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THREE THEOREMS ON A CLASS OF NÖRLUND MEANS

BY

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Let

(A) $$a_0+a_1+\ldots+a_n+\ldots$$

be a given series with partial sums $\{s_n\}$. Further, let $\{p_n\}$ be a sequence of real numbers and let us write

$$P_n = p_0+p_1+\ldots+p_n; \qquad P_{-1} = 0.$$

We call $\{t_n\}$ the *Nörlund transform* of the sequence $\{s_n\}$, if

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k \qquad (P_n \neq 0).$$

Series (A) is said to be *summable* (N, p_n) to the value s, if $\lim t_n = s$,

Series (A) will be said to be *absolutely summable* (N, p_n) or briefly $|N, p_n|$-*summable* provided that $\{t_n\}$ is of bounded variation, i.e. if the series

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}|$$

is convergent.

Obviously, the $|N, p_n|$-summability implies (N, p_n)-summability, but not conversely (1).

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(1) This follows from the following example: Let $p_n = 1$ $(n = 0, 1, \ldots)$, and let $a_n = (-1)^n (n = 0, 1, 2, \ldots)$. Then

$$t_k = \begin{cases} 1/(k+1) & \text{for } k \text{ even}, \\ 0 & \text{for } k \text{ odd}. \end{cases}$$

Clearly t_k converges to zero, but

$$|t_k - t_{k-1}| = \begin{cases} 1/(k+1) & \text{for } k \text{ even}, \\ 1/k & \text{for } k \text{ odd}. \end{cases}$$

Hence, $\sum_{k=1}^{\infty} |t_k - t_{k-1}|$ diverges (see [1], p. 168-169).

The conditions for regularity of the method (N, p_n) are:

$$\text{(a)} \qquad \frac{p_n}{P_n} \to 0, \qquad\qquad \text{(b)} \qquad \sum_{v=0}^{n} |p_v| = O(P_n).$$

If $\{p_n\}$ is non-negative, then the condition

$$\lim_{n\to\infty} \frac{p_n}{P_n} = 0$$

is necessary and sufficient for the regularity of the method (N, p_n) of summation (see e.g. [5], p. 353).

In the present note we shall transfer some results due to Kogbetlianz [4] to the case of certain classes of Nörlund means. The most important of them are the following:

THEOREM A. *If a given series is* $|C, a|$*-summable* $(a > -1)$, *then it is also* $|C, a+\beta|$*-summable, with arbitrary* $\beta > 0$.

THEOREM B. *If two given series are summable* (C, a) $(a > -1)$ *to* A *and* (C, β) $(\beta > 0)$ *to* B, *respectively, and, moreover, if the first of them is* $|C, a|$*-summable, then their Cauchy-product is* $|C, a+\beta|$*-summable to* AB.

THEOREM C. *If two given series are* $|C, a|$*- and* $|C, \beta|$*-summable, respectively, then the series obtained by their multiplication is also absolutely summable, namely* $|C, a+\beta|$*-summable.*

The below presented theorems are similar to these of Kogbetlianz. However, they establish only a partial generalization of Kogbetlianz' results.

At first, we introduce the following classes of (N, p_n)-means:

A sequence $\{p_n\}$ will be said to *belong to the class* M^a, if

$$\text{(i)} \qquad 0 < p_{n+1} < p_n \quad \text{ or } \quad 0 < p_n < p_{n+1} \qquad (n = 0, 1, 2, \ldots),$$

$$\text{(ii)} \qquad p_0 + p_1 + \ldots + p_n = P_n \nearrow +\infty,$$

$$\text{(iii)} \qquad \lim_{n\to\infty} \frac{(n+1)p_n}{P_n} = a \qquad (a \geqslant 0).$$

In particular, if we assume the condition

$$\text{(jjj)} \qquad \lim_{n\to\infty} \frac{(n+1)(p_n - p_{n-1})}{p_n} = a-1 \qquad (a \geqslant 0),$$

instead of (iii), preserving conditions (i) and (ii), then $\{p_n\}$ will be said to *belong to the class* $\overline{M}^a$, $a \geqslant 0$.

Obviously, if $\{p_n\} \in \overline{M}^a$, then a fortiori $\{p_n\} \in M^a$, but not conversely (see [7], Lemma 1, p. 239-244).

In the sequel, we shall require the following lemmas:

LEMMA 1. *If* $\{p_n\} \epsilon M^a$, $a > 0$, *then there exists a constant* C *such that*

$$\sum_{n=k+1}^{\infty} \frac{1}{nP_n} < C \frac{1}{P_k} \qquad (k = 0, 1, \ldots).$$

The proof follows immediately from the hypothesis and from a lemma of Pati [10].

In order to formulate the next lemma, we introduce the following notation:

$$P_n = p_0 + p_1 + \ldots + p_n; \qquad Q_n = q_0 + q_1 + \ldots + q_n; \qquad r_n = \sum_{k=0}^{n} p_k q_{n-k};$$

$$R_n = r_0 + r_1 + \ldots + r_n.$$

LEMMA 2. *The sequence* $\{P_n Q_n / R_n\}$ *is bounded in each of the following four cases*:

a) $\{p_n\} \epsilon M^a$, $0 < a < 1$; $\{q_n\} \epsilon M^\beta$, $\beta > 0$;

b) $\{p_n\} \epsilon M^a$, $a > 0$; $\{q_n\} \epsilon M^\beta$, $0 < \beta < 1$;

c) $\{p_n\} \epsilon M^a$, $1 < a < 2$; $\{q_n\} \epsilon M^\beta$, $1 < \beta < 2$;

d) $p_{n+1} p_n \searrow$, $\{p_n\} \epsilon \overline{M}^a$, $a > 1$; $q_{n+1}/q_n \searrow$, $\{q_n\} \epsilon \overline{M}^\beta$, $\beta > 1$.

Proof. Let us note that if $p_n > 0$ and $0 < q_n \searrow$, then

$$R_n > q_n \sum_{k=0}^{n} P_k,$$

and, similarly, if $q_n > 0$ and $0 < p_n \searrow$, then

$$R_n > p_n \sum_{k=0}^{n} Q_k.$$

Basing on these inequalities, we easily prove a) and b). Taking $m = [n/2]$, we write in the case c)

$$\frac{P_m}{P_n} = 1 - \frac{p_{m+1} + \ldots + p_n}{P_n} > 1 - \frac{1}{2} \frac{(n+2) p_n}{P_n} > \frac{1}{2}\left(1 - \frac{a}{2}\right) > 0$$

for $1 < a < 2$ and n large enough. Hence $P_n = O(P_m)$. In a similar manner, we find that $Q_n = O(Q_m)$. Since

$$R_n > P_m Q_m$$

for $p_n > 0$ and $q_n > 0$, we have $P_n Q_n / R_n < P_n Q_n / P_m Q_m = O(1)$. In the case d) we find for $n = 2k$

$$\frac{P_n Q_n}{R_n} < \frac{P_{2k} Q_{2k}}{P_k Q_k} = O(1) \frac{p_{2k} q_{2k}}{p_k q_k}.$$

Since $p_{n+1}/p_n \searrow$, we have

$$\frac{p_{2k}}{p_k} = \frac{p_{2k}}{p_{2k-1}}\,\frac{p_{2k-1}}{p_{2k-2}}\cdots\frac{p_{k+1}}{p_k} < \left(\frac{p_{k+1}}{p_k}\right)^k = \left(1+\frac{p_{k+1}-p_k}{p_k}\right)^k$$
$$= \left[\left(1+\frac{p_{k+1}-p_k}{p_k}\right)^{p_k/(p_{k+1}-p_k)}\right]^{k(p_{k+1}-p_k)/p_k}$$
$$< e^{k(p_{k+1}-p_k)/p_k} = O(1).$$

Remark. For $x > 0$ and $y > 0$ we have $e > (1+x/y)^{y/x}$ (see [3], p. 216, (η)). Thus $p_{2k} = O(p_k)$. In a similar manner, we find $q_{2k} = O(q_k)$. If $n = 2k+1$, then, according to $p_{n+1}/p_n \searrow$ and $q_{n+1}/q_n \searrow$, we have

$$\frac{P_nQ_n}{R_n} < \frac{P_{2k+1}Q_{2k+1}}{P_kQ_k} < \left(1+\frac{p_{2k+1}}{p_{2k}}\,\frac{p_{2k}}{P_{2k}}\right)\left(1+\frac{q_{2k+1}}{q_{2k}}\,\frac{q_{2k}}{Q_{2k}}\right)\frac{P_{2k}Q_{2k}}{P_kQ_k}$$
$$< \left(1+\frac{p_1}{p_0}\right)\left(1+\frac{q_1}{q_0}\right)\frac{P_{2k}Q_{2k}}{P_kQ_k}$$

and this case reduces to the previous one. Therefore $P_nQ_n = O(R_n)$ in both cases.

Let us write

$$a_n = \frac{(n+1)p_n}{P_n}, \qquad \beta_n = \frac{(n+1)q_n}{Q_n}.$$

We define: a sequence $\{p_n\}$ will be said to *belong to the class* $BV\bar{M}^a$, $a > 0$, if $\{p_n\}\in\bar{M}^a$ and if $\{a_n\}$ is a sequence of bounded variation.

LEMMA 3. *Let* $\{p_n\}\in BV\bar{M}^a$, $a > 0$, $\{q_n\}\in BV\bar{M}^a$, $\beta > 0$. *If* $a \geqslant 2$ (*or* $\beta \geqslant 2$), *we suppose additionally that* $p_{n+1}/p_n \searrow$ (*or* $q_{n+1}/q_n \searrow$). *Then*

$$\{r_n\}\in BV\bar{M}^{a+\beta} \qquad \left(r_n = \sum_{k=0}^{n} p_kq_{n-k}\right) \text{ (}^2\text{)}.$$

Proof. Since $R_n > P_mQ_m \to +\infty$ as $n \to +\infty$, $\{r_n\}$ satisfies condition (ii) of the class $\bar{M}^{a+\beta}$ for arbitrary positive a and β. Now, we shall prove that

$$\lim_{n\to\infty}\frac{(n+2)r_n}{R_n} = a+\beta. \tag{3}$$

(2) In the case when $0 < p_n \searrow$ and $0 < q_n \searrow$ simultaneously, the relation $\{r_n\}\in\bar{M}^{a+\beta}$ will mean that $\{r_n\}$ satisfies condition (i) only for n large enough. It seems interesting to find out whether this relation holds without that restriction. A similar lemma, but less general, was already proved by the author of this note (see [7], Lemma 3, p. 248-252, and also the proof of Theorem 1, p. 257-258).

Taking

$$\gamma_n = \frac{(n+2)r_n}{R_n},$$

we have

$$\gamma_n = \frac{1}{R_n}\sum_{k=0}^{n} a_k P_k q_{n-k} + \frac{1}{R_n}\sum_{k=0}^{n} \beta_k Q_k p_{n-k}. \tag{4}$$

Therefore and by (4), in order to prove formula (3), it is sufficient to show that the methods $\|a_{nk}\|$ and $\|b_{nk}\|$ are regular, where

$$a_{nk} = \begin{cases} P_k q_{n-k}/R_n & \text{for} \quad 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise;} \end{cases}$$

$$b_{nk} = \begin{cases} Q_k p_{n-k}/R_n & \text{for} \quad 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

This follows by means of easy calculation, applying Lemma 2.

Next, we will prove that

$$\lim_{n\to\infty} \frac{(n+2)(r_n - r_{n-1})}{r_n} = a+\beta-1, \tag{5}$$

i.e. that the sequence $\{r_n\}$ satisfies condition (jjj) of the class $\overline{M}^{a+\beta}$. By substituting

$$\bar{a}_n = \frac{(n+1)(p_n - p_{n-1})}{p_n}, \quad \bar{\beta}_n = \frac{(n+1)(q_n - q_{n-1})}{q_n},$$

and multiplying both sides of the equality

$$r_n - r_{n-1} = \sum_{k=0}^{n} p_{n-k}(q_n - q_{k-1})$$

by $(n+2)^2$, easy calculation shows that

$$\frac{(n+2)^2(r_n - r_{n-1})}{R_n} = \frac{2}{R_n}\sum_{k=0}^{n} a_k \bar{\beta}_{n-k} P_k q_{n-k} + \frac{1}{R_n}\sum_{k=0}^{n} \beta_k \bar{\beta}_k p_{n-k} Q_k + \tag{6}$$
$$+ \frac{1}{R_n}\sum_{k=0}^{n}\left[\left(\frac{k}{k+1}\right)^2 a_k \bar{a}_k + \frac{2k+1}{k+1} a_k\right] P_k q_{n-k}.$$

Now, we will investigate the limit of the first term on the right-hand side of formula (6). Namely, we will show that

$$\lim_{n\to\infty} \frac{1}{R_n}\sum_{k=0}^{n} a_k \bar{\beta}_{n-k} P_k q_{n-k} = a(\beta-1). \tag{7}$$

We write $a_n = a+\varepsilon_n$, $\bar{\beta}_n = \beta-1+\varepsilon'_n$, where $\varepsilon_n\to 0$, $\varepsilon'_n \to 0$. Let N denote a natural number such that $|\varepsilon_k| < \varepsilon$ and $|\varepsilon'_k| < \varepsilon$ for $k > N$. Breaking up the sum appearing in (7), we get

$$\sum_{k=0}^{n} = \sum_{k=0}^{N} + \sum_{k=N+1}^{n-N} + \sum_{k=n-N+1}^{n}.$$

We estimate

$$\Big|\sum_{k=0}^{N} q_{n-k}P_k a_k \bar{\beta}_{n-k}\Big| \leqslant O(Q_n)(|1-\beta|+\varepsilon),$$

$$\sum_{k=N+1}^{n-N} = \sum_{k=N+1}^{n-N} (a+\varepsilon_k)(\beta-1+\varepsilon'_{n-k})q_{n-k}P_k$$

$$= a(\beta-1)R_n+\varepsilon O(R_n)+O(P_n)+O(Q_n),$$

$$\Big|\sum_{k=n-N+1}^{n}\Big| \leqslant O(P_n)(a+\varepsilon).$$

Hence

$$\sum_{k=1}^{N} + \sum_{N+1}^{n-N} + \sum_{n-N+1}^{n} = a(\beta-1)R_n+\varepsilon\, O(R_n)+O(P_n)+O(Q_n).$$

Let us note that in all cases of monotonicity of the sequences $\{p_n\}$ and $\{q_n\}$ the following relations hold:

$$P_n = o(R_n) \quad \text{and} \quad Q_n = o(R_n).$$

Taking into consideration these relations, we deduce the validity of formula (7). This, together with formula (6), gives the relation

$$\lim_{n\to\infty} \frac{(n+2)^2(r_n-r_{n-1})}{R_n} = (a+\beta)(a+\beta-1).$$

Combining the last relation with (3), we get relation (5).

Thus, we have proved that the sequence $\{r_n\}$ satisfies condition (jjj) of the class $\vec{M}^{a+\beta}$. It remains still to examine condition (i) of this class. If at least one of the sequences $\{p_n\}$ or $\{q_n\}$ is monotonically increasing, then the sequence $\{r_n\}$ satisfies condition (i). This follows immediately from the fact that the expressions

$$r_n-r_{n-1} = \sum_{k=0}^{n} q_{n-k}(p_k-p_{k-1}) \quad \text{and} \quad r_n-r_{n-1} = \sum_{k=0}^{n} p_{n-k}(q_k-q_{k-1})$$

are positive.

If both sequences $\{p_n\}$ and $\{q_n\}$ are monotonically decreasing, then, in view of (5), the sequence $\{r_n\}$ decreases if $\alpha+\beta-1<0$ and increases if $\alpha+\beta-1>0$ for n large enough.

It remains still to show that if the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are of bounded variation, then $\{\gamma_n\}$ is also of bounded variation. The proof is based on formula (4).

Writing

$$A_n=\frac{1}{R_n}\sum_{k=0}^{n}P_k q_{n-k}\alpha_k, \qquad B_n=\frac{1}{R_n}\sum_{k=0}^{n}Q_k p_{n-k}\beta_k,$$

we have

$$\begin{aligned}
&A_n-A_{n-1}\\
&=\frac{1}{R_n}\sum_{k=0}^{n}P_k q_{n-k}\alpha_k-\frac{R_{n-1}+r_n}{R_nR_{n-1}}\sum_{k=0}^{n}P_k q_{n-k}\alpha_{k-1}+\frac{1}{R_{n-1}}\sum_{k=0}^{n}p_k q_{n-k}\alpha_{k-1}\\
&=\frac{1}{R_n}\sum_{k=0}^{n}P_k q_{n-k}(\alpha_k-\alpha_{k-1})+\frac{1}{R_{n-1}}\sum_{k=0}^{n}p_k q_{n-k}\alpha_{k-1}-\\
&\qquad\qquad-\frac{r_n}{R_nR_{n-1}}\sum_{k=0}^{n}P_k q_{n-k}\alpha_{k-1}.
\end{aligned}$$

Applying the Abel transformation, we find the formula

$$\begin{aligned}
(8)\qquad A_n-A_{n-1}=&\frac{1}{R_n}\sum_{k=0}^{n}P_k q_{n-k}(\alpha_k-\alpha_{k-1})+\\
&+\frac{1}{R_{n-1}}\sum_{k=0}^{n-1}\Big(\sum_{v=0}^{k}p_v q_{n-v}\Big)(\alpha_{k-1}-\alpha_k)-\\
&-\frac{r_n}{R_nR_{n-1}}\sum_{k=0}^{n-1}\Big(\sum_{v=0}^{k}P_v q_{n-v}\Big)(\alpha_{k-1}-\alpha_k)\qquad(\alpha_{-1}=0).
\end{aligned}$$

If $0<q_n\searrow$, then

$$|A_n-A_{n-1}|<O(1)\left\{\frac{1}{R_n}\sum_{k=0}^{n}P_k q_{n-k}|\alpha_k-\alpha_{k-1}|+\frac{r_n}{R_nR_{n-1}}\sum_{k=0}^{n}R_k|\alpha_k-\alpha_{k-1}|\right\}.$$

If $0<q_n\nearrow$, then

$$|A_n-A_{n-1}|<O(1)\left\{\frac{q_n}{R_n}\sum_{k=0}^{n}P_k|\alpha_k-\alpha_{k-1}|+\frac{q_n}{nR_n}\sum_{k=1}^{n}kP_k|\alpha_k-\alpha_{k-1}|\right\}.$$

First, we get

$$\sum_{n=1}^{\infty}|A_n-A_{n-1}|<O(1)\Big\{\sum_{k=1}^{\infty}P_k|a_k-a_{k-1}|\Big(\frac{Q_k}{R_k}+\sum_{n=k}^{\infty}\frac{1}{nP_n}\Big)+$$

$$+\sum_{k=1}^{\infty}R_k|a_k-a_{k-1}|\sum_{n=k}^{\infty}\frac{r_n}{R_nR_{n-1}}\Big\}=O(1)\sum_{k=1}^{\infty}|a_k-a_{k-1}|<\infty.$$

Next, in view of relation $q_n/R_n=O(1/nP_n)$, we find that

$$\sum_{n=1}^{\infty}|A_n-A_{n-1}|$$

$$<O(1)\Big\{\sum_{k=1}^{\infty}P_k|a_k-a_{k-1}|\sum_{n=k}^{\infty}\frac{q_n}{R_n}+\sum_{k=1}^{\infty}kP_k|a_k-a_{k-1}|\sum_{n=k}^{\infty}\frac{q_n}{nR_n}\Big\}$$

$$=O(1)\sum_{k=1}^{\infty}|a_k-a_{k-1}|<\infty.$$

Finally, we have

$$\sum_{n=1}^{\infty}|A_n-A_{n-1}|<\infty. \tag{9}$$

In a similar manner, we find the formula

$$B_n-B_{n-1}=\frac{1}{R_n}\sum_{k=1}^{n}Q_kp_{n-k}(\beta_k-\beta_{k-1})+\frac{1}{R_{n-1}}\sum_{k=1}^{n-1}\Big(\sum_{v=0}^{k}q_vp_{n-v}\Big)\times \tag{8a}$$

$$\times(\beta_{k-1}-\beta_k)-\frac{r_n}{R_nR_{n-1}}\sum_{k=1}^{n-1}\Big(\sum_{v=0}^{k}Q_vp_{n-v}\Big)(\beta_{k-1}-\beta_k).$$

If $0<p_n\searrow$, then

$$|B_n-B_{n-1}|<O(1)\Big\{\frac{1}{R_n}\sum_{k=1}^{n}Q_kp_{n-k}|\beta_k-\beta_{k-1}|+\frac{r_n}{R_nR_{n-1}}\sum_{k=1}^{n}R_k|\beta_k-\beta_{k-1}|\Big\}.$$

If $0<p_n\nearrow$, then

$$|B_n-B_{n-1}|<O(1)\Big\{\frac{p_n}{R_n}\sum_{k=1}^{n}Q_k|\beta_k-\beta_{k-1}|+$$

$$+\frac{p_n}{R_n}\sum_{k=1}^{n}Q_k|\beta_k-\beta_{k-1}|+\frac{p_n}{nR_n}\sum_{k=1}^{n}kQ_k|\beta_k-\beta_{k-1}|\Big\}.$$

By a similar estimation as in the case of convergence of the series (9), we conclude that

$$\sum_{n=1}^{\infty} |B_n - B_{n-1}| < \infty. \tag{10}$$

Since $\gamma_n - \gamma_{n-1} = (A_n - A_{n-1}) + (B_n - B_{n-1})$, it follows from (9) and (10) that

$$\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty.$$

This together with (5) proves that $\{r_n\} \epsilon BV\overline{M}^{\alpha+\beta}$, for $\{r_n\}$ satisfies conditions (i) and (ii).

LEMMA 4. *Let the partial sums of a series $\sum u_n$ be bounded, let σ_n denote the n-th (N, p_n)-mean of this series, and let a_n have the same meaning as above. Then*

$$\left| nP_n |\sigma_n - \sigma_{n-1}| - \left| \sum_{k=1}^{n} kp_{n-k} u_k \right| \right| < M \sum_{k=1}^{n} P_k \left| a_k - a_{k+1} \right|. \tag{11}$$

Proof. We write $(s_{-1} = 0)$

$$\sigma_n - \sigma_{n-1} = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k - \frac{P_{n-1} + p_n}{P_n P_{n-1}} \sum_{k=0}^{n} p_{n-k} s_{k-1}$$
$$= \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} u_k - \frac{p_n}{P_n P_{n-1}} \sum_{k=0}^{n-1} p_{n-k-1} s_k.$$

Hence

$$\sigma_n - \sigma_{n-1} = \frac{1}{nP_n} \sum_{k=1}^{n} kp_{n-k} u_k + \frac{1}{nP_n} \sum_{k=0}^{n} a_{n-k} P_{n-k} u_k - \frac{p_n}{P_n} \sigma_{n-1},$$

where a_n has the same meaning as above. Applying the Abel transformation to the last sum we get

$$\frac{1}{nP_n} \sum_{k=1}^{n} kp_{n-k} u_k = \frac{P_{n-1}}{P_n} (\sigma_n - \sigma_{n-1}) + \frac{1}{nP_n} \sum_{k=1}^{n} \left(\sum_{v=0}^{k} P_v u_{n-v} \right) (a_k - a_{k+1}).$$

Hence the boundedness of partial sums of the series $\sum u_k$ implies Lemma 4.

In order to formulate the next theorems, we introduce the following notation:

$$\sigma_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k, \quad t_n = \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} s_k, \quad T_n = \frac{1}{R_n} \sum_{k=0}^{n} r_{n-k} s_k,$$

where

$$r_n = \sum_{k=0}^{n} p_k q_{n-k}, \qquad R_n = \sum_{k=0}^{n} P_k q_{n-k}, \qquad \nu_n = \sigma_n - \sigma_{n-1}, \qquad \mu_n = t_n - t_{n-1}.$$

THEOREM 1. *Let $\{p_n\} \in \overline{M}^{\alpha}$, $\alpha \geqslant 0$, and let $\{q_n\}$ be a convex or concave sequence such that $\{q_n\} \in \overline{M}^{\beta}$, with arbitrary $\beta > 0$. If $\sum u_n$ is $|N, p_n|$-summable, then it is $|N, r_n|$-summable, with $\{r_n\} \in \overline{M}^{\alpha+\beta}$.*

Proof. Since

$$\sum_{k=0}^{n} q_{n-k} \sum_{v=0}^{k} p_{k-v} s_v = \sum_{k=0}^{n} s_k \sum_{v=0}^{n-k} q_{n-k-v} p_v = \sum_{k=0}^{n} r_{n-k} s_k,$$

it suffices to prove the series $\sum_{n} |\sigma_n - \sigma_{n-1}|$ to be convergent if

$$\sum_{n} |T_n - T_{n-1}| < \infty.$$

Owing to formula (8), we can write

$$(13) \qquad T_n - T_{n-1} = \frac{1}{R_n} \sum_{k=0}^{n} P_k q_{n-k} \nu_k - \frac{1}{R_{n-1}} \sum_{k=0}^{n-1} \Big(\sum_{k=0}^{k} p_v q_{n-v} \Big) \nu_k + $$
$$+ \frac{r_n}{R_n R_{n-1}} \sum_{k=0}^{n-1} \Big(\sum_{v=0}^{k} P_v q_{n-v} \Big) \nu_k .$$

Decomposing each of the first two sums on the right-hand side into two sums from $k = 0$ to $k = m = [n/2]$ and from $k = m+1$ to $k = n$ or $k = n-1$, respectively, and applying the Abel transformation to the inner sum of the second term, we conclude that the absolute value of the expression obtained in this manner is less than

$$\frac{r_n}{R_n R_{n-1}} \sum_{k=0}^{m} P_k q_{n-k} |\nu_k| + \frac{1}{R_{n-1}} \sum_{k=0}^{m} |\nu_k| \sum_{v=0}^{k-1} |q_{n-v} - q_{n-v-1}| P_v +$$
$$+ O(1) \frac{1}{nR_n} \sum_{k=m+1}^{n} k P_k q_{n-k} |\nu_k| = A + B + C, \quad \text{say}.$$

In order to evaluate the expression B, we distinguish two cases: 1) $0 < q_n \searrow$ and 2) $0 < q_n \nearrow$.

In the first case the convexity of the sequence $\{q_n\}$ implies

$$0 < q_{n-v-1} - q_{n-v} \leqslant q_{n-v-2} - q_{n-v-1} \leqslant q_{n-m-2} - q_{n-m-1} \leqslant q_m - q_{m+1}.$$

Thus

$$B < \frac{m(q_m - q_{m+1})}{q_m} \frac{q_m}{mR_n} \sum_{k=0}^{m} |\nu_k| \sum_{v=0}^{k} P_v$$

$$= O(1) \frac{q_m}{mR_m} \sum_{k=0}^{m} |\nu_k| \sum_{v=0}^{k} P_v.$$

In the second case the convexity of the sequence $\{q_n\}$ implies

$$0 < q_{n-v} - q_{n-v-1} \leqslant q_{n-v+1} - q_{n-v} \leqslant q_n - q_{n-1}$$

and

$$B < O(1) \frac{q_n}{nR_n} \sum_{k=0}^{n} |\nu_k| \sum_{v=0}^{k} P_v.$$

However, the concavity of the sequence $\{q_n\}$ implies

$$B < \frac{1}{R_{n-1}} \sum_{k=0}^{m} (q_{n-k+1} - q_{n-k}) |\nu_k| \sum_{v=0}^{k} P_v$$

$$= O(1) \frac{1}{R_n} \sum_{k=0}^{m} \frac{q_{n-k}}{n-k} |\nu_k| \sum_{v=0}^{k} P_v = O(1) \frac{q_n}{nR_n} \sum_{k=0}^{n} |\nu_k| \sum_{v=0}^{k} P_v.$$

Now, we observe that in both cases

$$A + C = O(1) \frac{1}{nR_n} \sum_{k=1}^{n} kP_k q_{n-k} |\nu_k|.$$

Finally, in view of formula (13), we have

$$|T_n - T_{n-1}| = O(1) \frac{1}{nR_n} \sum_{k=1}^{n} kP_k q_{n-k} |\nu_k| + \frac{q_m}{mR_m} \sum_{k=1}^{m} kP_k |\nu_k| +$$

$$+ \frac{q_n}{nR_n} \sum_{k=1}^{n} kP_k |\nu_k| + \frac{r_n}{R_n R_{n-1}} \sum_{k=0}^{n} R_k |\nu_k|.$$

Hence we get

$$\sum_{n=1}^{\infty} |T_n - T_{n-1}| < O(1) \Big\{ \sum_{k=1}^{\infty} kP_k |\nu_k| \left(\frac{Q_k}{kR_k} + \sum_{n=k}^{\infty} \frac{1}{n^2 P_n} \right) +$$

$$+ \sum_{k=1}^{\infty} kP_k |\nu_k| \sum_{m=k}^{\infty} \frac{1}{m^2 P_m} + \sum_{k=1}^{\infty} kP_k |\nu_k| \sum_{n=k}^{\infty} \frac{1}{n^2 P_n} +$$

$$+ \sum_{k=0}^{\infty} R_k |\nu_k| \sum_{n=k}^{\infty} \frac{r_n}{R_n R_{n-1}} \Big\} = O(1) \sum_{k=0}^{\infty} |\nu_k| < \infty.$$

Therefore we have

$$\sum_{n=1}^{\infty} |T_n - T_{n-1}| < \infty,$$

which ends the proof of Theorem 1.

Before we formulate the next theorem, we introduce the following notation:

$$\sigma_n = \frac{1}{P_n} \sum_{k=0}^{n} P_{n-k} u_k, \qquad t_n = \frac{1}{Q_n} \sum_{k=0}^{n} Q_{n-k} v_k,$$

$$\sigma_n^* = \frac{1}{p_n} \sum_{k=0}^{n} p_{n-k} u_k, \qquad t_n^* = \frac{1}{q_n} \sum_{k=0}^{n} q_{n-k} v_k,$$

$$T_n = \frac{1}{R_n} \sum_{k=0}^{n} R_{n-k} w_k, \qquad T_n^* = \frac{1}{r_n} \sum_{k=0}^{n} r_{n-k} w_k, \qquad w_n = \sum_{k=0}^{n} u_k v_{n-k}.$$

We write (3)

$$J = \sum_n P_n x^n \sum_n q_n x^n \sum_n u_n x^n \sum_n v_n x^n$$

$$= \Big(\sum_n x^n \sum_{k=0}^{n} P_{n-k} u_k\Big)\Big(\sum_n x^n \sum_{k=0}^{n} q_{n-k} v_k\Big) = \Big(\sum_n P_n \sigma_n x^n\Big)\Big(\sum_n q_n t_n^* x^n\Big).$$

Hence

$$J = \sum_n x^n \sum_{k=0}^{n} q_k P_{n-k} t_k^* \sigma_{n-k}.$$

On the other hand,

$$J = \sum_n P_n x^n \sum_n q_n x^n \sum_n w_n x^n = \sum_n R_n x^n \sum_n w_n x^n.$$

Hence

$$J = \sum_n x^n \sum_{k=0}^{n} R_{n-k} w_k = \sum_n R_n T_n x^n.$$

(3) The above considered power series have a positive radius of convergence. This follows from the theorem stating that if (N, p_n) is regular and $\sum a_n = s\,(N, p_n)$, then the series $\sum a_n x^n$ has a positive radius of convergence and defines an analytic function $a(x)$ which is regular for $0 \leqslant x < 1$ and tends to s when $x \to 1$ through real values less than 1 (see [2], p. 65, Theorem 18), and from the fact that $p_n/P_n \to 0$ and $q_n/Q_n \to 0$, which implies that $P_{n-1}/P_n \to 1$ and $Q_{n-1}/Q_n \to 1$. Hence $P(x) = \sum P_n x^n$, $p(x) = \sum p_n x^n$, $Q(x) = \sum_n Q_n x^n$ and $q(x) = \sum_n q_n x^n$ are convergent for $|x| < 1$.

Comparing both power series for J we obtain the formula

$$T_n = \frac{1}{R_n} \sum_{k=0}^{n} q_k P_{n-k} t_k^* \sigma_{n-k}. \tag{14}$$

In a similar manner, we find the formula

$$T_n = \frac{1}{R_n} \sum_{k=0}^{n} p_k Q_{n-k} \sigma_k^* t_{n-k}. \tag{15}$$

Observe that

$$\sigma_n^* = \sigma_{n-1} + \frac{P_n}{p_n} v_n. \tag{16}$$

In fact,

$$\sigma_n - \sigma_{n-1} = \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n} (p_{n-k} P_n - p_n P_{n-k}) u_k$$

and

$$\sigma_n^* - \sigma_{n-1} = \frac{1}{p_n P_{n-1}} \sum_{k=0}^{n} (p_{n-k} P_n - p_n P_{n-k}) u_k.$$

Comparing the last equations we get formula (16).

In a similar way, we find the formula

$$t_n^* = t_{n-1} + \frac{Q_n}{q_n} \mu_n. \tag{17}$$

THEOREM 2. *Let $\{p_n\} \in \overline{M}^{\alpha}$, $\alpha > 0$, $\{q_n\} \in \overline{M}^{\beta}$, $\beta > 0$. If $\alpha \geqslant 2$ (or $\beta \geqslant 2$), we suppose additionally $p_{n+1}/p_n \searrow$ (or $q_{n+1}/q_n \searrow$). If the series $\sum_n u_n$ and $\sum_n v_n$ are summable (N, p_n) to A or (N, q_n) to B, respectively, and, moreover, if the series $\sum_n u_n$ is $|N, p_n|$-summable, then their product is summable (N, r_n) to AB, with $\{r_n\} \in \overline{M}^{\alpha+\beta}$.*

Proof. From formulae (15) and (16) it follows that

$$T_n = \frac{1}{R_n} \sum_{k=0}^{n} p_k Q_{n-k} \left(\sigma_{k-1} + \frac{P_k}{p_k} v_k\right) t_{n-k}$$

or

$$T_n = \frac{1}{R_n} \sum_{k=0}^{n} p_n Q_{n-k} \sigma_{k-1} t_{n-k} + \frac{1}{R_n} \sum_{k=0}^{n} P_k Q_{n-k} v_k t_{n-k}. \tag{18}$$

By a similar argument as in the proof of formula (7) we will prove that

$$\lim_{n\to\infty}\frac{1}{R_n}\sum_{k=1}^{n}p_kQ_{n-k}\sigma_{k-1}t_{n-k}=AB. \tag{19}$$

Let $\sigma_n=A+\varepsilon_n$, $t_n=B+\varepsilon_n'$ and let r denote a natural number such that $|\varepsilon_k|<\varepsilon$ and $|\varepsilon_k'|<\varepsilon$ for $k>r$. Writing

$$\sum_{k=1}^{n}=\sum_{k=1}^{r}+\sum_{k=r+1}^{n-r}+\sum_{k=n-r+1}^{n},$$

we have

$$\Big|\sum_{k=1}^{r}p_kQ_{n-k}\sigma_{k-1}t_{n-k}\Big|\leqslant O(1)\sum_{k=n-r}^{n-1}Q_k|t_k|<O(1)\sum_{k=n-r}^{n-1}Q_k(|B|+\varepsilon)=O(Q_n),$$

$$\sum_{k=r+1}^{n-r}p_kQ_{n-k}\sigma_{k-1}t_{n-k}=\sum_{k=r+1}^{n-r}p_kQ_{n-k}(A+\varepsilon_{k-1})(B+\varepsilon_{n-k}')$$

$$=AB\sum_{k=r+1}^{n-r}p_kQ_{n-k}+O(R_n)=AB\cdot R_n+\varepsilon O(R_n)+O(P_n)+O(Q_n),$$

$$\Big|\sum_{k=n-r+1}^{n}p_kQ_{n-k}\sigma_{k-1}t_{n-k}\Big|\leqslant O(1)\sum_{k=n-r+1}^{n}p_k(|A|+\varepsilon)=O(P_n).$$

Hence, because of the equations $Q_n=o(R_n)$ and $P_n=o(R_n)$, we get

$$\frac{1}{R_n}\sum_{k=1}^{n}p_kQ_{n-k}\sigma_{k-1}t_{n-k}=AB+O(\varepsilon)+O\left(\frac{P_n}{R_n}\right)+O\left(\frac{Q_n}{R_n}\right)$$

$$=AB+O(\varepsilon)+o(1),$$

whence (19) follows.

Let us denote by d_n the second term on the right-hand side of formula (18). Theorem 2 will be proved if we show that

$$\lim_{n\to\infty}d_n=0.$$

By hypothesis the series $\sum\limits_n u_n$ is $|N,p_n|$-summable. Hence the series $\sum\limits_{k=0}^{\infty}|v_k|$ is convergent. For the same reason the series $\sum\limits_n v_n$ is (N,q_n)-summable. Hence there exists a positive number M such that $|t_n|<M$ $(n=0,1,\ldots)$. Therefore for arbitrarily small positive number ε there exists a natural number N such that

$$\sum_{k=N+1}^{\infty}|v_k|<\varepsilon.$$

Since $Q_n = o(R_n)$ and $P_nQ_n = O(R_n)$, we have

$$|d_n| < M\left(\frac{Q_n}{R_n}\sum_{k=0}^{N} P_k|\nu_k| + \sum_{k=N+1}^{n}\frac{P_kQ_{n-k}}{R_n}|\nu_k|\right)$$

$$< O\left(\frac{Q_n}{R_n}\right) + O\left(\frac{P_nQ_n}{R_n}\right)\sum_{k=N+1}^{\infty}|\nu_k| = o(1)+O(\varepsilon) = O(\varepsilon).$$

THEOREM 3. *Let $\{p_n\}\epsilon BV\overline{M}^{\alpha}$, $\alpha > 0$, $\{q_n\}\epsilon BV\overline{M}^{\beta}$, $\beta > 0$. If $\alpha \geqslant 2$ (or $\beta \geqslant 2$), we suppose additionally that $p_{n+1}/p_n\searrow$ (or $q_{n+1}/q_n\searrow$). Further, let the series $\sum_n u_n$ and $\sum_n v_n$ and their product have bounded partial sums. If the series $\sum_n u_n$ and $\sum_n v_n$ are absolutely summable (N, p_n) and (N, q_n), respectively, then their product is also absolutely summable, namely $|N, r_n|$--summable, with $\{r_n\}\epsilon BV\overline{M}^{\alpha+\beta}\left(r_n = \sum_{k=0}^{n} p_kq_{n-k}\right)$.*

Proof. Let

$$w_n = \sum_{k=0}^{n} u_kv_{n-k}.$$

Since $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences, we can write

$$\sum_n x^n\sum_{k=0}^{n} kr_{n-k}w_k = \left(\sum_n nw_nx^n\right)\left(\sum_n r_nx^n\right) = \left(\sum_n w_nx^n\right)'\sum_n r_nx^n$$

$$= \left(\sum_n u_nx^n\sum_n v_nx^n\right)'\left(\sum_n p_nx^n\sum_n q_nx^n\right)$$

$$= \sum_n nu_nx^n\sum_n v_nx^n\sum_n p_nx^n\sum_n q_nx^n +$$

$$+\sum_n u_nx^n\sum_n nv_nx^n\sum_n p_nx^n\sum_n q_nx^n$$

$$= \left(\sum_n x^n\sum_{k=0}^{n} kp_{n-k}u_k\right)\left(\sum_n x^n\sum_{k=0}^{n} q_{n-k}v_k\right) +$$

$$+\left(\sum_n x^n\sum_{k=0}^{n} kq_{n-k}v_k\right)\left(\sum_n x^n\sum_{k=0}^{n} p_{n-k}u_k\right).$$

Hence, using the above introduced notation, we find the formula

$$\sum_{k=1}^{n} kr_{n-k}w_k = \sum_{k=1}^{n} q_{n-k}t^*_{n-k}\sum_{v=1}^{k} vp_{k-v}u_v + \sum_{k=1}^{n} p_{n-k}\sigma^*_{n-k}\sum_{v=1}^{k} vq_{k-v}v_v. \tag{20}$$

Owing to the estimations given in Lemma 4 and to the assumptions concerning the boundedness of partial sums of the investigated series, we see that

$$|T_n - T_{n-1}| \leqslant \frac{1}{nR_n} \sum_{k=1}^{n} kP_k q_{n-k} |\nu_k|\, |t^*_{n-k}| + \\ + \sum_{k=1}^{n} kQ_k p_{n-k} |\mu_k|\, |\sigma^*_{n-k}| + \sum_{k=1}^{n} q_{n-k} \sum_{v=1}^{k} P_v |a_v - a_{v-1}| + \\ + \sum_{k=1}^{n} p_{n-k} \sum_{v=1}^{k} Q_v |\beta_v - \beta_{v-1}| + \sum_{k=1}^{n} R_k |\gamma_k - \gamma_{k-1}|.$$

Since

$$\frac{P_n}{nR_n} = O\left(\frac{1}{nQ_n}\right) \quad \text{and} \quad \frac{Q_n}{nR_n} = O\left(\frac{1}{nP_n}\right),$$

we find, in view of formulas (16) and (17), that

$$\sum_{n=1}^{\infty} |T_n - T_{n-1}| < O(1) \Bigg\{ \sum_{n=1}^{\infty} \frac{1}{nR_n} \sum_{k=1}^{n} kP_k q_{n-k} |\nu_k| \left| t_{n-k-1} + \frac{Q_{n-k}}{q_{n-k}} \mu_{n-k} \right| + \\ + \sum_{n=1}^{\infty} \frac{1}{nR_n} \sum_{k=1}^{n} kQ_k p_{n-k} |\mu_k| \left| \sigma_{n-k-1} + \frac{P_{n-k}}{p_{n-k}} \nu_{n-k} \right| + \\ + \sum_{k=1}^{\infty} P_k |a_k - a_{k-1}| \sum_{n=k}^{\infty} \frac{1}{nP_n} + \sum_{k=1}^{\infty} Q_k |\beta_k - \beta_{k-1}| \sum_{n=k}^{\infty} \frac{1}{nQ_n} + \\ + \sum_{k=1}^{\infty} R_k |\gamma_k - \gamma_{k-1}| \sum_{n=k}^{\infty} \frac{1}{nR_n} \Bigg\}.$$

By hypothesis the series $\sum\limits_n u_n$ and $\sum\limits_n v_n$ are absolutely summable, thence the series

$$\sum_{n=1}^{\infty} (\sigma_n - \sigma_{n-1}) \quad \text{and} \quad \sum_{n=1}^{\infty} (t_n - t_{n-1})$$

are convergent. Therefore there exist limits $\lim\limits_{n\to\infty} \sigma_n$ and $\lim\limits_{n\to\infty} t_n$. Hence there exists a positive number M such that

$$|\sigma_n| < M \quad \text{and} \quad |t_n| < M \quad (n = 1, 2, \ldots).$$

Consequently,

$$\sum_{n=1}^{\infty}|T_n-T_{n-1}| < O(1)\Bigg\{\sum_{k=1}^{\infty}kP_k|\nu_k|\left(\frac{Q_k}{kR_k}+\sum_{n=k}^{\infty}\frac{q_n}{nR_n}\right)+\sum_{n=1}^{\infty}\frac{P_nQ_n}{R_n}\sum_{k=1}^{n}|\nu_k|\times$$

$$\times|\mu_{n-k}|+\sum_{k=1}^{\infty}kQ_k|\mu_k|\left(\frac{P_k}{kR_k}+\sum_{n=k}^{\infty}\frac{p_n}{nR_n}\right)+$$

$$+\sum_{n=1}^{\infty}\frac{P_nQ_n}{R_n}\sum_{k=1}^{n}|\mu_k||\nu_{n-k}|+\sum_{k=1}^{\infty}|\alpha_k-\alpha_{k-1}|+$$

$$+\sum_{k=1}^{\infty}|\beta_k-\beta_{k-1}|+\sum_{k=1}^{\infty}|\gamma_k-\gamma_{k-1}|\Bigg\}.$$

Since

$$\sum_{k=1}^{\infty}|\nu_k| \text{ and } \sum_{k=1}^{\infty}|\mu_k|$$

are convergent, their Cauchy product, with the n-th term

$$\sum_{k=1}^{n}|\nu_k|\,|\mu_{n-k}|,$$

is also convergent. In consequence, the second and the fourth series on the right-hand side of the last inequality are convergent. Both remaining series occurring in this inequality are also convergent, because

$$P_nQ_n = O(R_n), \quad \frac{q_n}{nR_n} = O\left(\frac{1}{n^2P_n}\right) \quad \text{and} \quad \frac{p_n}{nR_n} = O\left(\frac{1}{n^2Q_n}\right).$$

Finally, we have

$$\sum_{n=1}^{\infty}|T_n-T_{n-1}| < \infty$$

which ends the proof of Theorem 3.

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