COLLOQUIUM MATHEMATICUM

XVI DÉDIÉ À M. FRANCISZEK LEJA 1967

A NOTE ON THE DIFFERENTIABILITY OF INTEGRALS

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1. This is a postscript to the paper [1] written more than 30 years ago and concerning the differentiability of integrals. The main result of the paper was that if $f(x) = f(x_1, x_2, \ldots, x_n)$ is in the class $L(\log^+ L)^{n-1}$ in the n-dimensional unit cube

$$0 \leqslant x_j \leqslant 1, \quad j = 1, 2, \ldots, n, \tag{Q_0}$$

then the indefinite integral

$$F(E) = \int_E f dx$$

is strongly differentiable almost everywhere in Q_0; more specifically, if I denotes any n-dimensional interval (with sides parallel to the axes) which contains x, then almost everywhere in Q_0 we have

$$\frac{F(I)}{|I|} \to f(x) \tag{1.1}$$

as I shrinks to x.

The paper also contains a proof of a somewhat more general result, namely, if instead of assuming that all the sides of I tend to 0 independently of one another, we impose the condition that r of them are kept equal ($1 \leqslant r \leqslant n$), then the condition $f \in L(\log^+ L)^{n-1}$ (which was shown in [2] to be best possible) can be replaced by $f \in L(\log^+ L)^{n-r}$.

In this note we give a generalization of the latter result; this generalization may be justified by the increasing significance of theorems on the differentiability of integrals for the theory of singular integrals and, indirectly, partial differential equations. Moreover, the argument here is somewhat different from that of [1], and the new argument may be of some interest.

THEOREM 1. *Let $1 \leqslant s \leqslant n$ and consider only intervals I whose sides have no more than s different sizes. Then if $f \in L(\log^+ L)^{s-1}$ in Q_0, we have* (1.1) *almost everywhere in Q_0 as I shrinks to x.*

It is clear that this result contains the preceding one as a special case.

2. In what follows we write $\|f\|_p$ for $\big(\int\limits_{Q_0} |f|^p dx\big)^{1/p}$.

LEMMA 1. *Let* $f \in L(Q_0)$ *and*

$$\bar{f}(x) = \operatorname*{Sup}_{Q \supset x} \frac{1}{|Q|} \int\limits_Q |f(y)|\,dy, \tag{2.1}$$

where $Q \subset Q_0$ *is an* n*-dimensional cube. Then if* $\xi > 0$ *and* $E(\xi) = \{x \in Q_0\colon \bar{f}(x) > \xi\}$, *we have*

$$|E(\xi)| \leqslant A \frac{\|f\|_1}{\xi}, \tag{2.2}$$

where A *depends on* n *only.*

This is immediate: each $x \in E(\xi)$ is contained in a cube $Q \subset Q_0$ such that $\int\limits_Q |f|\,dy > |Q|\,\xi$, and by the elementary version of Vitali's covering lemma we can find a finite number of such Q's non-overlapping and of total measure $> |E(\xi)|/A$; adding the inequalities for these Q's we are led to (2.2).

LEMMA 2. *If* $f \in L^p(Q_0)$, $1 < p < \infty$, *and* $\bar{f}(x)$ *is given by* (2.1), *then* $\bar{f} \in L^p$ *and*

$$\|\bar{f}\|_p \leqslant \frac{A}{p-1} \|f\|_p, \tag{2.3}$$

where the constant A *depends only on* n *and* p, *but is bounded over any finite range of* p (¹)

Lemma 2 is a corollary of Lemma 1 if one uses Marcinkiewicz's theorem on the interpolation of operations (see [4_{II}], p. 111). For the operation $\bar{f} = Tf$ is sublinear (i.e., $|T(f_1+f_2)| \leqslant |Tf_1| + |Tf_2|$) and by Lemma 1 it is of weak type (1,1). Since it is clearly of type (∞, ∞), it is also of type (p, p), $1 < p < \infty$, and the best constant K_p in the inequality $\|\bar{f}\|_p \leqslant K_p \|f\|_p$ is $O\{1/(p-1)\}$ (see [4_{II}], p. 116, equation (4.21)).

LEMMA 3. *Let* $f \in L^p(Q_0)$, $1 < p < \infty$, $1 \leqslant s \leqslant n$, *and let*

$$f^*(x) = \operatorname*{Sup}_{I \supset x} \frac{1}{|I|} \int\limits_I |f(y)|\,dy, \tag{2.4}$$

(¹) If we replace $A/(p-1)$ by $Ap/(p-1)$ on the right of (2.3), the new A will be independent of p, but this point is without importance to us since we are only interested in the values of p close to 1.

Of course both Lemma 1 and Lemma 2 can be found in [3], but we could not refer the reader to any proof of (2.3) with the coefficient $A/(p-1)$ on the right (which is important for us).

where I is an n-dimensional interval whose sides have s (or less) different sizes. Then $f^* \in L^p$ and

$$\|f^*\|_p \leqslant \frac{A}{(p-1)^s}\|f\|_p, \tag{2.5}$$

where A depends on n and p, but is bounded over any finite range of p.

We may assume that $f \geqslant 0$. The proof is by induction in s. Suppose that $s > 1$ and that Lemma 3 is valid if s is replaced by $s-1$. We may also assume that the various sizes $h_1, h_2, \ldots, h_s$ of the sides of I correspond to fixed groups of coordinates $x_1, x_2, \ldots, x_n$. Write $x = (x', x'')$, where x' is the totality of coordinates corresponding to h_1, and x'' the remaining coordinates. Correspondingly, we may write $Q_0 = Q_0' \times Q_0''$, $I = Q' \times I''$, where Q' is a cube in the space of x' and I'' an interval (with $s-1$ different dimensions) in the space of x''. The condition $I \supset x$ means $x' \in Q'$, $x'' \in I''$ and if we set

$$f^*(x' \mid x'') = \operatorname*{Sup}_{I'' \supset x''} \frac{1}{|I''|} \int_{I''} f(x', y'')\,dy'', \tag{2.6}$$

$$f^{**}(x' \mid x'') = \operatorname*{Sup}_{Q' \supset x'} \frac{1}{|Q'|} \int_{Q'} f^*(y' \mid x'')\,dy', \tag{2.7}$$

then from

$$\frac{1}{|I|}\int_I f(y)\,dy = \frac{1}{|Q'|}\int_{Q'} \left[\frac{1}{|I''|}\int_{I''} f(y', y'')\,dy''\right] dy' \tag{2.8}$$

we deduce that

$$f^*(x) \leqslant f^{**}(x' \mid x'').$$

Hence using successively Lemma 2 and Lemma 3 with $s-1$ instead of s we obtain

$$\begin{aligned}
\int_{Q_0} \{f^*(x)\}^p\,dx &\leqslant \int_{Q_0''}\left\{\int_{Q_0'} [f^{**}(x' \mid x'')]^p\,dx'\right\}dx'' \\
&\leqslant \left(\frac{A}{p-1}\right)^p \int_{Q_0''}\left\{\int_{Q_0'} [f^*(x' \mid x'')]^p\,dx'\right\}dx'' \\
&= \left(\frac{A}{p-1}\right)^p \int_{Q_0'}\left\{\int_{Q_0''} f^*(x' \mid x'')^p\,dx''\right\}dx' \\
&\leqslant \left(\frac{A}{p-1}\right)^p \left(\frac{A}{p-1}\right)^{p(s-1)} \int_{Q_0'}\left[\int_{Q_0''} f^p(x', x'')\,dx''\right]dx' \\
&= \left(\frac{A}{p-1}\right)^{ps} \|f\|_p^p,
\end{aligned}$$

and Lemma 3 is established.

3. LEMMA 4. *Suppose $f \in L(\log^+ L)^s(Q_0)$, $1 \leqslant s \leqslant n$, and let $f^*(x)$ be defined as in Lemma 3. Then $f^* \in L$ and*

$$\int_{Q_0} f^* dx \leqslant A \int_{Q_0} |f| (\log^+ |f|)^s dx + B, \tag{3.1}$$

where A and B depend on n only (²).

We may assume that $f \geqslant 0$. For each $k = 1, 2, \ldots$ we denote by $f_k(x)$ the function equal to $f(x)$ wherever $2^{k-1} \leqslant f(x) < 2^k$ and equal to 0 elsewhere; by $f_0(x)$ we denote the function equal to $f(x)$ wherever $f < 1$ and equal to 0 elsewhere. Hence $f = \sum f_k$ and clearly $f^* \leqslant \sum f_k^*$. Applying Lemma 3 to f_k we have

$$\int_{Q_0} f_k^* \, dx \leqslant \|f_k^*\|_p \leqslant \frac{A}{(p-1)^s} \|f_k\|_p \leqslant \frac{A}{(p-1)^s} e_k^{1/p} 2^k,$$

where e_k denotes the measure of the set of points where $f_k \neq 0$. If we set $p = 1+1/(k+1)$ for $k = 0, 1, \ldots$, we obtain

$$\int_{Q_0} f^* dx \leqslant \sum_k \int_{Q_0} f_k^* dx \leqslant A \, \Sigma (k+1)^s 2^k e_k^{(k+1)/(k+2)}.$$

Observe now that those k for which $e_k \leqslant 3^{-k}$ contribute a finite sum, depending on n only, to the last sum while for the remaining k we have $e_k^{(k+1)/(k+2)} \leqslant A e_k$. It follows that

$$\int_{Q_0} f^* dx \leqslant A \sum_0^\infty (k+1)^s 2^k e_k + B \leqslant A \int_{Q_0} f (\log^+ f)^s dx + B,$$

which completes the proof of the lemma.

LEMMA 5. *Let $1 \leqslant s \leqslant n$ and let*

$$f_*(x) = \limsup_{I \supset x} \frac{1}{|I|} \int_I |f(y)| \, dy, \tag{3.2}$$

where the interval I shrinking to x has only s distinct dimensions. Then, if $f \in L(\log^+ L)^{s-1}$, the function f_* is integrable and

$$\int_{Q_0} f_* dx \leqslant A \int_{Q_0} |f| (\log^+ |f|)^{s-1} dx + B, \tag{3.3}$$

where A and B depend on n only.

(²) This is actually a special case of Theorem (4. 41ii) due to Yano in Chapter XII of [4_{II}] except for the unfortunate fact that the latter was formulated for linear operations while our operation $f^* = Tf$ is sublinear. Rather than leaving to the reader the task of verifying that the proof is actually valid for sublinear operations we repeat the argument in the special case that interests us.

We may suppose that $f \geqslant 0$. Under the hypotheses of the lemma, $f(x) = f(x', x'')$, *qua* function of x'', is in the class $L(\log^+ L)^{s-1}$ for almost every $x' \epsilon Q_0'$. For any such x' consider the function $f^*(x' \mid x'')$ defined by the equation (2.6). By Lemma 4, with $s-1$ in place of s,

$$\int_{Q_0''} f^*(x' \mid x'')\, dx'' \leqslant A \int_{Q_0''} f(x', x'')\{\log^+ f(x', x'')\}^{s-1} dx'' + B,$$

and integrating this inequality over $x' \epsilon Q_0'$, we see that

$$\int_{Q_0} f^*(x' \mid x'')\, dx'\, dx'' \leqslant A \int_{Q_0} f(x)\{\log^+ f(x)\}^{s-1} dx + B \tag{3.4}$$

It follows that for almost every $x'' \epsilon Q_0''$ the function $f^*(x' \mid x'')$ is integrable in x' over Q_0'. Let us fix an x'' for which this occurs. If $x = (x', x'') \subset I = Q' \times I''$, then, by (2.8),

$$\frac{1}{|I|} \int_I f(y)\, dy \leqslant \frac{1}{|Q'|} \int_{Q'} f^*(y' \mid x'')\, dy',$$

so that

$$f_*(x) \leqslant f^*(x' \mid x'') \tag{3.5}$$

for almost all x', by Lebesgue's theorem on the differentiability of integrals. It follows that (3.5) holds at almost all points of the cube Q_0, and (3.4) leads to (3.3). This concludes the proof of Lemma 5.

4. The proof of Theorem 1 is now easy. We apply (3.2) to the function kf, where k is a positive constant, and obtain

$$\int_{Q_0} f_*\, dx \leqslant A \int |f| \{\log^+ |kf|\}^{s-1} dx + \frac{B}{k}.$$

If k is sufficiently large, the term B/k is arbitrarily small. A decomposition of f into a sum of two functions $f_1 + f_2$, where f_1 is continuous and the integral of $f_2(\log^+ |kf_2|)^{s-1}$ small gives the theorem in a routine way.

5. We conclude the note by remarks which do not require detailed proofs.

Revert to Lemma 1. Its proof is based on the elementary form of Vitali's covering lemma. But the latter is valid in a more general case. For suppose that we have a system of n functions

$$\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t), \qquad 0 \leqslant t \leqslant 1, \tag{5.1}$$

which are increasing, vanishing and continuous at $t = 0$, and positive for $0 < t \leqslant 1$. Suppose that each point of a set E of finite measure (or even only of finite outer measure, for we need not assume that E is measurable) is included in an interval I whose sides have lengths $\varphi_1(t), \ldots, \varphi_n(t)$ for some t in (0,1). Then we can find a finite number of such intervals

I, non-overlapping and of total measure $\geqslant |E|/A$, where A depends on n, but not on the functions $\varphi_1, \ldots, \varphi_n$ (see [1] or [4_{II}], p. 309, Lemma (3.2)). It follows that Lemma 1 holds if by $Q = Q(t)$ in (2.1) we mean not cubes but n-dimensional intervals of the form just described. Hence, as the proof shows, Lemma 2 is also valid in this case. This in turn extends Lemma 3 to the case when the sides of I can be split into s disjoint groups, the sides of the j-th group being given by n_j functions from (5.1) ($n_1 + n_2 + \ldots + n_s = n$) depending on the variable t_j; the variables $t_1, t_2, \ldots, t_s$ are supposed to be independent of one another. We will call for brevity such I's *intervals of type* (φ, s). Lemmas 3 and 4 hold for such intervals, and so does Lemma 5 if the interval I in (3.2) is of the form (φ, s) (which now means that each of the variables $t_1, t_2, \ldots, t_s$ tends to 0) and we arrive at the following generalization of Theorem 1:

THEOREM 2. *If* $f \in L(\log^+ L)^{s-1}$, $1 \leqslant s \leqslant n$, *then we have* (1.1) *almost everywhere in* Q_0, *provided the intervals* I *are of the form* (φ, s).

It is obvious that this immediately leads to a somewhat stronger conclusion in which instead of (1.1) we have the relation

$$|I|^{-1} \int_I |f(y) - f(x)|\, dy \to 0 .$$

Also the condition that $I \supset x$ can be replaced by a somewhat weaker one: I can be enclosed in an interval $\bar{I}$, $\bar{I} \supset x$, $\bar{I}$ is of type (φ, s) and $|\bar{I}|/|I|$ is bounded by a constant (depending possibly on x).

REFERENCES

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Reçu par la Rédaction le 13. 12. 1965