COLLOQUIUM MATHEMATICUM

XVI DÉDIÉ À M. FRANCISZEK LEJA 1967

ALMOST PERIODIC EXTENSIONS OF FUNCTIONS, III

BY

S. HARTMAN AND C. RYLL-NARDZEWSKI (WROCŁAW)

We recall the definition of $\boldsymbol{I}$-sets and $\boldsymbol{I}_0$-sets (see [1] and [2]) in locally compact abelian groups (LCA groups):

A is an *$\boldsymbol{I}$-set* if every bounded real or complex valued function on A which is uniformly continuous on A with respect to the uniform group structure in G can be extended to an almost periodic function over G.

To define the (stronger) property $\boldsymbol{I}_0$ we simply omit the assumption of uniform continuity of the function which has to be extended.

In [2], we proved that if E is an $\boldsymbol{I}_0$-set in a separable non-discrete LCA group G and $\tilde{E}$ means the (weak) closure of E in the Bohr compactification $\tilde{G}$ of G, then $\mu(\tilde{E}) = 0$, μ denoting the Haar measure in $\tilde{G}$. This was an answer to the first question in problem P 452 which was raised in [1] for $G = R$ (the real line) and reformulated in [2] for arbitrary LCA groups. This result was extended and strengthened by Kahane [3], in particular, we know at present that $\mu(\tilde{E}) = 0$ for $E \in \boldsymbol{I}_0$, whatever be the LCA group G. Here we intend to answer the second question in P 452 for $G = R$ by proving the following

THEOREM. *If $E \subset R$ is an $\boldsymbol{I}$-set, then $\mu(\tilde{E}) = 0$.*

Denote by G_d a group G with discrete topology, and by $\hat{G}$ or $G^{\wedge}$ the character group of G.

LEMMA. *If H is a dense subgroup of R and φ an isomorphic continuous imbedding of R into $\tilde{R} = (R_d)^{\wedge}$, then $\varphi(R) \cap (R_d/H_d)^{\wedge} = (0)$.*

In fact, $(R_d/H_d)^{\wedge}$ is the annihilator of H_d and so, in view of the den sity of H, it does not contain any non-trivial continuous character of R, thus any non-zero element of $\varphi(R)$.

Let us observe that $\tilde{R}$ is the cartesian product of $2^{\aleph_0}$ copies of the (solenoidal) group $\hat{S}$, $S = S_d$ denoting the group of rationals. We therefore can regard $\tilde{R}$ as the product of two compact groups: the metric group $\hat{S}$ (one copy) and a non-metric complementary factor T (isomorphic to the group $\tilde{R}$ itself). Whatever be such splitting, the lemma gives

$\varphi(R) \cap \hat{S} = (0)$. To see this we may take for H a summand complementary to S, so that $R_d = S_d + H_d$ and $S_d = R_d/H_d$.

We now proceed to the proof of the Theorem. It is obvious that there is an $\boldsymbol{I}_0$-set $A \subset E$ and a compact set (closed interval) K such that $A+K \supset E$. We shall prove that $\mu\big((A+K)^{\sim}\big) = \mu(\tilde{A}+K) = 0$. (Here we identify every subset of R with its φ-image.) Let μ_1 be the Haar measure in $\hat{S}$ and μ_2 that in T. Then $\mu = \mu_1 \times \mu_2$ is the Haar measure in $\tilde{R}$. It is enough to prove that for each $y \in T$ we have $\mu_1(\{x\colon (x, y) \in \tilde{A}+K\}) = 0$ and to apply Fubini's theorem. Actually, we will show that for each $y \in T$ there is only a finite number of x's such that $(x, y) \in \tilde{A}+K$. In fact, all such x's make a compact metric set in $\hat{S}$, hence, if there were infinitely many of them for some y, there would be a convergent sequence $\{(x_n, y)\}$ of distinct elements in $\tilde{A}+K$. Putting $x_n = \xi_n' + \xi_n''$ and $y = \eta_n' + \eta_n''$ with $(\xi_n', \eta_n') \in \tilde{A}$ and $(\xi_n'', \eta_n'') \in K$ we could select a convergent subsequence $\{(\xi_{n_k}'', \eta_{n_k}'')\}$, owing to the compactness and metrizability of K. Then $(\xi_{n_k}', \eta_{n_k}')$ would equally converge. But since $\tilde{A}$ is homeomorphic to the Čech compactification $\beta(N)$ of the set of integers (see [1]), it does not contain any non-trivial convergent sequence. So we would have $\xi_{n_k}' = \text{const}$, $\eta_{n_k}' = \text{const}$ and $\eta_{n_k}'' = y - \eta_{n_k}' = \text{const} = \eta$ for $k > k_0$. Hence, for those k, $\{(\xi_{n_k}'', \eta)\}$ would consist of distinct elements of K. This, however, is impossible, because the "axis" $\hat{S}$ having no non-zero element in common with $R \supset K$, no set $\{(x, y)\}$ with a fixed y contains more than one point from K. The proof is thus complete.

The Theorem can be reformulated in the intrinsic language of R as follows:

Is E an $\boldsymbol{I}$-set in R, then, for every $\varepsilon > 0$, there is an almost periodic function on R, equal 1 *on E, non-negative and of mean value less than ε*.

REFERENCES

[1] S. Hartman and C. Ryll-Nardzewski, *Almost periodic extensions of functions*, Colloquium Mathematicum 12 (1964), p. 23-39.

[2] — *Almost periodic extensions of functions II*, ibidem 15 (1966), p. 79-86.

[3] J.-P. Kahane, *Ensembles de Ryll-Nardzewski et ensembles de Helson*, ibidem, p. 87-92.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 28. 12. 1965