

SOME REMARKS ON CHROMATIC GRAPHS

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A graph G is said to be k -chromatic if its vertices can be split into k classes so that two vertices of the same class are not joined (by an edge) and such a splitting into $k - 1$ classes is not possible. The chromatic number will be denoted by $H(G)$. A graph is called complete if any two of its vertices are joined. Denote by $K(G)$ the number of vertices of the largest complete subgraph of G . The complementary graph \bar{G} of G is defined as follows: \bar{G} has the same vertices as G and two vertices are joined in \bar{G} if and only if they are not joined in G . A set of vertices of G is called independent if no two of them are joined. $I(G)$ denotes the greatest integer for which there is a set of $I(G)$ independent vertices of G . We evidently have

$$H(G) \geq K(G) = I(\bar{G}).$$

Throughout this paper G_n will denote a graph of n vertices, c_1, c_2, \dots will denote positive absolute constants. Vertices of G will be denoted by X_1, X_2, \dots , $G - X_1 - \dots - X_r$ will denote the graph G from which the vertices X_1, \dots, X_r and all the edges incident to them have been omitted. $G(X_1, \dots, X_m)$ denotes the subgraph of G spanned by the vertices X_1, \dots, X_m .

Tutte and Ungar (see [2]) and Zykov [10] were the first to show that for every l there is a graph G with $K(G) = 2$ (i.e. G contains no triangle) and $H(G) = l$. I proved [6] that for every n there is a G_n with $K(G_n) = 2$ and $H(G_n) > cn^{1/2}/\log n$. On the other hand, it easily follows from a result of Szekeres and myself [7] that if $K(G_n) = 2$, then $H(G_n) < c_1 n^{1/2}$.

It is an open and difficult problem to decide if for every n there is G_n with $K(G_n) = 2$ and $H(G_n) > c_2 n^{1/2}$ (**P 573**).

In the present note we prove the following

THEOREM. For every n there is a G_n satisfying

$$(1) \quad \frac{H(G_n)}{K(G_n)} > \frac{c_3 n}{(\log n)^2}.$$

But, on the other hand, for every G_n we have

$$(2) \quad \frac{H(G_n)}{K(G_n)} < \frac{c_4 n}{(\log n)^2}.$$

It seems likely that

$$(3) \quad \lim_{n \rightarrow \infty} \left(\max_{G_n} \left(\frac{H(G_n)}{K(G_n)} \right) / \frac{n}{(\log n)^2} \right) = C$$

exists (**P 574**), but I have not been able to prove (3). By the methods of this note it would be easy to prove that

$$\begin{aligned} \frac{(\log 2)^2}{4} &\leq \liminf_{n \rightarrow \infty} \left(\max_{G_n} \left(\frac{H(G_n)}{K(G_n)} \right) / \frac{n}{(\log n)^2} \right) \\ &\leq \limsup_{n \rightarrow \infty} \left(\max_{G_n} \left(\frac{H(G_n)}{K(G_n)} \right) / \frac{n}{(\log n)^2} \right) \leq (\log 2)^2 \end{aligned}$$

First we prove (1). It is known [5] that for every $n > n_0$ there is a graph G_n so that

$$(4) \quad K(G_n) \leq \frac{2 \log n}{\log 2}, \quad K(\bar{G}_n) \leq \frac{2 \log n}{\log 2}.$$

From the definition of the chromatic number we immediately obtain that for every graph G_n

$$(5) \quad H(G_n) \geq \frac{n}{I(G_n)} = \frac{n}{K(\bar{G}_n)}.$$

The proof of (5) is immediate since the vertices of G_n can be decomposed into $H(G_n)$ independent sets or $n \leq H(G_n) I(G_n)$.

(4) and (5) immediately imply (1).

To prove (2) we first prove two simple lemmas.

LEMMA 1. Let $\binom{u+v}{v} \geq n$. Then $uv > c_5(\log n)^2$.

Without loss of generality we can assume $u \geq v$. We then have

$$(6) \quad n \leq \binom{u+v}{v} \leq \binom{2u}{v} \leq \frac{(2u)^v}{v!} < \left(\frac{2eu}{v} \right)^v.$$

$uv > c_5(\log n)^2$ follows from (6) by a simple computation.

In fact, with somewhat more trouble we could prove the following stronger result:

If $\binom{u+v}{v} \geq n$, then

$$(7) \quad \min(uv) = \left\lceil \frac{t}{2} \right\rceil \left\lceil \frac{t+1}{2} \right\rceil,$$

where t is the smallest integer for which $\binom{t}{\lfloor t/2 \rfloor} \geq n$.

From (7) we obtain by a simple computation

$$uv \geq (1+o(1)) \left(\frac{\log n}{\log 4} \right)^2.$$

LEMMA 2. Let $n \geq m \geq N$. Assume that for every m and every subgraph $G(X_1, \dots, X_m)$ of G_n we have $I(G(X_1, \dots, X_m)) \geq l$. Then

$$H(G_n) \leq \frac{n}{l} + N.$$

Let $X_1^{(1)}, \dots, X_{n_1}^{(1)}$ be a maximal system of independent vertices of G_n ($n_1 = I(G_n)$). $X_1^{(2)}, \dots, X_{n_2}^{(2)}$ is a maximal system of independent vertices of $G_n - X_1^{(1)} - \dots - X_{n_1}^{(1)}$; $X_1^{(3)}, \dots, X_{n_3}^{(3)}$ — a maximal system of independent vertices of $G_n - X_1^{(1)} - \dots - X_{n_1}^{(1)} - X_1^{(2)} - \dots - X_{n_2}^{(2)}$ etc. We continue this process until

$$\sum_{i=1}^r n_i > n - N.$$

By our assumption $n_i \geq l$ for all i , $1 \leq i \leq r$. Thus $r \leq n/l$. The $X_j^{(i)}$, $1 \leq j \leq n_i$, $1 \leq i \leq r \leq n/l$, are the vertices of the i -th colour and the remaining fewer than N vertices all get different colours. Thus Lemma 2 is proved.

Now we are ready to prove (2). It is known [7] that

$$(8) \quad \binom{K(G_m) + K(\bar{G}_m) - 2}{K(G_m) - 1} \geq m.$$

Thus by Lemma 1

$$(9) \quad K(G_m) K(\bar{G}_m) > c_5 (\log m)^2.$$

From (9) we infer that if $m \geq n/(\log n)^2$, then for every subgraph $G(X_1, \dots, X_m)$ we have

$$(10) \quad I(G(X_1, \dots, X_m)) > \frac{c_5 (\log m)^2}{K(G(X_1, \dots, X_m))} > \frac{c_6 (\log n)^2}{K(G_n)}.$$

Now apply Lemma 2 with $N = n/(\log n)^2$, $l = c_6 (\log n)^2 / K(G_n)$. We then obtain

$$(11) \quad H(G_n) < \frac{nK(G_n)}{c_6 (\log n)^2} + \frac{n}{(\log n)^2}$$

and (2) immediately follows from (11). This completes the proof of our theorem.

Finally we state some more problems. Denote by $G(n; m)$ a graph of n vertices and m edges. It is easy to see that if $H(G(n; m)) = k$, then $m \geq \binom{k}{2}$ and if $m = \binom{k}{2}$, then $n = k$, i.e. we have the complete graph

of k vertices. Determine the smallest integer $f(l, k)$ for which there exists a graph G having $f(l, k)$ edges and satisfying $K(G) \leq l$, $H(G) = k$. As we just stated, $f(k, k) = \binom{k}{2}$ and Dirac showed that $f(k-1, k) = \binom{k+2}{2} - 5$ (see [3] and [4]). It seems to be very difficult to determine $f(2, k)$. The graph constructed in [6] shows that $f(2, k) < c_7 k^3 (\log k)^3$ and it is easy to see that $f(2, k) > c_8 k^3$. Perhaps

$$\lim_{k \rightarrow \infty} \frac{f(2, k)}{k^3} = C < \infty$$

exists (P 575).

Denote by $g(n; l)$ the smallest integer for which there is a $G(n; g(n; l))$ satisfying $I(G(n; g(n; l))) = l$. Turán [9] determined $g(n; l)$ for every n and l . Let $g(n; k, l)$ be the smallest integer for which there is a $G(n; g(n; k, l))$ satisfying

$$I(G(n; g(n; k, l))) = l, \quad K(G(n; g(n; k, l))) = k.$$

By (8) we must have $\binom{k+l-2}{k-1} \geq n$. I have not succeeded in determining $g(n; k, l)$ (P 576).

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