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ON CERTAIN CURVATURE CONDITIONS 'ON RIEMANNIAN MANIFOLDS

BY

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1. Introduction. Let M be a connected n-dimensional ($n \geqslant 3$) Riemannian manifold with not necessarily definite metric g. In this paper we consider generalized curvature tensors $\tilde{B}$ satisfying on M the condition

$$\underset{X,Y,Z}{\mathscr{S}} w(X)\tilde{B}(Y, Z) = 0, \tag{1}$$

where w is a 1-form, $\mathscr{S}$ denotes the cyclic sum and $X, Y, Z \in \mathfrak{X}(M)$, $\mathfrak{X}(M)$ being the Lie algebra of vector fields on M. In Theorem 1 we prove that if a generalized curvature tensor $\tilde{B}$ and a 1-form w fulfil condition (1), then at every point of M at which $w \neq 0$ the equality

$$\tilde{B} \cdot B = Q(\mathrm{Ric}(\tilde{B}), B) \tag{2}$$

holds. Further results of this paper are concerned with generalized curvature tensors fulfilling (1) on a Riemannian manifold admitting 1-form v such that

$$\tilde{B} \cdot v = hQ(g, v), \tag{3}$$

h being a function on M. In Theorem 2 we prove, among others, that if an analytic manifold M admits a non-zero generalized curvature tensor B and non-zero 1-forms w and v satisfying (1) and (3) (with a non-zero function h), respectively, then the relations

$$C(\tilde{B}) = 0 \quad \text{and} \quad \tilde{B} \cdot B = hQ(g, B)$$

hold on M. From this theorem it follows (see Theorem 3) that if M is an analytic Riemannian manifold of dimension $\geqslant 4$ and admitting a certain concircular vector field and the curvature tensor R satisfies (1) with a non-zero 1-form w, then M is a conformally flat pseudo-symmetric manifold. At the end of this paper we give examples illustrating our theorems.

Throughout this paper all manifolds are assumed to be connected Hausdorff manifolds of class C^∞. Whenever analyticity is supposed, it concerns all objects involved.

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2. Preliminaries. Let M be a Riemannian manifold. We denote by ∇, $\tilde{R}$, R, S, $\tilde{C}$ and K the Levi-Cività connection, the curvature tensor, the Riemann–Christoffel curvature tensor, the Ricci tensor, the Weyl conformal curvature tensor and the scalar curvature of M, respectively.

A tensor field $\tilde{B}$ of type (1, 3) on M is said to be a *generalized curvature tensor* [12] if

$$\underset{X_1,X_2,X_3}{\mathcal{S}} \tilde{B}(X_1, X_2)X_3 = 0, \qquad \tilde{B}(X_1, X_2)+\tilde{B}(X_2, X_1) = 0$$

and

$$B(X_1, X_2, X_3, X_4) = B(X_3, X_4, X_1, X_2),$$

where

$$B(X_1, X_2, X_3, X_4) = g(\tilde{B}(X_1, X_2)X_3, X_4) \quad \text{and} \quad X_1, \ldots, X_4 \in \mathfrak{X}(M).$$

The *Ricci tensor* $\operatorname{Ric}(\tilde{B})$ of $\tilde{B}$ is the trace of the linear mapping

$$X_1 \to \tilde{B}(X_1, X_2)X_3.$$

The tensor fields $X \wedge Y$, $\widetilde{\operatorname{Ric}}(\tilde{B})$ and $\operatorname{Ric}^2(\tilde{B})$ are defined by the following relations:

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y,$$

$$\operatorname{Ric}(\tilde{B})(X, Y) = g(\widetilde{\operatorname{Ric}}(\tilde{B})X, Y),$$

$$\operatorname{Ric}^2(\tilde{B})(X, Y) = g(\widetilde{\operatorname{Ric}}(\tilde{B})X, \widetilde{\operatorname{Ric}}(\tilde{B})Y),$$

respectively, where $X, Y, Z \in \mathfrak{X}(M)$. Further, we define the *Weyl curvature tensor* $C(\tilde{B})$ associated with $\tilde{B}$ by the formula

$$C(\tilde{B})(X, Y) = \tilde{B}(X, Y) - \frac{1}{n-2}(\widetilde{\operatorname{Ric}}(\tilde{B})X \wedge Y + X \wedge \widetilde{\operatorname{Ric}}(\tilde{B})Y) + \frac{K(\tilde{B})}{(n-1)(n-2)} X \wedge Y.$$

As a generalization of the definition of the tensor $X \wedge Y$ we define for a tensor A of type (0, 2) the tensor $X \wedge_A Y$ by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y.$$

For a tensor field T of type $(0, p)$, $p \geqslant 1$, we define the tensor fields $\tilde{B} \cdot T$ and $Q(A, T)$ by the formulas

$$(\tilde{B} \cdot T)(X_1, \ldots, X_p; X, Y) = -T(\tilde{B}(X, Y)X_1, X_2, \ldots, X_p) - \ldots - T(X_1, \ldots, X_{p-1}, \tilde{B}(X, Y)X_p)$$

and

$$Q(A, T)(X_1, \ldots, X_p; X, Y)$$
$$= T((X \wedge_A Y)X_1, X_2, \ldots, X_p)+\ldots+T(X_1, \ldots, X_{p-1}, (X \wedge_A Y)X_p),$$

respectively, where $X_1, \ldots, X_p, X, Y \in \mathfrak{X}(M)$.

A 1-form w on a Riemannian manifold M is said to have *property* (P) (cf. [7]) if the relation

$$\tilde{R} \cdot v = Q(g, w) \tag{4}$$

holds at every point $x \in M$, v being a suitably chosen 1-form at x. A vector field V on a manifold M is said to be *concircular* [16] if the equation

$$\nabla V = f \cdot \mathrm{Id} \tag{5}$$

holds on M, where f is a function and Id is the identity transformation on M. It is clear that if a manifold M admits a vector field V satisfying (5), then the 1-form $w = -df$ satisfies (4) with v given by $v(X) = g(V, X)$, $X \in \mathfrak{X}(M)$. Moreover, if $df \neq 0$ at a point $x \in M$, then at x we have $v \neq 0$, $-df = \tau v$ with $\tau \in \boldsymbol{R}-\{0\}$ and

$$\tilde{R} \cdot v = \tau Q(g, v). \tag{6}$$

We see that condition (6) is just (3) with $\tilde{B} = \tilde{R}$.

A manifold M is said to be *pseudo-symmetric* (see [5]) if at every point of M the tensors $\tilde{R} \cdot R$ and $Q(g, R)$ are linearly dependent. Recently, pseudo-symmetric manifolds have been studied by many authors (for references see [4]).

Some results on generalized curvature tensors on Riemannian manifolds admitting concircular vector fields are given in [5]–[7].

3. Auxiliary lemmas.

LEMMA 1. *Let $\tilde{B}$ be a generalized curvature tensor field on a Riemannian manifold M. If the conditions*

$$\tilde{B} \cdot B = \alpha Q(\mathrm{Ric}(\tilde{B}), B), \qquad \tilde{B} \cdot v = \tau Q(g, V) \tag{7}$$

hold at $x \in M$, where $\alpha \in \boldsymbol{R}$, $0 \neq v \in T_x^* M$ and $\tau \in \boldsymbol{R}-\{0\}$, then $C(\tilde{B}) = 0$.

Proof. (i) Suppose that $\alpha = 0$. Transvecting the equality

$$(\tilde{B} \cdot B)_{rstuvw} = 0$$

with $V^r = g^{rs} v_s$ and using (7) we obtain

$$A_{wvstu} = A_{vwstu}, \tag{8}$$

where

$$A_{wvstu} = v_w(B_{vstu} - \tau G_{vstu}), \qquad G_{vstu} = g_{vu} g_{st} - g_{vt} g_{su},$$

and $(\tilde{B}\cdot B)_{rstuvw}$, v_r, B_{rstu} and g_{st} are local components of $\tilde{B}\cdot B$, v, B and g, respectively, and $r, s, t, u, v, w \in \{1, 2, \ldots, n\}$. But (8) and the equality

$$A_{wvstu} = -A_{wsvtu},$$

in view of Lemma 1 in [3], give $A_{wvstu} = 0$. Thus

$$B_{vstu} = \tau G_{vstu}$$

and, consequently, $C(\tilde{B}) = 0$.

(ii) Now, let $\alpha \neq 0$. The equality

$$(\tilde{B}\cdot B)_{rstuvw} = \alpha Q(\operatorname{Ric}(\tilde{B}), B)_{rstuvw}, \tag{9}$$

by transvection with V^r and making use of (7), gives

$$\begin{aligned}((n-1)\alpha-1)(v_w B_{vstu} - v_v B_{wstu}) + \tau(v_w G_{vstu} - v_v G_{wstu})\\ + \alpha\big(Z_{ws}(v_u g_{vt} - v_t g_{vu}) + Z_{wt}(v_u g_{sv} - v_t g_{su})\\ + Z_{wu}(v_v g_{st} - v_t g_{su}) - Z_{vs}(v_u g_{wt} - v_t g_{wu})\\ - Z_{vt}(v_u g_{sw} - v_w g_{su}) - Z_{vu}(v_w g_{st} - v_t g_{sw})\big) = 0,\end{aligned} \tag{10}$$

where $Q(\operatorname{Ric}(\tilde{B}), B)_{rstuvw}$ and Z_{uv} are local components of $Q(\operatorname{Ric}(\tilde{B}), B)$ and $\operatorname{Ric}(\tilde{B})$, respectively. This, by contraction with g^{st} and application of (7), turns into

$$(\alpha-1)\big(v_w(Z_{uv} - (n-1)\tau g_{uv}) - v_v(Z_{uw} - (n-1)\tau g_{uw})\big) = 0, \tag{11}$$

whence we obtain easily

$$(\alpha-1)\left(\tau - \frac{K(\tilde{B})}{n(n-1)}\right) = 0 \tag{12}$$

and

$$(\alpha-1)\left(Z_{uv} - \frac{K(\tilde{B})}{n} g_{uv} - \beta v_u v_v\right) = 0. \tag{13}$$

Further, transvecting (9) with V^w and using (7) we get

$$\begin{aligned}((n-1)\alpha-1)(v_r B_{vstu} + v_s B_{rvtu} + v_t B_{rsvu} + v_u B_{rstu})\\ + \tau(v_r G_{vstu} + v_s G_{rvtu} + v_t G_{rsvu} + v_u G_{rstv})\\ + v_r(g_{us} Z_{vt} - g_{ts} Z_{vu}) + v_s(g_{tr} Z_{vu} - g_{ur} Z_{vt})\\ + v_u(g_{rt} Z_{vs} - g_{ts} Z_{vr}) + v_t(g_{su} Z_{vr} - g_{ur} Z_{vs}) = 0,\end{aligned} \tag{14}$$

whence, by contraction with g^{rv}, we find

$$(n\alpha-1)(v_u Z_{ts} - v_t Z_{us}) + ((n-1)\tau - \alpha K(\tilde{B}))(v_u g_{st} - v_t g_{su}) = 0$$

or, interchanging the indices,

(15) $$(n\alpha-1)(v_w Z_{vu}-v_v Z_{uw})+((n-1)\tau-\alpha K(\tilde{B}))(v_w g_{uv}-v_v g_{uw}) = 0.$$

Now, comparing (15) and (11) and applying the assumption $\alpha \neq 0$, we obtain

$$v_w Z_{vu}-v_v Z_{wu}+\left(\tau-\frac{K(\tilde{B})}{n-1}\right)(v_w g_{uv}-v_v g_{uw}) = 0,$$

whence

(16) $$Z_{uv} = \left(\frac{K(\tilde{B})}{n-1}-\tau\right)g_{uv}+\gamma v_u v_v.$$

Furthermore, by (16), equation (14) turns into

$$v_r D_{vstu}+v_s D_{rvtu}+v_t D_{vurs}+v_u D_{tvrs} = 0,$$

whence, in view of Lemma 3 in [15], we obtain

(17) $$D_{rstu} = 0,$$

where

$$D_{rstu} = ((n-1)\alpha-1)B_{rstu}+\left(2\tau-\frac{K(\tilde{B})}{n-1}\alpha\right)G_{rstu}$$
$$-\alpha\gamma(v_u v_r g_{ts}+v_t v_s g_{ur}-v_t v_r g_{us}-u_u v_s g_{vt}).$$

It is easy to verify, by standard calculations, that if $\alpha \neq 1/(n-1)$, then from (17) it follows that $C(\tilde{B}) = 0$.

Suppose now that $\alpha = 1/(n-1)$. Applying this in (12) we get

(18) $$\tau = \frac{K(\tilde{B})}{n(n-1)}.$$

Equality (17), together with the last two equalities and the condition $\alpha \neq 0$, yields

$$\frac{n-2}{n(n-1)}K(\tilde{B})G_{rstu} = \gamma(v_u v_r g_{ts}+v_t v_s g_{ur}-v_t v_r g_{us}-v_s v_u g_{rt}),$$

whence, by contraction with g^{st} and g^{ru}, we find

(19) $$(n-2)K(\tilde{B}) = 2(n-1)\gamma V^s v_s.$$

Moreover, substituting (18) into (16), we obtain

$$Z_{ur} = \frac{1}{n}K(\tilde{B})g_{ur}+\gamma v_u v_r,$$

whence, by contraction with g^{ur}, we get $\gamma V^s v_s = 0$. But this, together with (19), gives $K(\tilde{B}) = 0$, which, by (18), yields $\tau = 0$, a contradiction. Our lemma is thus proved.

LEMMA 2. *Let B be a generalized curvature tensor on a Riemannian manifold M. If $C(\tilde{B}) = 0$ at a point $x \in M$, then at this point the following three identities are equivalent to each other:*

$$\tilde{B} \cdot B = \varrho Q(g, B),$$

$$\tilde{B} \cdot \mathrm{Ric}(\tilde{B}) = \varrho Q(g, \mathrm{Ric}(\tilde{B})),$$

$$\left(\varrho + \frac{K(\tilde{B})}{(n-1)(n-2)}\right)\left(\mathrm{Ric}(\tilde{B}) - \frac{K(\tilde{B})}{n} g\right) = \frac{1}{n-2}\left(\mathrm{Ric}^2(\tilde{B}) - \frac{1}{n}\mathrm{tr}(\mathrm{Ric}^2(\tilde{B}))\, g\right),$$

where $\varrho \in \boldsymbol{R}$.

To prove this lemma we can use the method of the proof of Lemma 1.2 in [2].

LEMMA 3. *Let $\tilde{B}$ be a generalized curvature tensor on a Riemannian manifold M. If the conditions $C(\tilde{B}) = 0$ and (7) hold at a point $x \in M$, then at x the equalities*

$$(20) \quad B_{rstu} = \frac{1}{n-2}\left(\frac{K(\tilde{B})}{n-1} - 2\tau\right) G_{rstu}$$

$$+ \frac{\mu}{n-2}(v_r v_u g_{st} + v_s v_t g_{ru} - v_r v_t g_{su} - v_s v_u g_{rt}), \qquad \tau, \mu \in \boldsymbol{R},$$

$$Q(\mathrm{Ric}(\tilde{B}) - \tau g, B) = 0, \qquad \tilde{B} \cdot B = \tau Q(g, B)$$

are satisfied.

Proof. Let Z_{uv} be local components of $\mathrm{Ric}(\tilde{B})$. Transvecting the equality

$$(21) \quad B_{rstu} - \frac{1}{n-2}(g_{ru} Z_{st} - g_{rt} Z_{su} + g_{st} Z_{ru} - g_{su} Z_{rt}) + \frac{K(\tilde{B})}{(n-1)(n-2)} G_{rstu} = 0$$

with $V^r = g^{rs} v_s$ and using (7) we obtain

$$v_u Z_{ts} - v_t Z_{us} = \left(\frac{K(\tilde{B})}{n-1} - \tau\right)(v_u g_{ts} - v_t g_{us}),$$

which gives

$$(22) \qquad Z_{ts} = \left(\frac{K(\tilde{B})}{n-1} - \tau\right) g_{ts} + \mu v_t v_s.$$

Substituting (22) into (21) we get (20). The second equation of our assertion is a consequence of (20) and (22). Further, in view of (22), we can verify that the equality

$$\left(\tau+\frac{K(\tilde{B})}{(n-1)(n-2)}\right)\left(\operatorname{Ric}(\tilde{B})-\frac{K(\tilde{B})}{n}g\right)=\frac{1}{n-2}\left(\operatorname{Ric}^2(\tilde{B})-\frac{1}{n}\operatorname{tr}(\operatorname{Ric}^2(\tilde{B}))g\right)$$

holds at x. But this, in view of Lemma 2, completes the proof.

4. Main results.

THEOREM 1. *Let $\tilde{B}$ be a generalized curvature tensor on a Riemannian manifold M such that condition* (1) *is satisfied for $\tilde{B}$ and a 1-form w. If $w \neq 0$ at a point $x \in M$, then the relation*

$$\tilde{B}\cdot B = Q(\operatorname{Ric}(\tilde{B}), B)$$

holds at x.

Proof. Since w is non-zero at x, condition (1) yields

$$\begin{aligned} B(X_1, X_2, X_3, X_4) &= w(X_1)w(X_4)A(X_2, X_3)-w(X_1)w(X_3)A(X_2, X_4) \\ &\quad + w(X_2)w(X_3)A(X_1, X_4)-w(X_2)w(X_4)A(X_1, X_3), \end{aligned}$$

where A is a symmetric tensor of type (0, 2). Applying now the above equality in the definitions of the tensors $\tilde{B}\cdot B$ and $Q(\operatorname{Ric}(\tilde{B}), B)$, after straightforward calculations we obtain our assertion.

From the above theorem we obtain immediately:

COROLLARY 1. *Let M be a Riemannian manifold such that the curvature tensor $\tilde{R}$ (resp., the Weyl conformal curvature tensor $\tilde{C}$) satisfies condition* (1) *for a 1-form w. If $w \neq 0$ at a point $x \in M$, then the relation*

$$\tilde{R}\cdot R = Q(S, R) \quad (\textit{resp.}, \ \tilde{C}\cdot C = 0)$$

holds at x, where

$$g(\tilde{C}(X_1, X_2)X_3, X_4) = C(X_1, X_2, X_3, X_4).$$

THEOREM 2. *Let $\tilde{B}$ be a generalized curvature tensor on a Riemannian manifold M such that condition* (1) *is satisfied for $\tilde{B}$ and a 1-form w. Moreover, let the condition $\tilde{B}\cdot v = hQ(g, v)$ be satisfied on M, where h is a function on M. If at a point $x \in M$ the 1-forms w and v and the function h are non-zero, then the relations*

$$C(\tilde{B}) = 0, \tag{23}$$

$$\tilde{B}\cdot B = hQ(g, B), \tag{24}$$

$$Q(\operatorname{Ric}(\tilde{B})-hg, B) = 0 \tag{25}$$

hold on some neighbourhood U of x. Moreover, if M is an analytic manifold, then relations (23)–(25) *hold on M.*

Proof. Let U be a neighbourhood of x such that v and w and h are non-zero at every point $y \in U$. Thus, in view of Theorem 1, equation (2) holds on U. Furthermore, from Lemma 1 it follows that $C(\tilde{B}) = 0$. Now, Lemma 3 completes the proof.

From the above theorem we get

THEOREM 3. *Let M, $\dim M \geqslant 4$, be an analytic Riemannian manifold such that the curvature tensor $\tilde{R}$ and a non-zero 1-form w satisfy condition* (1). *Moreover, let M admit a non-zero concircular vector field V* (see (5)) *with a non-constant function f. Then M is a conformally flat pseudo-symmetric manifold satisfying the equality*

$$Q(S-hg,\, R) = 0,$$

where h is a function on M defined by

$$-df = hv \quad \text{and} \quad v(X) = g(V, X),\ X \in \mathfrak{X}(M).$$

5. Examples. In this section we give two examples of manifolds satisfying condition (1). The second one is a conformally flat pseudo-symmetric manifold admitting a concircular vector field.

EXAMPLE 1. Define the metric g on $M = \boldsymbol{R}^n$, $n \geqslant 4$, by the formula

$$g_{rs}\, dx^r dx^s = A(dx^1)^2 + k_{\alpha\beta}\, dx^\alpha dx^\beta + 2dx^1 dx^n,$$

where $[k_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constants, A is a function independent of x^n, and $\alpha, \beta, \gamma, \delta \in \{2, 3, \ldots, n-1\}$. The only components of R and C not identically zero are those related to (see [14])

$$R_{1\alpha\beta 1} = \tfrac{1}{2} A_{.\alpha\beta}$$

and

$$C_{1\alpha\beta 1} = \tfrac{1}{2} A_{.\alpha\beta} - \frac{1}{2(n-1)} k_{\alpha\beta} k^{\gamma\delta} A_{.\gamma\delta},$$

where the dot denotes partial differentiation. Now, it is easy to verify that the tensors R and C and the 1-form w with components $w_1 \neq 0$, $w_2 = w_3 = \ldots = w_n = 0$ satisfy condition (1). Let $w_1 \neq 0$ at every point of M. Then Theorem 1 implies $\tilde{R} \cdot R = Q(S, R)$. On the other hand, the manifold M is semi-symmetric, i.e., $\tilde{R} \cdot R = 0$ (see [8]). Thus, by the above equality, we have on M the relation

$$Q(S, R) = 0.$$

EXAMPLE 2. Let I be an open interval of $\boldsymbol{R}$ with the standard metric $g_{11} = \varepsilon$, $\varepsilon \in \{-1, 1\}$, F a positive smooth function on I, $\bar{M}$ with the metric $\bar{g}$ a manifold of constant curvature and $n-1 = \dim \bar{M} \geqslant 3$. Then the manifold

$M = I \times \bar{M}$ with the warped product metric g (see [1] and [9]) given by the formula

$$g_{rs}\, dx^r\, dx^s = g_{11}(dx^1)^2 + F(x^1)\, \bar{g}_{\alpha\beta}\, dx^\alpha\, dx^\beta,$$

where α, $\beta \in \{2, 3, \ldots, n\}$, is conformally flat ([11], pp. 176 and 179) and pseudo-symmetric (see [2]). Let

$$f = \varepsilon a \partial_1 (F^{1/2}), \qquad a \in \boldsymbol{R} - \{0\},$$

and V be the vector field with local components $V^s = g^{rs} v_r$, where $v_1 = aF^{1/2}$, $v_2 = \ldots = v_n = 0$. Then V and f satisfy identity (5) on M (cf. [10], p. 115). Therefore, if the function F does not satisfy on M the equation

$$2F_{.11}\, F - (F_{.1})^2 = 0,$$

then the manifold M admits a concircular vector field with the non-zero 1-form df. Now, using relation (20) with $B = R$, we see that the tensor R and the 1-form $w = v$ satisfy condition (1) on M if and only if the equation

$$\frac{K}{n-1} - 2h = 0 \tag{26}$$

holds on M, where h is the function on M defined by

$$-df = hv \tag{27}$$

and K is the scalar curvature of M. In virtue of (27) and the standard formula concerning scalar curvature of a warped product (see [13], Lemma 4), relation (26) is equivalent to

$$\frac{4\varepsilon}{(n-1)(n-2)} \bar{K} = \frac{1}{F}(F_{.1})^2,$$

where $\bar{K}$ is the scalar curvature of $\bar{M}$.

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