^3 with $D^3 \cap (\partial M^3 \cup D^2) = \partial D^3$ (*proper* means $\partial M^3 \cap D^2 = \partial D^2$).

2.9. LEMMA. *If P^2 is a 2-sided projective plane in a 3-manifold M and M is not irreducible (respectively, M is not congruent to an irreducible manifold), then there is an embedding $S^2 \hookrightarrow \mathrm{int}(M)$ such that $S^2 \cap P^2 = \emptyset$ and S^2 does not bound a 3-cell (respectively, a homotopy 3-cell) in M. Moreover, if $S_1^2 \hookrightarrow \mathrm{int}(M)$ does not bound a 3-cell (respectively, a homotopy 3-cell) in M, then we may require $S^2 \subset V$ for any fixed regular neighborhood V of $S_1^2 \cup P^2$.*

Proof is similar to that of [13] and [23]. Let M be not an irreducible manifold. There exists an embedding $S_1^2 \hookrightarrow \mathrm{int}(M)$ not bounding a 3-cell. We may assume that S_1^2 is in a general position with respect to P^2. Then $S_1^2 \cap P^2$ is a collection of pairwise disjoint, simple, closed curves $\gamma_1, \ldots, \gamma_k$ in P^2. At least one of these curves (say, γ_k) bounds a 2-cell D_1 in P^2 with $D_1 \cap \bigcup_i \gamma_i = \gamma_k$. To prove this we use the fact that γ_i is 2-sided in P^2 for each i. Otherwise, γ_i changes the orientation in P^2, so γ_i changes the orientation in M, and this contradicts the fact that $\gamma_i \subset S_1^2$ is contractible in M. Let γ_k separate S_1^2 into disks E_1 and E_2. Since S_1^2 does not bound a 3-cell in M, either $D_1 \cup E_1$ or $D_1 \cup E_2$ does not bound a 3-cell in M. We move slightly this sphere and obtain a 2-sphere S_2^2 which does not bound a 3-cell in M and $S_2^2 \cap P^2$ consists of at most $k-1$ simple closed curves. By repeating this process we obtain the desired embedding $S^2 \hookrightarrow \mathrm{int}(M)$. Similarly we can prove Lemma 2.9 for M congruent to an irreducible manifold.

2.10. COROLLARY. *If $\{P_i^2\}_{i=1}^n$ is a finite system of pairwise disjoint, 2-sided projective planes in M and $M_1, \ldots, M_s$ are connected 3-manifolds obtained by cutting M along $\bigcup_i P_i^2$, then*

(a) *M is an irreducible manifold iff each M_i is an irreducible manifold;*

(b) *M is a manifold congruent to an irreducible manifold iff each M_i is congruent to an irreducible manifold.*

Proof. One way is easy, and the other follows from Lemma 2.9.

2.11. LEMMA. *If P^2 is 2-sided in a 3-manifold M and M is not ∂-irreducible (respectively, M is not congruent to a ∂-irreducible manifold), then there are two possibilities:*

(a) *there is an embedding $S^2 \hookrightarrow \mathrm{int}(M)$ such that $S^2 \cap P^2 = \emptyset$ and S^2 does not bound a 3-cell (respectively, homotopy 3-cell);*

(b) *there is a proper embedding $D^2 \hookrightarrow M$ such that $D^2 \cap P^2 = \emptyset$ and ∂D^2 does not bound a 2-cell in ∂M.*

Proof. If (a) is not satisfied, then M is irreducible (respectively, congruent to an irreducible manifold) by Lemma 2.9. Thus there exists a proper embedding $D_0^2 \hookrightarrow M$ such that ∂D_0^2 does not bound a 2-cell in ∂M. Now we use the "cut and paste" technique (similarly as in Lemma 2.9) to obtain a disk D^2 with $\partial D^2 = \partial D_0^2$ which satisfies (b).

The next corollary follows similarly as 2.10.

2.12. Corollary. *If $\{P_i^2\}_{i=1}^n$ is a finite system of pairwise disjoint, 2-sided projective planes in M and $M_1, \ldots, M_s$ are connected 3-manifolds obtained by cutting M along $\bigcup_{i=1}^n P_i^2$, then the following holds:*

(a) *M is either an irreducible manifold without boundary or a ∂-irreducible, irreducible manifold iff each M_i is an irreducible, ∂-irreducible manifold;*

(b) *M is either congruent to an irreducible manifold without boundary or to a ∂-irreducible, irreducible manifold iff each M_i is congruent to an irreducible, ∂-irreducible manifold.*

Conclusions analogous to Lemmas 2.9 and 2.11 hold also for S^2 and D^2.

2.13. Lemma. *If D^2 is a proper disk embedded into a 3-manifold M and M is not irreducible (respectively, congruent to an irreducible manifold), then there is an embedding $S^2 \hookrightarrow \mathrm{int}(M)$ such that $S^2 \cap D^2 = \emptyset$ and S^2 does not bound a 3-cell (respectively, a homotopy 3-cell).*

2.14. Corollary. *If $\{D_i^2\}_{i=1}^n$ is a finite system of pairwise disjoint disks properly embedded in M and $M_1, \ldots, M_s$ are connected manifolds obtained by cutting M along $\bigcup_{i=1}^n D_i^2$, then the following equivalences are satisfied:*

(a) *M is irreducible iff each M_i is irreducible;*

(b) *M is congruent to an irreducible manifold iff each M_i is congruent to an irreducible manifold.*

3. Basic constructions. We define first a connected sum of Z_n-manifolds as a generalization of an ordinary connected sum.

3.1. Definition. Let (M_i, Z_{n_i}) be an n-dimensional manifold M_i with an action of Z_{n_i} generated by T_i $(i = 1, 2)$. Let $F_i \subset \partial_{a_i} M_i$ be a compact manifold embedded in some component $\partial_{a_i} M_i$ of ∂M_i such that $T_i^j(F_i) \cap \cap F_i = F_i$ or $\emptyset$ for each j and that there exists a number j such that $Z_j = \{g \in Z_{n_i}\colon g(F_i) = F_i\}$ $(i = 1, 2)$. Let $f\colon F_1 \to F_2$ be a Z_j-equivariant homeomorphism (which reverses orientation if $\partial_{a_1} M_1$ and $\partial_{a_2} M_2$ are oriented). We define the *connected sum*

$$(M, Z_n) = (M_1, Z_{n_1}) \,\hat{\#}_{(F_1, F_2, f)} (M_2, Z_{n_2})$$

as follows: Let

$$j_0 = \frac{\text{g.c.d.}(n_1, n_2)}{j}, \quad s_1 = \frac{n_2}{jj_0}, \quad s_2 = \frac{n_1}{jj_0},$$

$$n = \text{l.c.m.}(n_1, n_2) = jj_0 s_1 s_2, \quad \text{and} \quad k = j_0 s_1 s_2.$$

Let

$$M_i \overset{h_{i,1}}{=} M_{i,2} \overset{h_{i,2}}{=} M_{i,3} \overset{h_{i,3}}{=} \dots \overset{h_{i,s_i-1}}{=} M_{i,s_i} \overset{h_{i,s_i}}{=} M_i,$$

where $h_{i,s_i} \dots h_{i,1} = T_i$ $(i = 1, 2)$.

For each j we identify $h_{1,j} \dots h_{1,1}(F_1)$ with $h_{2,j} \dots h_{2,1}(F_2)$ using the homeomorphism $h_{2,j} \dots h_{2,1} f h_{1,1}^{-1} \dots h_{1,j}^{-1}$. The Z_n-action on the obtained manifold M is determined by the maps $h_{1,1}, \dots, h_{1,s_1}, h_{2,1}, \dots, h_{2,s_2}$ (Fig. 1).

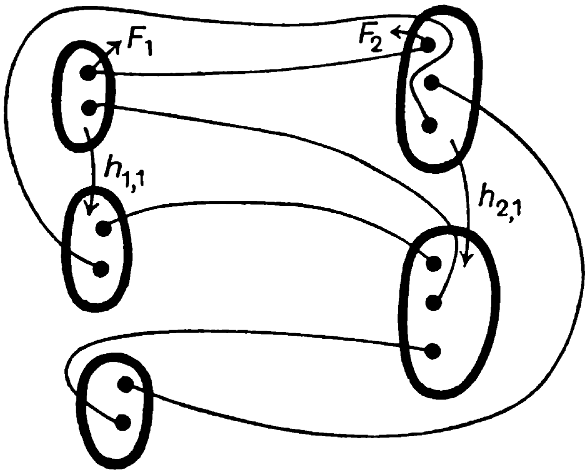

Fig. 1. Case $j_0 = 1$, $s_1 = 3$, $s_2 = 2$

If $j = 1$ in the connected sum of Z_n-manifolds, then we will write $\overline{\#}$ instead of $\hat{\#}$.

"Multiple" of Z_n-manifold is the second useful method of constructing new Z_n-actions.

3.2. Definition. Let (M_1, Z_{n_1}) be an n-dimensional manifold M_1 with an action of Z_{n_i} generated by g. Let $F_i \subset \partial M_1$ $(i = 1, 2)$ be a compact manifold embedded in ∂M such that, for each j, $g^j(F_i) \cap F_i = F_i$ or $\emptyset$ and $g^j(F_1) \cap F_2 = \emptyset$. Suppose there exists a number j such that $Z_j = \{g \in Z_{n_1}: g(F_i) = F_i\}$. Let $f\colon F_1 \to F_2$ be a Z_j-equivariant homeomorphism. Let s and r be coprime natural numbers $((s, r) = 1)$. We define the *multiple*

$$(M, Z_n) = (M_1, Z_{n_1})^s_{(F_1, F_2, f, r)}$$

as follows:

Let $n = n_1 s$ and $j_0 = n_1/j$. Let

$$M_1 \overset{h_1}{=} M_2 \overset{h_2}{=} \dots \overset{h_{s-1}}{=} M_s \overset{h_s}{=} M_1,$$

where $h_s \dots h_1 = g$. For each i we identify $h_{r+i} \dots h_r \dots h_1(F_1)$ with $h_i \dots h_1(F_2)$ using the homeomorphism $h_i \dots h_1 f h_1^{-1} \dots h_{r+i}^{-1}$. The Z_n-action on the obtained manifold M is determined by the maps $h_1, \dots, h_s$ (Fig. 2).

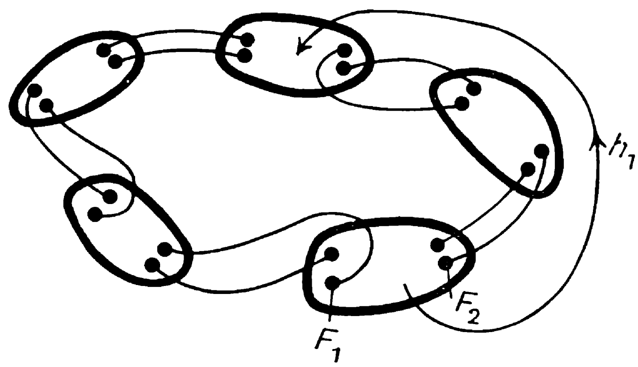

Fig. 2. Case $s = 5$, $r = 3$, $j_0 = 2$

3.3. Remarks. (a) We define $(M_1, Z_{n_1}) \hat{\#}_{(x_1,x_2,f)} (M_2, Z_{n_2})$, where $x_i \in \operatorname{int}(M_i)$ $(i = 1, 2)$, isotropy groups of x_1 and x_2 are isomorphic to Z_j, and f maps Z_j-equivariantly some Z_j-invariant, regular neighborhoods of x_1 and x_2 $(f(x_1) = x_2)$, as

$$\overline{\left(M_1 - \bigcup_{i=1}^{n_1} T_1^i(D_{x_1}), Z_{n_1}\right)} \hat{\#}_{(\partial D_{x_1}, \partial D_{x_2}, f|\partial D_{x_1})} \overline{\left(M_2 - \bigcup_{i=1}^{n_2} T_2^i(D_{x_2}), Z_{n_2}\right)},$$

where T_i is the generator of the Z_{n_i}-action and D_{x_i} is a sufficiently small, Z_j-invariant, regular neighborhood of x_i $(i = 1, 2)$.

Let $\operatorname{Iz}(M)$ denote the set of points with nontrivial isotropy group (for a given G-action on M). If $x_i \in \operatorname{int}(M_i) - \operatorname{Iz}(M_i)$ $(i = 1, 2)$, then we write $(M_1, T_1) \bar{\#}_{(x_1,x_2,f)} (M_2, T_2)$ instead of $(M_1, T_1) \hat{\#}_{(x_1,x_2,f)} (M_2, T_2)$. Moreover, if $(M_i - \operatorname{Iz}(M_i))$ is orientable, then we assume that it is oriented and f reverses orientation.

(b) We define $(M_1, Z_{n_1})^s_{(x_1,x_2,f,r)}$ by a slight modification of Definition 3.2 (similarly as (a) is a modification of Definition 3.1).

From the definitions of Z_n-connected sum and multiple we obtain immediately the following facts:

3.4. Proposition. I. *Let* $(M, Z_n) = (M_1, Z_{n_1}) \hat{\#}_{(F_1,F_2,f)} (M_2, Z_{n_2})$. *Then*:

(a) *If the manifolds M_1 and M_2 are oriented and T_1, T_2 preserve orientation, then Z_n acts on the oriented manifold M preserving orientation.*

(b) *If the manifolds M_1 and M_2 are oriented and T_1, T_2 satisfy the condition: T_i reverses orientation iff s_i is odd, then Z_n acts on the oriented manifold M reversing orientation.*

(c) *If conditions* (a) *and* (b) *are not satisfied, then M is a nonorientable manifold.*

II. *Let* $(M, Z_n) = (M_1, Z_{n_1})^s_{(F_1, F_2, f, r)}$. *Then:*

(a) *If the manifold* M_1 *is oriented, g preserves orientation, and f reverses orientation, then* Z_n *acts on the oriented manifold* M *preserving orientation.*

(b) *If the manifold* M_1 *is oriented and g, f satisfy the conditions*

(i) *g preserves orientation iff s is even,*

(ii) *f reverses orientation iff r is even,*

then Z_n *acts on the oriented manifold* M *reversing orientation.*

(c) *If conditions* (a) *and* (b) *are not satisfied, then* M *is a nonorientable manifold.*

3.5. PROPOSITION. I$_1$. *If* $(M, Z_n) = (M_1, Z_{n_1}) \hat{\#}_{(x_1, x_2, f} (M_2, Z_{n_2})$, *then*

$$M = \underbrace{M_1 \# \dots \# M_1}_{s_1\ times} \# \underbrace{M_2 \# \dots \# M_2}_{s_2\ times} \# \underbrace{S^1 \times S^2 \# \dots \# S^1 \times S^2}_{k-s_1-s_2+1\ times}$$
$$= (M_1)_{s_1} \# (M_2)_{s_2} \# (S^1 \times S^2)_{k-s_1-s_2+1}$$

if Z_n *preserves orientation;*

$$M = (M_1)_{[(s_1+1)/2]} \# (-M_1)_{[s_1/2]} \# (M_2)_{[(s_2+1)/2]} \# (-M_2)_{[s_2/2]} \# (S^1 \times S^2)_{k-s_1-s_2+1}$$

($[x]$ *denotes the integer part of* x) *if* Z_n *reverses orientation;*

$$M = (M_1)_{s_1} \# (M_2)_{s_2} \# (N)_{k-s_1-s_2+1}$$

(N *denotes the nonorientable* S^2*-bundle over* S^1) *if* M *is nonorientable.*

I$_2$. *If* $(M, Z_n) = (M_1, Z_{n_1}) \hat{\#}_{(P_1^2, P_2^2, f)} (M_2, Z_{n_2})$, *then*

$$\tilde{M} = (\tilde{M}_1^d)_{s_1} \# (\tilde{M}_2^d)_{s_2} \# (S^1 \times S^2)_{k-s_1-s_2+1},$$

where $\tilde{M}_i^d$ *is obtained from* $\tilde{M}_i$ *by capping off each 2-sphere component of* $\partial \tilde{M}_i$ *which covers the projective plane from the construction.*

II$_1$. *If* $(M, Z_n) = (M_1, Z_{n_1})^s_{(x_1, x_2, f, r)}$, *then*

$$M = \begin{cases} (M_1)_s \# (S^1 \times S^2)_{(j_0-1)s+1} & \text{if } Z_n \text{ preserves orientation,} \\ (M_1)_{[(s+1)/2]} \# (-M_1)_{[s/2]} \# (S^1 \times S^2)_{(j_0-1)s+1} & \text{if } Z_n \text{ reverses orientation,} \\ (M_1)_s \# (N)_{(j_0-1)s+1} & \text{if } M \text{ is not orientable.} \end{cases}$$

II$_2$. *If* $(M, Z_n) = (M_1, Z_{n_1})^s_{(P_1^2, P_2^2, f, r)}$, *then*

$$\tilde{M} = (\tilde{M}_1^d)_s \# (S^1 \times S^2)_{(j_0-1)s+1}.$$

The following easy proposition allows us to decrease the number of constructions.

3.6. PROPOSITION. *Assume that* (M', Z_n) *and* (M'', Z_n) *are defined as*

(a)
$$(M', Z_n) = (M_1, Z_{n_1}) \mathbin{\hat{\#}}_{(F_1', F_2', f')} (M_2, Z_{n_2}),$$
$$(M'', Z_n) = (M_1, Z_{n_1}) \mathbin{\hat{\#}}_{(F_1'', F_2'', f'')} (M_2, Z_{n_2})$$

and that there exists a T_i*-equivariant homeomorphism* $h_i\colon M_i \to M_i$ *with* $h_i(F_i') = F_i''$ $(i = 1, 2)$ *and* $(h_2|F_2')f' = f''(h_1|F_1')$, *or as*

(b) $$(M', Z_n) = (M_1, Z_{n_1})^s_{(F_1', F_2', f', r)}, \qquad (M'', Z_n) = (M_1, Z_{n_1})^s_{(F_1'', F_2'', f'', r)}$$

and that there exists a Z_{n_1}*-equivariant homeomorphism* $h\colon M_1 \to M_1$ *with* $h(F_1') = F_i''$ $(i = 1, 2)$ *and* $(h|F_2')f' = f''(h|F_1')$.

Then the following conditions hold:

(1) *if* M' *is nonorientable, then there exists a* Z_n*-equivariant homeomorphism* $M' \to M''$;

(2) *if* M' *is orientable and* h *(respectively,* h_1 *and* h_2*) preserves orientation, then there exists a* Z_n*-equivariant, orientation-preserving homeomorphism* $M' \to M''$;

(3) *if* M' *is orientable and* h *(respectively,* h_1 *and* h_2*) reverses orientation, then there is a* Z_n*-equivariant, orientation-reversing homeomorphism* $M' \to M''$.

Proposition 3.6 enables us to establish the following

3.7. PROPOSITION. (a) *Let* $(M, Z_n) = (M_1, Z_{n_1})^s_{(x_1, x_2, f, r)}$, *where* $\dim M_1 \geqslant 2$, $M_1 - \mathrm{Iz}(M_1)$ *is connected, and* $x_1, x_2 \in \mathrm{int}(M_1) - \mathrm{Iz}(M_1)$. *Then* (M, Z_n) *is independent of the choice of* $x_1, x_2 \in \mathrm{int}(M_1) - \mathrm{Iz}(M_1)$. *Moreover, if* $M_1 - \mathrm{Iz}(M_1)$ *is nonorientable, then* (M, Z_n) *is independent of the choice of* f, *and if* $M_1 - \mathrm{Iz}(M_1)$ *is orientable, then* (M, Z_n) *possibly depends on whether or not* f *reverses orientation.*

(b) *Let* $(M, Z_n) = (M_1, Z_{n_1}) \mathbin{\overline{\#}}_{(x_1, x_2, f)} (M_2, Z_{n_2})$, *where* $M_i - \mathrm{Iz}(M_i)$ *is connected* $(i = 1, 2)$. *Then* (M, Z_n) *does not depend on the choice of the points* x_1, x_2 *and the homeomorphism* f.

From now on, to avoid excessive notation, we use the simplified notation (e.g., $\overline{\#}$ or $\overline{\#}_f$ instead of $\overline{\#}_{(x_1, x_2, f)}$) if it does not lead to misunderstanding.

Proposition 3.7 follows from Proposition 3.6 and Lemma 3.8.

3.8. LEMMA. *Let* (M, T) *be a* Z_n*-manifold and* $x_1, y_1 \in \mathrm{int}(M) - \mathrm{Iz}(M)$. *If there exists an arc* γ *joining* x_1 *and* y_1 *and disjont with* $\mathrm{Iz}(M)$, *then there* *exists a Z_n-equivariant homeomorphism $h\colon M \to M$ with $h(x_1) = y_1$, where h preserves orientation if $M - \mathrm{Iz}(M)$ is orientable.*

The proof is fairly similar to that for the nonequivariant case.

3.9. PROPOSITION. *If $\{M_a\}_{a\in A}$ are manifolds such that $M_a - \mathrm{Iz}(M_a)$ is connected, then the operation $\overline{\#}$ is associative and commutative.*

The proof is easy (cf. Proposition 3.7 and Lemma 3.8).

3.10. Definition. An action of Z_n on $S^1 \hat{\times} S^2$ is said to be *standard* if one of the following conditions is satisfied:

(a) Z_n preserves orientation of $S^1 \times S^2$ and the action is given by

$$(z, y) \to (e^{2\pi i r/n} z, y), \quad \text{where } (r, n) = 1;$$

(b) Z_n reverses orientation of $S^1 \times S^2$ and the action is given by

$$(z, y) \to (e^{2\pi i r/n} z, -y), \quad \text{where } (r, n) = 1 \text{ and } n \text{ is even};$$

(c) Z_n acts on $S^1 \hat{\times} S^2 = N = R \times S^2/\sim$, $(t, y) \sim (t+1, -y)$, and the action is given by

$$(t, y) \to \big(1 + r/n, (-1)^r y\big), \quad \text{where } (r, n) = 1 \text{ and } n \text{ is odd}.$$

$S^1 \hat{\times} F$ denotes an F-bundle over S^1.

3.11. PROPOSITION (multiple as a special case of a connected sum). *If $(M, T) = (M_1, g)^s_{(x_1, x_2, f, r)}$, where g generates a Z_{n_1}-action on a 3-manifold M_1, $M_1 - \mathrm{Iz}(M_1)$ is connected, and $x_1, x_2 \in \mathrm{int}(M_1) - \mathrm{Iz}(M_1)$, then*

$$(M, T) = (M_1, g) \mathbin{\overline{\#}}_{(f')} (S^1 \hat{\times} S^2, T_2),$$

where T_2 generates a standard action of Z_{n_2} on $S^1 \hat{\times} S^2$. Moreover, n_2 and s_2 are any natural numbers which satisfy $s_2 n_2 = s n_1 = n$ and $(s_2, s) = 1$.

In order to prove Proposition 3.11 we use the following fact which can be easily shown:

3.12. FACT. *If $(s, r) = 1$, then for each n_1 the sequence*

$$\text{g.c.d.}(r, n_1),\ \text{g.c.d.}(s + r, n_1),\ \text{g.c.d.}(2s + r, n_1), \ldots$$

contains each natural number a such that a divides n_1 and $(a, s) = 1$.

Proof of Proposition 3.11. We use the notation of 3.1-3.3. We may join ∂D_{x_1} with $g^i(\partial D_{x_2})$ (for each i) using an arc γ such that γ is properly embedded in

$$M_{1,0} = M_1 - \bigcup_{j=1}^{n} g^j(\mathring{D}_{x_1} \cup \mathring{D}_{x_2})$$

and $\gamma, g(\gamma), \ldots, g^{n_1-1}(\gamma)$ are pairwise disjoint. Let V_γ be a small regular neighborhood of $\partial D_{x_1} \cup \gamma \cup g^i(\partial D_{x_2})$ in $M_{1,0}$ and ∂V_γ be its boundary in $\mathrm{int}(M_{1,0})$. Then ∂V_γ is a 2-sphere in M and $\bigcup_{j=1}^{n} T^j(\partial V_\gamma)$ defines some splitting of M. It is easy to see that (M, Z_n) is obtained as a connected sum (M_1, g) and $M_2 = S^1 \hat{\times} S^2$ with a standard action. s_2 is the number of connected components of the manifold $\bigcup_{j=1}^{n} T^j(V_\gamma)$. Since the component containing V_γ is equal to

$$V_\gamma \cup T^{si+r}(V_\gamma) \cup T^{2(si+r)}(V_\gamma) \cup \ldots = \bigcup_{j=0}^{s-1} T^{j(si+r)}\big(V_\gamma \cup g^{si+r}(V_\gamma) \cup \ldots\big),$$

we have $s_2 = \text{g.c.d.}(si+r, n_1)$. Now, we obtain Proposition 3.11 from Fact 3.12.

Propositions similar to those of 3.7-3.11 are also true for multiple and connected sum along a disk.

3.13. PROPOSITION. *Let*

$$(M, Z_n) = (M_1, Z_{n_1})^s_{(D_{x_1}, D_{x_2}, f, r)} \text{ or } (M, Z_n) = (M_1, Z_{n_1}) \mathbin{\overline{\#}}_{(D_{x_1}, D_{x_2}, f)} (M_2, Z_{n_2}),$$

where $x_1 \in \partial_j M_1 - \mathrm{Iz}(M_1)$ for some component $\partial_j M_1$ of ∂M_1, and D_{x_i} is a sufficiently small, regular neighborhood of x_i in ∂M_i $(i = 1, 2)$. If $\dim M_1 \geqslant 3$ and $\partial_j M_1 - \mathrm{Iz}(M_1)$ is connected, then (M, Z_n) does not depend on the choice of the point $x_1 \in \partial_j M_1$.

The proof is very similar to that of Proposition 3.7.

3.14. Definition. An action of Z_n on $S^1 \hat{\times} D^2$ is said to be *standard* if one of the following conditions is satisfied:

(a) Z_n preserves orientation of $S^1 \times D^2$ and the action is given by

$$(z_1, z_2) \to (e^{2\pi i r/n} z_1, z_2), \quad \text{where } (r, n) = 1;$$

(b) Z_n reverses orientation of $S^1 \times D^2$ and the action is given by

$$(z_1, z_2) \to (e^{2\pi i r/n} z_1, \bar{z}_2), \quad \text{where } (r, n) = 1 \text{ and } n \text{ is even};$$

(c) Z_n acts on $S^1 \hat{\times} D^2 = R \times D^2/\sim$, $(t, z) \sim (t+1, \bar{z})$, and the action is given by

$$(t, z) \to \big(t + r/n, \mathrm{Re}(z) + (-1)^r i \mathrm{Im}(z)\big), \quad \text{where } (r, n) = 1 \text{ and } n \text{ is odd}.$$

3.15. PROPOSITION. *Let $(M, T) = (M_1, g)^s_{(D_{x_1}, D_{x_2}, f, r)}$, where g generates a Z_{n_1}-action on a 3-manifold M_1, and $x_1, T^i(x_2)$ lie in the same component $\partial_j M_1$ of ∂M_1. Moreover, $x_i \in M_i - \mathrm{Iz}(M_i)$ and D_{x_i} is a sufficiently small,* *regular neighborhood of* x_i *in* ∂M_1 $(i = 1, 2)$. *If* $\partial_j M_1 - \mathrm{Iz}(M_1)$ *is connected, then*

$$(M, T) = (M_1, g) \,\bar{\#}_{(D_{y_1}, D_{y_2}, f_1)} (S^1 \hat{\times} D^2, T_2),$$

where T_2 *generates a standard action of* Z_{n_2} *on* $S^1 \hat{\times} D^2$. *Moreover,*

$$n_2 = \frac{sn_1}{\text{g.c.d.}(si + r, n_1)}.$$

The proof is similar to that of Proposition 3.11.

4. Splitting Z_n-action along a compact hypersurface. In this section we prove Theorem 4.1 which is a main tool in reduction of Z_n-actions on manifolds.

4.1. THEOREM. *Let T be a generator of an effective Z_n-action on a k-manifold M. Let F be a compact, connected, 2-sided $(k-1)$-manifold, properly embedded into M and such that for each j either $T^j(F) \cap F = \emptyset$ or $T^j(F) = F$. Then either*

$$(M, Z_n) = (M_1, Z_{n_1}) \,\hat{\#}_{(F_1, F_2, f)} (M_2, Z_{n_2})$$

or

$$(M, Z_n) = (M_1, Z_{n_1})^s_{(F_1, F_2, f, r)},$$

where n_i $(i = 1, 2)$ divides n, M_1 is one of the connected manifolds obtained by cutting M along $\bigcup_{j=1}^{n} T^j(F)$, and M_2 is either $F \times [-1, 1]$ with action $(x, t) \to (h(x), -t)$ (h generates a cyclic action on F) or one of the connected manifolds obtained by cutting M along $\bigcup_{j=1}^{n} T^j(F)$.

First we prove the following

4.2. LEMMA. *Let M_1 be one of the connected manifolds obtained by cutting M along $\bigcup_{i=1}^{n} T^i(F)$ with $F \subset M_1$. Then one of the following conditions holds:*

(a) *$\{T^i(M_1)\}_{i=1}^{n}$ contains each component obtained by cutting M;*

(b) *there exists a second component M_2 such that $M_1 \cap M_2 \supset F$, $M_2 \neq T^i(M_1)$ for each i, and $\{T^i(M_1)\}_{i=1}^{n} \cup \{T^i(M_2)\}_{i=1}^{n}$ is the set of all components obtained by cutting M along $\bigcup_{i=1}^{n} T^i(F)$.*

Proof of Lemma 4.2. Suppose (a) is not satisfied. Then there exists a component M_0 such that $M_0 \neq T^i(M_1)$ for each i. Since M is connected and we cut M only along $\bigcup_i T^i(F)$, we have $M_0 \supset T^i(F)$ for some i. Thus

$T^{-i}(M_0) \cap M_1 \supset F$. We put $M_2 = T^{-1}(M_0)$. It remains to verify that $\{T^i(M_1)\}_{i=1}^n \cup \{T^i(M_2)\}_{i=1}^n$ is the set of all components. Suppose P is not of the form $T^i(M_1)$ $(i = 1, \ldots, n)$. Then $P \supset T^j(F)$ for some j, and so $\big(P \cap T^j(M_1)\big) \supset T^j(F)$. Since $T^j(M_2) \cap T^j(M_1) \supset T^j(F)$, we have $P = T^j(M_2)$.

Proof of Theorem 4.1. First we consider the possibility (b) from Lemma 4.2. Let s_i be the smallest natural number which satisfies $T^{s_i}(M_i) = M_i$ $(i = 1, 2)$. Since M is connected, $(s_1, s_2) = 1$. Let $n_i = n/s_i$. Then $T_i = T^{s_i}$ generates a Z_{n_i}-action on M_i $(i = 1, 2)$. Let k be the smallest natural number which satisfies $T^k(F) = F$. In our case, k is a multiple of s_1 and s_2 (T^k preserves locally the sides of F). Thus $k = j_0 s_1 s_2$ for some j_0. Let $j = n/k$. Then $n_1 = jj_0 s_2$ and $n_2 = jj_0 s_1$. The surfaces $F, T(F), \ldots \ldots, T^{k-1}(F)$ cut M into $s_1 + s_2$ components

$$M_1,\ M_{1,2} = T(M_1), \ldots, M_{1,s_1} = T^{s_1-1}(M_1),\ M_2,\ M_{2,2} = T(M_2), \ldots \ldots, M_{2,s_2} = T^{s_2-1}(M_2).$$

Let

$$h_{1,1} = T|M_1,\ h_{1,2} = T|M_{1,2}, \ldots, h_{1,s_1} = T|M_{1,s_1},\ h_{2,1} = T|M_2, \ldots \ldots, h_{2,s_2} = T|M_{2,s_2}.$$

We may describe this as follows:

$$M_i \overset{h_{i,1}}{=} M_{i,2} \overset{h_{i,2}}{=} M_{i,3} \overset{h_{i,3}}{=} \ldots \overset{h_{i,s_i-1}}{=} M_{i,s_i} \overset{h_{i,s_i}}{=} M_i,$$

where $h_{i,s_i} \ldots h_{i,1} = T_i$ $(i = 1, 2)$. Moreover, let

$$F \overset{i_1}{=} F_1 \hookrightarrow M_1, \quad F \overset{i_2}{=} F_2 \hookrightarrow M_2, \quad \text{and} \quad f = i_2 i_1^{-1}\colon F_1 \to F_2.$$

Then f is $T^{j_0 s_1 s_2}$-equivariant.

Consequently, we obtain $(M, Z_n) = (M_1, Z_{n_1}) \mathbin{\hat{\#}}_{(F_1, F_2, f)} (M_2, Z_{n_2})$.

Now we consider the possibility (a) of Lemma 4.2. The manifold M splits into s connected components $M_1, T(M_1) = M_2, \ldots, T^{s-1}(M_1) = M_s$, so $T^s(M_1) = M_1$. Let k be the smallest natural number such that $T^k(F) = F$. We have two possibilities:

(a) T^k changes locally the sides of F. Let V_F be a small, regular, T^k-invariant neighborhood of F in M. We may assume that $V_F = F \times \times [-1, 1]$, $F = F \times \{0\}$, and on V_F we have $T^k(x, t) = \big(T^k(x), -t\big)$. Let $F' = F \times \{-1\}$. Then M splits along $\bigcup_{i=1}^{n} T^i(F')$ into k manifolds $Z_{n/k}$-homeomorphic to $(F \times [-1, 1], T^k)$ and s manifolds $Z_{n/s}$-homeomorphic to manifolds obtained in the last decomposition. We have reduced the problem to the case (b) of Lemma 4.2.

(b) T^k preserves locally the sides of F. Thus $k = j_0 s$ for some j_0. Let $j = n/k$. We put

$$h_1 = T|M_1,\ h_2 = T|M_2,\ \ldots,\ h_{s-1} = T|M_{s-1},\ h_s = T|M_s.$$

Let $g = h_s \ldots h_1 = T^s|M_1$, $F_2 = F \hookrightarrow M_1$, and $F_1 = T^{-r}(F) \hookrightarrow M_1$, where $F \subset M_1 \cap T^r(M_1)$ ($(r, s) = 1$ because M is connected). Let

$$\begin{array}{ccccc} f\colon F_1 & \longrightarrow & & & F_2 \\ \| & & & & \| \\ T^{-r}(F) & \overset{\approx}{\leftarrow} & F & = & F \subset M_1 \\ & & \cap & & \\ & & M_r & & \end{array}$$

Now we obtain $(M, Z_n) = (M, Z_{jj_0})^s_{(F_1, F_2, f, r)}$. This completes the proof of Theorem 4.1.

4.3. Corollary. *Let T be a generator of an effective action of Z_n on a decomposable 3-manifold M. If there is an embedding $S^2 \hookrightarrow \mathrm{int}(M)$ such that S^2 does not bound a 3-cell in M and $T^i(S^2) \cap S^2 = \emptyset$ or S^2 for each i, then either*

$$(M, Z_n) = (M_1, Z_{n_1}) \mathbin{\hat{\#}}_{(x_1, x_2, f)} (M_2, Z_{n_2})$$

or

$$(M, Z_n) = (M_1, Z_{n_1})^s_{(x_1, x_2, f, r)},$$

where n_j divides n and $i(M_j) < i(M)$ $(j = 1, 2)$.

Proof. We may obtain Corollary 4.3 similarly as Theorem 4.1. The decrease of $i(\cdot)$ follows from the form of M in multiple and connected sum (Proposition 3.5).

4.4. Corollary. *Let T be a generator of an effective action of Z_n on a 3-manifold M with $i(\tilde{M}) < \infty$. If M contains a 2-sided projective plane $P^2 \hookrightarrow \mathrm{int}(M)$, not parallel to the boundary, such that $T^j(P^2) \cap P^2 = \emptyset$ or P^2 for each j, then either*

$$(M, Z_n) = (M_1, Z_{n_1}) \mathbin{\hat{\#}}_{(P_1^2, P_2^2, f)} (M_2, Z_{n_2})$$

or

$$(M, Z_n) = (M_1, Z_{n_1})^s_{(P_1^2, P_2^2, f, r)},$$

where n_j divides n and $i(\hat{M}_j) < i(\hat{M})$ $(j = 1, 2)$.

Proof. It remains to verify $i(\hat{M}_j) < i(\hat{M})$ $(j = 1, 2)$ and this follows from the form of $\tilde{M}$ in multiple and connected sum (Proposition 3.5) and from the theorem of Livesay [12].

4.5. Corollary. *Let T be a generator of an effective action of Z_n on a decomposable 3-manifold M with a compact boundary. If there is a proper* *embedding* $(D^2, \partial D^2) \hookrightarrow (M, \partial M)$ *such that* D^2 *does not cut out a 3-cell from* M *and* $T^j(D^2) \cap D^2 = \emptyset$ *or* D^2 *for each* j, *then either*

$$(M, Z_n) = (M_1, Z_{n_1}) \mathbin{\hat{\#}}_{(D_1^2, D_2^2, f)} (M_2, Z_{n_2})$$

or

$$(M, Z_n) = (M_1, Z_{n_1})^s_{(D_1^2, D_2^2, f, r)},$$

where n_j *divides* n *and one of the following conditions is satisfied* $(j = 1, 2)$:

(a) $\mathrm{gen}(M_j) < \mathrm{gen}(M)$,

(b) $\mathrm{gen}(M_j) = \mathrm{gen}(M)$ *and* $i(M_j) < i(M)$,

(c) $\mathrm{gen}(M_j) = \mathrm{gen}(M)$, $i(M_j) = i(M)$, *and* $k_{\partial M_j} > k_{\partial M}$.

Proof. It remains to verify that one of the conditions (a), (b) or (c) is satisfied. If ∂D^2 does not disconnect ∂M, then (a) is satisfied. If D^2 disconnects M and (a) is not satisfied, then (b) must be satisfied. If D^2 does not disconnect M and ∂D^2 disconnects some component $\partial_i M$ of ∂M and (a) and (b) are not satisfied, then (c) is satisfied.

5. Possibilities of reducing down a Z_n-action along S^2, P^2, and D^2. The authors of the papers [22], [23], [8], [6] have studied a possibility of reducing involution of a compact 3-manifold along a 2-sphere. Tollefson and Kim studied in [8] the possibility of reducing involution of a compact 3-manifold along D^2. We study here the possibility of reducing a Z_n-action along S^2, P^2, and D^2. Lemma 1 from [8] has the following slight generalization:

5.1. THEOREM. *Let* T *be an involution on a 3-manifold* M. *Suppose that* M *is not irreducible. Then there exists a 2-sphere* S^2 *in* M *not bounding a 3-cell and such that either* $T(S^2) \cap S^2 = \emptyset$ *or* $T(S^2) = S^2$, *and* S^2 *is in a general position with respect to* $\mathrm{Fix}(T)$.

5.2. THEOREM. *Let* M *be a 3-manifold admitting an effective* Z_3*-action. Let* T *be a homeomorphism on* M *generating the given* Z_3*-action. If* $\mathrm{int}(M)$ *contains a 2-sphere disjoint from* $\mathrm{Fix}(T)$ *bounding no 3-cell, then there exists a 2-sphere* S^2 *in* $\mathrm{int}(M)$ *bounding no 3-cell such that* $S^2 \cap \mathrm{Fix}(T) = \emptyset$ *and* $T(S^2) \cap S^2 = \emptyset$.

We will prove similar results for a Z_n-action with $\dim \mathrm{Iz}(\cdot) \leqslant 0$.

5.3. THEOREM. *Assume that a finite group* G *acts on a 3-manifold* M *which satisfies the following condition: there is an embedding* $i\colon S_0^2 \hookrightarrow \mathrm{int}(M)$ *such that* S_0^2 *does not bound a homotopy 3-cell in* M *and* $S_0^2 \cap \mathrm{Iz}(M) = \emptyset$. *Then there is an embedding* $S^2 \hookrightarrow \mathrm{int}(M)$ *such that* $S^2 \cap \mathrm{Iz}(M) = \emptyset$, S^2 *does not bound any 3-cell in* M, *and* $g(S^2) \cap S^2 = \emptyset$ *or* S^2 *for each* $g \in G$.

First we prove some lemmas.

5.4. LEMMA. *Assume that a finite group G acts freely on a 3-manifold M. Let N_0 be a $\pi_1(M)$- and G-invariant subgroup of $\pi_2(M)$ and let $\pi_2(M)-N_0 \neq \emptyset$ (G acts on $\pi_2(M)$ up to the action of $\pi_1(M)$ on $\pi_2(M)$). Then one of the following possibilities holds:*

(a) *there is an embedding $i\colon S^2 \hookrightarrow \operatorname{int}(M)$ such that $[i] \notin N_0$ and $g(S^2) \cap S^2 = \emptyset$ or S^2 for each $g \in G$;*

(b) *there is a 2-sided embedding $i\colon P^2 \hookrightarrow \operatorname{int}(M)$ such that $i_*(\pi_2(P^2)) \notin N_0$ and $g(P^2) \cap P^2 = \emptyset$ for each $g \in G$, $g \neq 1$.*

Proof. Consider the projection $p\colon M \to M^* = M|G$. We have $\pi_2(M^*)-p_*(N_0) \neq \emptyset$ and $p_*(N_0)$ is a $\pi_1(M^*)$-invariant subgroup of $\pi_2(M^*)$. Now, by the projective plane theorem, there are two possibilities:

(a) there is an embedding $i\colon S^2 \hookrightarrow \operatorname{int}(M^*)$ with $[i] \in \pi_2(M^*)-p_*(N_0)$: thus a lifting of i to M satisfies condition (a) of Lemma 5.4;

(b) there exists a 2-sided embedding $i\colon P^2 \hookrightarrow \operatorname{int}(M^*)$ such that

$$i_*\big(\pi_2(P^2)\big) \in \big(\pi_2(M^*)-p_*(N_0)\big);$$

thus a lifting of P^2 to M satisfies (a) or (b) of Lemma 5.4.

5.5. LEMMA. *Let N_0 be a $\pi_1(M^3)$-invariant subgroup of $\pi_2(M^3)$ generated by projective planes embedded into M^3. Then an embedding $i\colon S^2 \hookrightarrow \operatorname{int}(M^3)$ such that either S^2 does not disconnect M^3 or S^2 generates the connected sum $M = M_1 \# M_2$, where M_j $(j = 1, 2)$ is not a closed manifold, satisfies $[i] \notin N_0$.*

Proof. Let $h\colon \pi_2(M^3) \to H_2(M^3)$ be the Hurewicz homomorphism. Lemma 5.5 follows from the observation that $h[i] \neq 0$ and $h(N_0) = 0$.

Proof of Theorem 5.3. Assume first that M is compact. Thus $\tilde{M}$ is compact and decomposable. Let $N_1 \subset \pi_2(M)$ be the $\pi_1(M)$-invariant subgroup of $\pi_2(M)$ generated by all projective planes embedded into ∂M. By Lemma 5.5 and Proposition 2.4, $[i] \notin N_1$. Let $N_0 = \varphi_*^{-1}(N_1)$, where $\varphi\colon M-\operatorname{Iz}(M) \hookrightarrow M$. Thus $[i] \notin N_0$. On the other hand, G acts freely on $M-\operatorname{Iz}(M)$. Now we use Lemma 5.4. If (a) of 5.4 is satisfied, Theorem 5.3 is proved. If (b) of 5.4 is satisfied, then let $i_0\colon P^2 \hookrightarrow \operatorname{int}(M)-\operatorname{Iz}(M)$ be the projective plane obtained in (b). Since $i_{0*}\big(\pi_2(P^2)\big) \notin N_0$, P^2 is not parallel to the boundary. It follows from Lemma 2.9 that there is an embedding $S_1^2 \hookrightarrow M-\operatorname{Iz}(M)$ such that S_1^2 does not bound any homotopy 3-cell and

$$S_1^2 \cap \bigcup_{g\in G} g(P^2) = \emptyset.$$

We cut out M along $\bigcup_{g\in G} g(P^2)$ and repeat the procedure. Since $\tilde{M}$ is decomposable, the decomposing process terminates in finitely many steps (we use Lemma 5.4 and the fact that each step decreases $i(\hat{\tilde{M}})$).

Now assume that M is not compact. If S_0^2 does not disconnect M or disconnects M into $M_1 \# M_2$, where M_1 and M_2 are not closed, then Theorem 5.3 follows from Lemmas 5.4 and 5.5. Let S_0^2 disconnect M into $M_1 \# M_2$,where M_1 is not compact and M_2 is closed. We may assume that for each $g_1, g_2 \in G$ either $g_1(S_0^2) = g_2(S_0^2)$ or $g_1(S_0^2)$ is in a general position with respect to $g_2(S_0^2)$. Let V_G be an equivariant regular neighborhood of $\bigcup_{g\in G} g(S_0^2)$. Thus V_G cuts M into a finite number of components. Let E_1 be a sum of compact components and V_G. Then S_0^2 does not cut off a homotopy 3-cell from E_1. Since E_1 is compact, we can use the just proved part of Theorem 5.3. Let $S_1^2 \hookrightarrow \operatorname{int}(E_1)$ be a 2-sphere as in the conclusion of Theorem 5.3. If S_1^2 does not disconnect E_1, then S_1^2 is the desired sphere. Otherwise, S_1^2 generates the decomposition $E_1 = E_{11} \# E_{12}$, where E_{11} and E_{12} are not homotopy spheres. Thus, after glueing cutted off parts of M, E_{1i} $(i = 1, 2)$ is either not modified or is not compact. So S_1^2 is the desired 2-sphere in M.

From Lemma 5.4 we obtain

5.6. COROLLARY. *Assume that a finite group G acts on a 3-manifold M which satisfies the following conditions*:

(i) *there is a 2-sided embedding $P_0^2 \hookrightarrow \operatorname{int}(M) - \operatorname{Iz}(M)$ such that P_0^2 is not homotopy parallel to the boundary*;

(ii) *the assumption of Theorem 5.3 is not satisfied.*

Then there is a 2-sided embedding $P^2 \hookrightarrow \operatorname{int}(M) - \operatorname{Iz}(M)$ such that P^2 is not homotopy parallel to the boundary and $g(P^2) \cap P^2 = \emptyset$ for each $g \in G$, $g \neq 1$.

Proof. We use Lemma 5.4 and the fact that if P_0^2 is not homotopy parallel to the boundary, then P_0^2 does not belong to the $\pi_1(M)$-invariant subgroup of $\pi_2(M)$ generated by projective planes embedded into ∂M. We also use the fact that each self-homeomorphism of P^2 has a fixed point.

Kim and Tollefson studied [8] the problem of a possibility of reducing down an action along D^2, solving this problem for involutions.

We shall now prove Theorem 5.7 which is a partial generalization of Lemma 2 from [8] to actions of a finite group.

5.7. THEOREM. *Let a finite group G act on a 3-manifold M. Suppose that there is a disk D^2 embedded in M such that ∂D^2 lies in a preferred component $\partial_i M$ of ∂M, $D^2 \cap \operatorname{Iz}(M) = \emptyset$, and ∂D^2 does not bound a disk in $\partial_i M$. Then there exists a disk D_1^2 properly embedded in M with the following properties*:

(i) $\partial D_1^2 \subset \partial_i M$,

(ii) ∂D_1^2 *does not bound a disk in* $\partial_i M$,

(iii) $g(D_1^2) \cap D_1^2 = \emptyset$ *for each* $g \in G$, $g \neq 1$,

(iv) $D_1^2 \cap \operatorname{Iz}(M) = \emptyset$.

Proof. Assume that N_0 is a normal subgroup of $\pi_1(\partial_i M - \mathrm{Iz}(M))$ equal to $\ker(\pi_1(\partial_i M - \mathrm{Iz}(M)) \to \pi_1(\partial_i M))$. Of course, $[\partial D^2] \notin N_0$. Consider the covering

$$p: (M - \mathrm{Iz}(M)) \to M_0^* = (M - \mathrm{Iz}(M))|G.$$

Since N_0 is a G-invariant normal subgroup of $\pi_1(\partial_i M - \mathrm{Iz}(M))$, $p_*(N_0)$ is a normal subgroup of $\pi_1(M_0^*)$. We use the loop theorem for $[p(\partial D^2)] \notin p_*(N_0)$ to obtain a proper embedding

$$(D_0^2, \partial D_0^2) \hookrightarrow (M_0^*, p(\partial_i M - \mathrm{Iz}(M)))$$

with $[\partial D_0^2] \notin p_*(N_0)$. A lifting D_1^2 of $(D_0^2, \partial D_0^2)$ to $(M - \mathrm{Iz}(M), \partial_i M - \mathrm{Iz}(M))$ satisfies $[\partial D_1^2] \notin N_0$. Thus we have $0 \neq [\partial D_1^2] \in \pi_1(\partial_i M)$, and so ∂D_1^2 bounds no disk in $\partial_i M$. Of course, D_1^2 satisfies (iii) and (iv) of Theorem 5.7.

5.8. COROLLARY. *Let a finite group G act on a 3-manifold M with nonempty boundary. Suppose that M is congruent to an irreducible, not ∂-irreducible manifold. If* $\dim \mathrm{Iz}(M) \leqslant 0$, *then there exists a disk D^2 properly embedded in M with the following properties*:

(i) *∂D^2 does not bound a disk in ∂M,*

(ii) *$g(D^2) \cap D^2 = \emptyset$ for each $g \in G$, $g \neq 1$,*

(iii) *$D^2 \cap \mathrm{Iz}(M) = \emptyset$.*

For the proof it is easy to see that the assumptions of Theorem 5.7 are satisfied.

6. Reduction theorems. Kim and Tollefson [8] proved the theorem describing the decomposition of a closed manifold with involution into a Z_2-connected sum. This result is also true for decomposable manifolds. In this section we prove the theorem of this type for Z_n-manifolds.

6.1. Definition. A Z_n-action described in Definition 3.10 (x) is called a *standard action of type* $((\mathrm{x});\ n, r, 0)$. A *standard action of type* $((\mathrm{x});\ n, r, k)$ on $\mathop{\#}\limits_{i=1}^{1+kn} (S^1 \hat{\times} S^2)_i$ is defined as follows:

$$((\mathrm{a});\ n, r, k) = ((\mathrm{a});\ n, r, 0)\ \overline{\#}\ \Big(\mathop{\#}\limits_{j=1}^{k} (S^1 \times S^2)_j, Z_1\Big),$$

where Z_1 denotes a one-element group;

$$((\mathrm{b});\ n, r, k) = ((\mathrm{b});\ n, r, 0)\ \overline{\#}\ \Big(\mathop{\#}\limits_{j=1}^{k} (S^1 \times S^2)_j, Z_1\Big), \quad \text{where } n \text{ is even};$$

$$((\mathrm{c});\ n, r, k) = ((\mathrm{a});\ n, r, 0)\ \overline{\#}\ \Big(\mathop{\#}\limits_{j=1}^{k} (N)_j, Z_1\Big), \quad \text{where } k > 0.$$

6.2. THEOREM. *Let T be a generator of a Z_n-action on a decomposable 3-manifold M with* $\dim \mathrm{Iz}(M) \leqslant 0$. *Moreover, each 2-sphere S^2 embedded in* int(M), *such that* $S^2 \cap T^i(S^2) = \emptyset$ *for each* $0 < i < n/2$, $T^{n/2}(S^2) = S^2$, *and* $T^{n/2}$ *preserves locally the sides of* S^2, *bounds a* 3-*cell in* M. *Then*

$$(M, T) = \big(M_1, T(M_1)\big) \mathbin{\bar{\#}} \ldots \mathbin{\bar{\#}} \big(M_k, T(M_k)\big),$$

where each $T(M_i)$ *is a generator of a* Z_{n_i}-*action on* M_i, n_i *divides* n, *and*

(a) *if* $n \leqslant 3$, *then* M_i $(i > 1)$ *is irreducible and* M_1 *is either irreducible or equal to* $\# S^1 \hat{\times} S^2$ *with a standard* Z_n-*action*,

(b) *if* $n > 3$, *then* M_i $(i > 1)$ *is congruent to an irreducible manifold and* M_1 *is either congruent to an irreducible manifold or is equal to* $\# S^1 \hat{\times} S^2$ *with a standard* Z_n-*action*.

Proof for $n > 3$. If M is not congruent to an irreducible manifold, then using Corollary 4.3, Propositions 3.11, 3.7, and Theorem 5.3 we obtain

$$(M, T) = \big(M_1', T(M_1')\big) \mathbin{\bar{\#}} \big(M_2', T(M_2')\big),$$
$$\text{where } i(M_j') < i(M) \ (j = 1, 2).$$

Repeating the procedure (using the commutativity and associativity of $\bar{\#}$ (Propositions 3.9 and 3.11), we obtain

$$(M, T) = \big(M_1, T(M_1)\big) \mathbin{\bar{\#}} \ldots \mathbin{\bar{\#}} \big(M_{k_1}, T(M_{k_1})\big) \mathbin{\bar{\#}} \big(M_{k_1+1}, T(M_{k_1+1})\big) \mathbin{\bar{\#}} \ldots$$
$$\ldots \mathbin{\bar{\#}} \big(M_{k_2}, T(M_{k_2})\big),$$

where the first k_1 manifolds are homeomorphic to $S^1 \hat{\times} S^2$ with standard action and other manifolds are congruent to irreducible ones with Z_{n_i}-actions (n_i divides n).

Now consider

$$(M', T') = \mathop{\bar{\#}}_{i=1}^{k_1} \big(M_i, T(M_i)\big).$$

We show that (M', T') is a standard action.

We know that

(1) T' generates a free action of $Z_{n'}$ on M', where n' divides n;

(2) $M'/Z_{n'}$ is homeomorphic to $\mathop{\#}_{i=1}^{k_1} (S^1 \hat{\times} S^2)_i$.

We must classify actions on $\# S^1 \hat{\times} S^2$ which satisfy (1) and (2). The classification of Z_n-actions on $\# S^1 \hat{\times} S^2$ with $\dim \mathrm{Iz}(\cdot) \leqslant 0$ will be done in [16]. We give only the outline of the proof of our special case. The proof (with assumptions (1) and (2)) that the action is standard is similar to that for free actions on handlebodies [17]. That is, we show that we may change a connected sum by multiple and multiple leads from standard actions to standard ones. To prove our theorem it is sufficient to show that we can obtain $n' = n$. Up to now, we have proved that

$$(M, T) = \big((\mathrm{x});\ n', r_1, k_1\big) \bar{\#} (M'', T''),$$

where (M'', T'') can be described as a connected sum of manifolds congruent to irreducible manifolds. To complete the proof of Theorem 6.2, we must show that if

$$(M, T) = \big((\mathrm{x});\ n_1, r_1, 0\big) \bar{\#} \big(M_0, T(M_0)\big),$$

where T generates a Z_n-action on M, then

$$(M, T) = \big((\mathrm{y});\ n, r_2, 0\big) \bar{\#} \big(M_0, T(M_0)\big).$$

Indeed, there is a 2-sphere $S^2 \hookrightarrow \mathrm{int}(M)$ such that $T^i(S^2) \cap S^2 = \emptyset$ for each i $(0 < i < n)$, S^2 lies in the summand $S^1 \hat{\times} S^2$ (i.e., $\big((\mathrm{x});\ n_1, r_1, 0\big)$), and S^2 does not disconnect $S^1 \hat{\times} S^2$. Such an S^2 yields the expression for (M, T) as a multiple of $\big(M_0, T(M_0)\big)$. Thus we use Proposition 3.11 to obtain the desired decomposition.

This completes the proof of Theorem 6.2 (b) (we use Proposition 3.9). Since the proof of Theorem 6.2 (a) is very similar to that of 6.2 (b) (using Theorems 5.1 and 5.2 in place of Theorem 5.3), we omit it.

We summarize the results of the paper as follows.

6.3. THEOREM. *Let T be a generator of a Z_n-action on a 3-manifold M with compact boundary (possibly empty). We assume that* $\dim \mathrm{Iz}(M) \leqslant 0$ *and $i(\tilde{M}) < \infty$. Then we can obtain (M, T) using multiple and connected sum from Z_{n_i}-actions on manifolds M_i, where n_i divides n and one of the following possibilities holds:*

(a) $n \leqslant 3$ *and* (i) $\dim \mathrm{Iz}(M_i) \leqslant 0$ *for each i,* (ii) *M_i is irreducible,* (iii) *each projective plane, 2-sided in* $\mathrm{int} M_i$, *is homotopy parallel to the boundary, and* (iv) *if $\partial M_i \neq \emptyset$, then M_i is ∂-irreducible;*

(b) $n > 3$ *and* (i) $\dim \mathrm{Iz}(M_i) \leqslant 0$ *for each i,* (ii) *M_i is congruent to an irreducible manifold,* (iii) *each projective plane, 2-sided in* $\mathrm{int} M_i$, *is homotopy parallel to the boundary, and* (iv) *if $\partial M_i \neq \emptyset$, then M_i is congruent to a ∂-irreducible manifold.*

First we note that M is decomposable.

6.4. PROPOSITION. *If M is a nonorientable 3-manifold, then $i(\tilde{M}) < \infty$ implies $i(M) < \infty$ or, more exactly, $i(\tilde{M}) \geqslant 2i(M) - 1$.*

The proof is easy and we omit it.

Proof of Theorem 6.3 (a). (1) We may reduce our action down to actions on irreducible manifolds using Theorems 5.1 and 5.2, Corollary 4.3, and Proposition 6.4.

(2) We may reduce an action on a manifold satisfying the conditions (i) and (ii) of Theorem 6.3 (a) down to actions on manifolds satisfying the conditions (i), (ii), and (iv) of Theorem 6.3 (a) using Corollaries 5.8, 4.5, and 2.14.

(3) We may reduce an action on a manifold satisfying the conditions (i), (ii), and (iv) of Theorem 6.3 (a) down to actions on manifolds satisfying the conditions (i)-(iv) of Theorem 6.3 (a) using Corollaries 5.6, 4.4, and 2.12.

Theorem 6.3 (a) follows immediately from (1)-(3).

Since the proof of Theorem 6.3 (b) is very similar to that of 6.3 (a) (using Theorem 5.3 in place of Theorems 5.1 and 5.2), we omit it.

7. Corollaries and remarks. We may use the results of this paper to classify Z_n-actions with $\dim \mathrm{Iz}(\cdot) \leqslant 0$ on handlebodies, compact surfaces, and $\# S^1 \hat{\times} S^2$ (up to free actions of Z_n on S^3). The author presents this classification in [17] and [16]. Most of the results contained in this paper can be probably extended to actions of finite groups on 3-manifolds. We could improve Theorems 6.2 and 6.3 if the following conjecture is true:

7.1. CONJECTURE. Let a finite group G act effectively on a 3-manifold M. Suppose that M is not irreducible. Then there is an embedding $S^2 \hookrightarrow \mathrm{int}(M)$ such that S^2 does not bound a 3-cell in M, $g(S^2) \cap S^2 = \emptyset$ or S^2 for each $g \in G$, and S^2 is in a general position with respect to $\mathrm{Iz}(M)$. (**P 1270**)

7.2. Remark. Conjecture 7.1 for $\dim \mathrm{Iz}(M) \leqslant 0$ follows from the Poincaré Conjecture.

7.3. Remark. Conjecture 7.1 for $\dim \mathrm{Iz}(M) \leqslant 0$ is true for any (Z_2, Z_3)-solvable group G. This follows from Theorems 5.1 and 5.2. G is (Z_2, Z_3)-solvable iff there is a sequence $1 = G_0 \subset G_1 \subset \ldots \subset G_n = G$ such that G_i is normal in G_{i+1} and $G_{i+1}/G_i = Z_2$ or Z_3.

REFERENCES

[1] G. E. Bredon, *Introduction to compact transformation groups*, New York 1972.

[2] J. L. Gross, *A unique decomposition theorem for 3-manifolds with connected boundary*, Transactions of the American Mathematical Society 147 (1969), p. 191-199.

[3] — *The decomposition of 3-manifolds with several boundary components*, ibidem 147 (1970), p. 561-572.

[4] A. J. S. Hamilton, *The triangulation of 3-manifold*, The Quarterly Journal of Mathematics 27 (1976), p. 63-70.

[5] J. Hempel, *3-manifolds*, Annals of Mathematics Studies No. 86, Princeton University Press, Princeton 1976.

[6] P. K. Kim, *Periodic homeomorphisms of the 3-sphere and related spaces*, The Michigan Mathematical Journal 21 (1974), p. 1-6.

[7] — *PL-involution on the nonorientable 2-sphere bundle over S^1*, Proceedings of the American Mathematical Society 55 (1976), p. 449-452.

[8] — and J. L. Tollefson, *Splitting the PL-involutions on nonprime 3-manifolds*, The Michigan Mathematical Journal (to appear).

[9] K. W. Kwun, *Piecewise linear involutions of $S^1 \times S^2$*, ibidem 16 (1969), p. 93-96.

[10] — and J. L. Tollefson, *PL involutions of $S^1 \times S^1 \times S^1$*, Transactions of the American Mathematical Society 203 (1975), p. 97-106.

[11] G. R. Livesay, *Fixed point free involutions on the 3-sphere*, Annals of Mathematics 72 (1960), p. 603-611.

[12] — *Involutions with two fixed points on the three-sphere*, ibidem 78 (1963), p. 582-593.

[13] J. Milnor, *A unique decomposition theorem for 3-manifolds*, American Journal of Mathematics 84 (1962), p. 1-7.

[14] J. R. Munkres, *Obstruction to smoothing piecewise-differentiable homeomorphisms*, Annals of Mathematics 72 (1960), p. 521-554.

[15] P. Orlik and F. Raymond, *Action of* SO(2) *on 3-manifolds*, Proceedings of the Conference on Transformation Groups, Springer-Verlag, 1968.

[16] J. H. Przytycki, *Actions of Z_n on connected sum of some 3-manifolds*, in preparation.

[17] — *Free actions of Z_n on handlebodies and surfaces*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 7 (1978), p. 617-624.

[18] G. X. Ritter, *Free actions of Z_{2n} on S^3* (to appear).

[19] — *Actions of Z_{2n} on S^3* (to appear).

[20] C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*, Berlin 1972.

[21] G. A. Swarup, *Some properties of 3-manifolds with boundary*, The Quarterly Journal of Mathematics 21 (1970), p. 1-24.

[22] J. L. Tollefson, *Free involutions on non-prime 3-manifolds*, Osaka Journal of Mathematics 7 (1970), p. 161-164.

[23] — *Involutions on $S^1 \times S^2$ and other 3-manifolds*, Transactions of the American Mathematical Society 183 (1973), p. 139-152.

[24] — *Homotopically trivial periodic homeomorphism on 3-manifold*, Annals of Mathematics 97 (2) (1973), p. 14-26.

[25] F. Waldhausen, *Über Involutionen der 3-Sphäre*, Topology 8 (1969), p. 81-91.

[26] J. H. C. Whitehead, *Manifolds with transverse fields in Euclidean space*, Annals of Mathematics 73 (1961), p. 154-212.

INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY
WARSZAWA

Reçu par la Rédaction le 15. 7. 1979