COLLOQUIUM MATHEMATICUM

XVI DÉDIÉ À M. FRANCISZEK LEJA 1967

ON THE METHODS OF REPRESENTATION OF SOME ALGEBRAS

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Introduction. In this paper we give a method of description of some algebras with the aid of idempotent algebras, i.e. algebras in which each fundamental operation f_i is idempotent, which means that it satisfies the equation $f_i(x, \ldots, x) = x$. This method is useful especially if some representation of the corresponding idempotent algebra is known.

By a *retraction* of an algebra $\mathfrak{A}$ we mean every endomorphism r of $\mathfrak{A}$ such that $r(r(x)) = r(x)$.

I. Let $\boldsymbol{K}$ be an equational class of algebras and U be a system of axioms of $\boldsymbol{K}$. Let each algebra $\mathfrak{A} \in \boldsymbol{K}$ be of the form

$$\mathfrak{A} = (X; \{f_i\}_{i \in I}).$$

We denote by U^* the set of axioms obtained from U by adding to U all equations

$$f_i(x, \ldots, x) = x \quad (i \in I) \tag{1}$$

and by $\boldsymbol{K}^*$ the equational class of algebras corresponding to U^*.

THEOREM 1. *The following two conditions are equivalent*:

(c_1) $\mathfrak{A} \in \boldsymbol{K}$ *and* $\mathfrak{A}$ *satisfies the equations*

$$\begin{aligned} f_i(x_1, \ldots, x_n) &= f_j(f_i(x_1, \ldots, x_n), \ldots, f_i(x_1, \ldots, x_n)) \\ &= f_i(f_j(x_1, \ldots, x_1), \ldots, f_j(x_n, \ldots, x_n)) \quad \text{for all } i, j \in I; \end{aligned} \tag{2}$$

(c_2) *There exists a retraction r of $\mathfrak{A}$ into $\mathfrak{A}$ such that $r(\mathfrak{A}) \in \boldsymbol{K}^*$ and for every $x_1, \ldots, x_n \in X$ we have*

$$f_i(x_1, \ldots, x_n) = f_i(r(x_1), \ldots, r(x_n)) \quad \text{for any } i \in I. \tag{3}$$

Proof. (c_1) $\Rightarrow$ (c_2). From (2) we obtain

$$\begin{aligned} f_i(x, \ldots, x) &= f_j(f_i(x, \ldots, x), \ldots, f_i(x, \ldots, x)) \\ &= f_i(f_j(x, \ldots, x), \ldots, f_j(x, \ldots, x)) \\ &= f_j(x, \ldots, x) \quad (i, j \in I). \end{aligned}$$

We put

$$r(x) = f_i(x, \ldots, x) \quad (i \in I). \tag{4}$$

It follows from (4) and (2) that r is a homomorphism of $\mathfrak{A}$ into $\mathfrak{A}$. Further, $r\big(r(x)\big) = f_i\big(f_i(x, \ldots, x), \ldots, f_i(x, \ldots, x)\big) = f_i(x, \ldots, x) = r(x)$, hence r is a retraction.

If $x \in r(\mathfrak{A})$, then $f_i(x, \ldots, x) = r(x) = x$ for any $i \in I$. Consequently, $r(\mathfrak{A})$ is a subalgebra of $\mathfrak{A}$ in which all axioms from U^* are fulfilled. Formula (3) follows easily from (4) and (2).

$(c_2) \Rightarrow (c_1)$. Consider an arbitrary equation $\Phi_1 = \Phi_2$ from U. Let Φ_k^* $(k = 1, 2)$ denote the expression obtained from Φ_k by replacing any variable x_j occurring in Φ_k by $r(x_j)$. Since $r(\mathfrak{A}) \in \boldsymbol{K}^*$, U is a subset of U^* and (3) holds in $\mathfrak{A}$, we have $\Phi_1 = \Phi_1^* = \Phi_2^* = \Phi_2$ in $\mathfrak{A}$. Hence $\mathfrak{A} \in \boldsymbol{K}$.

It remains to prove that (2) holds in $\mathfrak{A}$. If $x \in r(\mathfrak{A})$, then (1) holds and hence, by (3), we have in $\mathfrak{A}$

$$r(x) = f_i\big(r(x), \ldots, r(x)\big) = f_i(x, \ldots, x)$$

and

$$\begin{aligned}
&f_i(f_j(x_1, \ldots, x_1), \ldots, (f_j(x_n, \ldots, x_n)) = f_i\big(r(x_1), \ldots, r(x_n)\big) \\
&= r\big(f_i(x_1, \ldots, x_n)\big) = f_j\big(f_i(x_1, \ldots, x_n), \ldots, f_i(x_1, \ldots, x_n)\big) \\
&= f_j\Big(r\big(f_i(x_1, \ldots, x_n)\big), \ldots, r\big(f_i(x_1, \ldots, x_n)\big)\Big) = r(f_i(x_1, \ldots, x_n)) \\
&= f_i\big(r(x_1), \ldots, r(x_n)\big) = f_i(x_1, \ldots, x_n).
\end{aligned}$$

This completes the proof.

II. Examples. 1. Let us consider a generalized diagonal algebra, i.e. an algebra $\widehat{\mathfrak{D}} = \big(X; d(x_1, \ldots, x_n)\big)$, where the operation d satisfies the axiom $d\big(d(x_1^1, \ldots, x_n^1), \ldots, d(x_1^n, \ldots, x_n^n)\big) = d(x_1^1, \ldots, x_n^n)$. It is easy to verify that d satisfies (2). Thus $\widehat{\mathfrak{D}}$ is of the form described in (c_2), where $r(\widehat{\mathfrak{D}})$ is a diagonal algebra (see [1]).

2. Let $\widehat{\mathfrak{S}} = (X; +)$ be an algebra, where $+$ satisfies the axioms

$$x+y = y+x, \qquad (x+y)+z = x+(y+x), \qquad x+x+y = x+y.$$

We obtain such an algebra if X is the set of real numbers and $x+y = [\max(x, y)]$ ($[x]$ denotes the integral part of x). It is easy to verify that (2) holds in $\widehat{\mathfrak{S}}$, and hence $\widehat{\mathfrak{S}}$ is of the form described in (c_2), where $r(\widehat{\mathfrak{S}})$ is a semilattice, i.e. an algebra of sets with union.

3. Let $\widehat{\mathfrak{P}} = (X; \circ)$ be an algebra in which $\circ$ satisfies the axioms

$$(x \circ y) \circ z = x \circ (y \circ z), \qquad x \circ y \circ z = x \circ z \circ y, \qquad x \circ x \circ y = x \circ y \circ y = x \circ y.$$

It follows from theorem 1 that $\widehat{\mathfrak{P}}$ is of the form described in (c_2), where $r(\mathfrak{D})$ is a quasi-abelian semigroup (see [2] and [3]).

4. Let $\widehat{\mathfrak{Q}} = (X; +, \cdot)$ be an algebra, where $+$ and $\cdot$ satisfy the axioms

$$x+y = y+x, \quad x\cdot y = y\cdot x,$$
$$(x+y)+z = x+(y+z), \quad (x\cdot y)\cdot z = x\cdot(y\cdot z),$$
$$x+x+y = x+y, \quad x\cdot x\cdot y = x\cdot y,$$
$$x\cdot(y+z) = x\cdot y+x\cdot z, \quad x+y\cdot z = (x+y)\cdot(x+z),$$
$$x+x = x\cdot x.$$

It is easy to verify that (2) holds in $\widehat{\mathfrak{Q}}$, and hence $\widehat{\mathfrak{Q}}$ is of the form described in (c_2), where $r(\widehat{\mathfrak{Q}})$ is a distributive quasi-lattice (see [2]).

REFERENCES

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Reçu par la Rédaction le 7. 3. 1966