COLLOQUIUM MATHEMATICUM

VOL. XXII 1971 FASC. 2

ON A GENERALIZATION OF SMOOTH AND SYMMETRIC FUNCTIONS

BY

N. C. MANNA (BURDWAN, WEST BENGAL)

1. Let f be a real function defined in some neighbourhood of the closed interval $I_0 = [a, b]$. Then f is termed *smooth* [*symmetric*] at $x \in I_0$, iff for $h \to 0$

$$f(x+h)+f(x-h)-2f(x) = o(h)$$

$$[f(x+h)+f(x-h)-2f(x) = o(1)].$$

Let E be the set of points of differentiability of f on I_0. Zygmund [5] has shown that if f is continuous and smooth on I_0, then E is of the power of continuum in every subinterval of I_0 and f' has Darboux property on E. Neugebauer [3] proved that if f is measurable and smooth on I_0 and if $|E \cap I| < |I|$ for every subinterval $I \subset I_0$, then f' has Darboux property on E and the set $\{x: D^+f(x) = +\infty \text{ and } D_+f(x) = -\infty\}$ is residual in I_0. He also proved that if f is measurable and smooth on I_0 and if the set $\{x: D^+f(x) = +\infty \text{ and } D_+f(x) = -\infty\}$ is of the first category, then there exists a dense open set G in I_0 such that f' exists almost everywhere on G, and that if a continuous and smooth function f is such that $f' \geqslant 0$ on E, then f is non-decreasing on I_0.

In this paper we have shown that if f is lower semicontinuous and smooth above, then f is continuous and that the results of Zygmund remain true even when smoothness of f is replaced by smoothness above. We have also obtained some results analogous to those of Neugebauer by considering nowhere monotone functions which are smooth above. It is also shown under certain conditions that if f is smooth above then f' takes up every real value in every subinterval of I_0.

2. A function f is said to be *smooth above at a point* x_0 iff

$$\overline{\lim_{h\to 0+}}\ \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h} \leqslant 0.$$

Similarly, f is said to be *smooth below at* x_0 iff

$$\varliminf_{h\to 0+} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h} \geqslant 0.$$

If f is smooth above and below at x_0, then f is smooth at x_0. If f is smooth above [below] at each point of I_0, then f is said to be *smooth above* [*below*] *on* I_0.

THEOREM 1. *Let f be smooth above at a point x_0. Then* (i) $D^+f(x_0) \leqslant D^-f(x_0)$ *and* (ii) $D_+f(x_0) \leqslant D_-f(x_0)$.

Proof. If possible, suppose that $l = D^+f(x_0) > D^-f(x_0) = m$. Choose ε, $0 < \varepsilon < (l-m)/3$. Since

$$\varlimsup_{h\to 0+} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h} \leqslant 0,$$

there is a $\delta > 0$ such that

$$\frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h} < \varepsilon \tag{1}$$

for all h, $0 < h < \delta$. Since

$$\varlimsup_{h\to 0+} \frac{f(x_0-h)-f(x_0)}{-h} = m,$$

there is a δ_1, $0 < \delta_1 < \delta$, such that

$$\frac{f(x_0-h)-f(x_0)}{-h} < m+\varepsilon$$

for all h, $0 < h < \delta_1$. Now since

$$\varlimsup_{h\to 0+} \frac{f(x_0+h)-f(x_0)}{h} = l,$$

there is an h', $0 < h' < \delta_1$, such that

$$\frac{f(x_0+h')-f(x_0)}{h'} > l-\varepsilon.$$

Hence

$$\frac{f(x_0+h')+f(x_0-h')-2f(x_0)}{h'}$$

$$= \frac{f(x_0+h')-f(x_0)}{h'} - \frac{f(x_0-h')-f(x_0)}{-h'}$$

$$> l-m-2\varepsilon > \varepsilon,$$

which contradicts (1). Hence $D^+f(x_0) \leqslant D^-f(x_0)$. Similarly, it can be shown that $D_+f(x_0) \leqslant D_-f(x_0)$.

THEOREM 2. *If f is smooth above at a point x_0 and if f has a minimum at x_0, then $f'(x_0)$ exists and $f'(x_0) = 0$.*

Proof. Since f has a minimum at x_0, we have $D_+f(x_0) \geqslant 0$ and $D^-f(x_0) \leqslant 0$. Hence, from Theorem 1, the results follows.

THEOREM 3. *If f is lower semicontinuous and smooth above at x_0, then f is continuous at x_0.*

Proof. If possible let f be not upper semicontinuous at x_0. Then for some $\varepsilon_0 > 0$ there is a sequence $\{h_n\}$, $h_n \to 0$, such that

$$f(x_0+h_n) \geqslant f(x_0)+\varepsilon_0.$$

Since f is lower semicontinuous at x_0, we have

$$f(x_0-h_n) \geqslant f(x_0)-\frac{\varepsilon_0}{2} \quad \text{for } n \geqslant N.$$

We may assume $h_n > 0$ for all n. Then for $n \geqslant N$

$$\frac{f(x_0+h_n)+f(x_0-h_n)-2f(x_0)}{h_n} \geqslant \frac{\varepsilon_0}{2h_n} \to +\infty,$$

which contradicts that f is smooth above at x_0. This proves the theorem.

Since from Theorem 3 lower semicontinuity and smoothness above imply continuity, henceforth we shall consider continuous functions. We recall that a real function f is *nowhere monotone* in an interval I, if there is no subinterval of I in which f is monotone [1]. A nowhere monotone function f is of the *first species* in I if there exists a real number r such that the function g, where $g(x) = f(x)+rx$, is monotone in I. A nowhere monotone function f is of the *second species* in I if the function g, where $g(x) = f(x)+rx$, remains nowhere montone in I for every real number r.

LEMMA 1. *Let f be continuous and smooth above in I_0. Let $-\infty < \lambda < \infty$ and let the function $f(x)-\lambda x$ be nowhere monotone in I_0. Then there exists a dense set D in I_0 such that $f'(x)$ exists and $f'(x) = \lambda$ for $x \in D$.*

Proof. Let $J = [p, q]$ be any subinterval of I_0. We shall show that there is a point ξ in J such that $f'(\xi) = \lambda$. Let $g(x) = f(x)-\lambda x$. We shall now show that g has a minimum in J. We may assume that $g(p) < g(q)$. If there is a point ξ, $p < \xi < q$, such that $g(\xi) < g(p)$, then, since g is continuous, g/J attains its lower bound at a point in (p, q). Hence g has a minimum in (p, q). If this is not the case, then $g(x) \geqslant g(p)$ for all x in (p, q). Let $g(p) < \eta < g(q)$. Since g is continuous, there is a point x_0, $p < x_0 < q$, such that $g(x_0) = \eta$. If there is a point x' in (p, x_0)

for which $g(x') > \eta$, then, since $g(q) > \eta$, by the above argument g has a minimum in (x', q). So we suppose $g(x) \leqslant \eta$ for all x in (p, x_0). Since g is nowhere monotone, there are two points $x_1, x_2, x_1 < x_2$, in (p, x_0) such that $g(x_1) > g(x_2)$. Also $g(x_0) > g(x_2)$. Then by the same argument g has a minimum in (x_1, x_0). Thus we conclude that g has a minimum at a point ξ in (p, q). Since g is smooth above, by Theorem 2 $g'(\xi)$ exists and $g'(\xi) = 0$, i.e., $f'(\xi) = \lambda$. This proves the lemma.

THEOREM 4. *Let f be a continuous nowhere monotone function of the second species in I_0. If f is smooth above in I_0, then in every subinterval of I_0 there are points where f' exists and takes up any prescribed value.*

Proof. Let r be any real number and J be any subinterval of I_0. Since the function $f(x) - rx$ is nowhere monotone, by Lemma 1 there is a point ξ in J such that $f'(\xi) = r$. This completes the proof.

THEOREM 5. *Let f be continuous and smooth above in I_0. Then the set of points of differentiability of f is of the power of continuum in every subinterval of I_0.*

Proof. Let $J = [p, q]$ be any subinterval of I_0. If for some real number r, the function $f(x) - rx$ is monotone in some subinterval J_0 of J, then f' exists almost everywhere in J_0 and hence the set of points of differentiability of f has the power of continuum in J. So, we suppose that for all real numbers r, the function $f(x) - rx$ is nowhere monotone in J. Then, by Theorem 4, for each real number r, there is a point ξ_r in J such that $f'(\xi_r) = r$. Also distinct real numbers r correspond to distinct points ξ_r. For, if $r_1 < r_2$, then $f'(\xi_{r_1}) = r_1 < r_2 = f'(\xi_{r_2})$ and so $\xi_{r_1} \neq \xi_{r_2}$. Since the set of r's is of the power of continuum, it follows that the set of points at which f' exists is of the power of continuum in J.

Note. It is interesting to observe that Zygmund [6] constructed a continuous function f such that

$$\frac{f(x+h)+f(x-h)-2f(x)}{h} \leqslant m < \infty$$

holds for all h uniformly in x, but f is nowhere differentiable; for $m = 0$ this no longer holds.

THEOREM 6. *Let f be a continuous nowhere monotone function of second species in I_0. If f is smooth above in I_0 and if $E = \{x : x \in I_0; f'(x)$ exists$\}$, then f' has Darboux property on E.*

Proof. Let $\alpha, \beta \in E$ and $\alpha < \beta$. Let $f'(\alpha) < \lambda < f'(\beta)$. Since the function f satisfies the hypothesis of Theorem 4, there is a point ξ in (α, β) such that $f'(\xi) = \lambda$. This proves the theorem.

THEOREM 7. *Let f be continuous and smooth above in I_0. If $f'(x) \geqslant 0$ for $x \in E$, where $E = \{x : x \in I_0; f'(x)$ exists$\}$, then f is non-decreasing in I_0.*

Proof. If possible, let there be two points α and β, $\alpha < \beta$, in I_0 for which $f(\beta) < f(\alpha)$. Consider the function

$$g(x) = f(x) - \frac{f(\beta)-f(\alpha)}{\beta-\alpha}\, x.$$

Then $g(\alpha) = g(\beta)$. We shall show that g has a minimum in (α, β). If there is a point ξ', $\alpha < \xi' < \beta$, such that $g(\xi') < g(\beta)$, then since g is continuous, g attains its lower bound at a point in (α, β). If there is no such point, then $g(x) \geqslant g(\beta)$ for all x in (α, β). Let $x' \epsilon (\alpha, \beta)$ be such that $g(x') > g(\beta)$. Choose η such that $g(x') > \eta > g(\beta)$. Then there is an x_0, $x' < x_0 < \beta$, for which $g(x_0) = \eta$. Now g cannot be non-increasing in (x_0, β). For if g is non-increasing in (x_0, β), then $g'(x) \leqslant 0$, i.e.,

$$f'(x) \leqslant \frac{f(\beta)-f(\alpha)}{\beta-\alpha} < 0$$

almost everywhere in (x_0, β), which is a contradiction. Hence there are points $x_1, x_2, x_0 < x_1 < x_2 < \beta$, such that $g(x_1) < g(x_2)$. So we conclude that g has a minimum in (x', β). Hence, by Theorem 2, there is a $\xi \epsilon (\alpha, \beta)$ such that $g'(\xi) = 0$, i.e.,

$$f'(\xi) = \frac{f(\beta)-f(\alpha)}{\beta-\alpha} < 0,$$

which is a contradiction.

For convenience we state the following lemma which is proved in [3]:

LEMMA 2. *Let f be continuous in I_0 and let there be a dense set D and a number λ, $-\infty < \lambda < \infty$, such that $D^+f(x) \leqslant \lambda$ for $x \epsilon D$. Then the set*

$$A = \{x: \lambda < D_+f(x)\}$$

is of the first category.

THEOREM 8. *Let f be continuous and smooth above in I_0. Let there be a sequence of positive numbers $\{s_n\}$, $s_n \to \infty$ as $n \to \infty$, such that the function $f(x) + s_n x$ is nowhere monotone for each n. Then the set*

$$A = \{x: D_+f(x) = -\infty\}$$

is a residual set in I_0.

Similarly, if there is $\{r_n\}$, $r_n \to \infty$ as $n \to \infty$, such that $f(x) - r_n x$ is nowhere monotone for each n, then the set

$$B = \{x: D^+f(x) = +\infty\}$$

is a residual set in I_0.

Proof. Let

$$A_n = \{x: x \epsilon I_0;\ -s_n < D_+f(x)\}$$

and

$$B_n = \{x: x \epsilon I_0;\ r_n > D^+f(x)\}.$$

By Lemmas 1 and 2, A_n is of the first category for each n. Hence $I_0 - A$ is of the first category. So the set A is residual in I_0. Similarly, the set B is residual in I_0.

THEOREM 9. *Let f be continuous and smooth above in I_0. Let the set*

$$A = \{x: D_+f(x) = -\infty;\ D^+f(x) = +\infty\}$$

be of the first category.

Then there exists a dense open set G in I_0 such that $f'(x)$ exists almost everywhere on G.

Proof. Let I_1 be a subinterval of I_0. Then, for some real number r, there is a subinterval $I_2 \subset I_1$ such that $f(x)+rx$ is monotone in I_2 and hence $f'(x)$ exists almost everywhere in I_2. Then the set $G = \bigcup I_2^0$, where I^0 denotes the interior of I, the union being extended over all $I_1 \subset I_0$, has the desired property.

3. A function f is said to be *symmetric above at a point* x_0 iff

$$\overline{\lim_{h\to 0}}\ [f(x_0+h)+f(x_0-h)-2f(x_0)] \leqslant 0.$$

Similarly, f is said to be *symmetric below at* x_0 iff

$$\underline{\lim_{h\to 0}}\ [f(x_0+h)+f(x_0-h)-2f(x_0)] \geqslant 0.$$

If f is symmetric above and below at a point x_0, then f is said to be *symmetric at* x_0. If f is symmetric above [below] at each point of I_0, then f is said to be *symmetric above* [*below*] *on* I_0.

LEMMA 3. *Let f be Borel measurable in I_0. Then D^+f, D^-f are Borel measurable in I_0 and for $r, s > 0$ the set*

$$A = \{x_0: x_0 \in I_0;\ f(x)-f(x_0) < r(x-x_0),\ x_0 < x < x_0+s\}$$

is Borel measurable.

This can be proved by a slight modification of the argument used in [3] with the help of [4].

LEMMA 4. *Let $B \subset I_0$ be a Borel set and assume that B is not of the first category in I_0. Then there exists an interval $I \subset I_0$ such that $B \cap I$ is residual in I.*

For the proof see [3].

THEOREM 10. *Let f be symmetric above in I_0 and be of Baire class* 1. *Then the set*

$$S = \{x: x \in I_0;\ D^+f(x) < D^-f(x)\} \cup \{x: x \in I_0;\ D_+f(x) < D_-f(x)\}$$

is of the first category.

Proof. We shall first show that the set $A = \{x: x \in I_0;\ D^+f(x) < D^-f(x)\}$ is of the first category. For r rational let

$$A_r = \{x: x \in I_0;\ D^+f(x) < r < D^-f(x)\}$$

and for every positive integer j,

$$A_{r,j} = \left\{x_0;\ x_0 \in A_r \text{ and } \frac{f(x)-f(x_0)}{x-x_0} < r,\ x_0 < x < x_0 + \frac{1}{j}\right\}.$$

By Lemma 3, $A_{r,j}$ is Borel measurable. Since $A = \bigcup_r \bigcup_j A_{r,j}$, it suffices to show that $A_{r,j}$ is of the first category for each r and j. If this is not the case, by Lemma 4 there is an interval $(\alpha, \beta) \subset I_0$ such that $A_{r,j} \cap (\alpha, \beta)$ is residual in (α, β). We may assume that $\beta - \alpha < 1/j$. Let x', x'' be any two points with $\alpha < x' < x'' < \beta$. Choose $\delta > 0$ such that $\alpha < x' - \delta < x' < x' + \delta < x''$. Let $A' = A_{r,j} \cap (x', x' + \delta)$ and $A'' = A_{r,j} \cap (x' - \delta, x')$. Then A' and A'' are residual in $(x', x' + \delta)$ and $(x' - \delta, x')$, respectively. Let $A_0 = \{y: y \in (x' - \delta, x') \text{ and } 2x' - y \in A'\}$, the reflection of A' across x'. Then A_0 is residual in $(x' - \delta, x')$. Since the complements of A'' and A_0 in $(x' - \delta, x')$ are sets of the first category, it follows that $A'' \cap A_0 \neq 0$. Hence we can construct sequences $\{x_n\}$ and $\{y_n\}$ in $A_{r,j}$ such that $\alpha < x_n < x' < y_n < x''$, $x_n + y_n = 2x'$, $x_n \to x'$ and $y_n \to x'$. Since $x_n, y_n \in A_{r,j}$ and $x_n < x'' < x_n + 1/j$, $y_n < x'' < y_n + 1/j$, we have

$$\frac{f(x'')-f(x_n)}{x''-x_n} < r \quad \text{and} \quad \frac{f(x'')-f(y_n)}{x''-y_n} < r.$$

Now $x_n - x'' < y_n - x''$. If $r \geqslant 0$, we have

$$\frac{f(y_n)-f(x'')}{x_n-x''} = \frac{f(y_n)-f(x'')}{y_n-x''} \cdot \frac{y_n-x''}{x_n-x''} < r. \tag{1}$$

Also

$$\frac{f(x_n)-f(x'')}{x_n-x''} < r.$$

So

$$\frac{f(x_n)+f(y_n)-2f(x'')}{x_n-x''} < 2r.$$

Since

$$\overline{\lim_{n\to\infty}}\{f(x_n)+f(y_n)\} \leqslant 2f(x'),$$

we have

$$\frac{f(x')-f(x'')}{x'-x''} \leqslant r, \qquad \alpha < x' < x'' < \beta. \tag{2}$$

Also if $x'' \in A_{r,j} \cap (\alpha, \beta)$, $D^- f(x'') > r$. So, there is $x' \in (\alpha, x'')$ such that

$$\frac{f(x') - f(x'')}{x' - x''} > r,$$

which contradicts (2). If $r < 0$, interchanging x_n and y_n beginning with inequality (1), we can arrive at the same contradiction. Hence A is of the first category. Similarly, it can be shown that the set

$$B = \{x\colon x \in I_0;\ D_+ f(x) < D_- f(x)\}$$

is of the first category.

COROLLARY 1. *Let f be lower semicontinuous and symmetric above in I_0. Then except a set of the first category in I_0*

$$D^+ f(x) = D^- f(x) \quad \textit{and} \quad D_+ f(x) = D_- f(x).$$

Proof. Since f is lower semicontinuous, the set

$$\{x\colon x \in I_0;\ D^- f(x) < D^+ f(x)\} \cup \{x\colon x \in I_0;\ D_- f(x) < D_+ f(x)\}$$

is of the first category [2]. Hence applying Theorem 10, the conclusion follows.

Remark. Neugebauer [3] proved that if f is measurable, and symmetric in I_0, then except a set of the first category in I_0.

$$D^+ f(x) = D^- f(x) \quad \text{and} \quad D_+ f(x) = D_- f(x).$$

COROLLARY 2. *Let f be smooth above in I_0 and be of Baire class* 1. *Then except a set of the first category in I_0*

$$D^+ f(x) = D^- f(x) \quad \textit{and} \quad D_+ f(x) = D_- f(x).$$

Proof. Since f is smooth above in I_0, it is symmetric above in I_0. So, by Theorem 10, the set

$$\{x\colon x \in I_0;\ D^+ f(x) < D^- f(x)\} \cup \{x\colon x \in I_0;\ D_+ f(x) < D_- f(x)\}$$

is of the first category. Hence the result follows from Theorem 1.

The author expresses earnest gratitudes to Dr. S. N. Mukhopadhyay for his kind help in the preparation of this paper.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BURDWAN
BURDWAN, WEST BENGAL, INDIA

Reçu par la Rédaction le 12. 9. 1969