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ISOMORPHIC EMBEDDINGS OF FREE PRODUCTS OF COMPACT GROUPS

BY

A. HULANICKI (WROCŁAW)

0. In a recent paper [1] K. Golema has noticed that the category of compact groups is closed with respect to taking free products in the category. The aim of this note* is to give some more information about the structure of such free products. We show that the free product in the category of compact groups is the Bohr compactification of the algebraic free product equipped with a natural topology(1). We shall prove this via a theorem of its own interest.

PROPOSITION 1. *The algebraic free product* $\mathop{*}\limits_{\alpha\in A} G_\alpha$ *of a family of compact groups* $\{G_\alpha\}_{\alpha\in A}$ *is isomorphically embeddable into a compact group by an isomorphism* φ *continuous on each of the groups* G_α, $\alpha\in A$.

1. Free products in the category of compact groups. Let Γ be the category of compact groups and let $\{G_\alpha\}_{\alpha\in A}$ be a family of groups in Γ. The free product of the groups $\{G_\alpha\}_{\alpha\in A}$ in the category Γ is a unique group in Γ, denoted by $\mathop{\bar{*}}\limits_{\alpha\in A} G_\alpha$, together with a system of monomorphisms

$$\iota_\alpha\colon G_\alpha \to \mathop{\bar{*}}\limits_{\alpha\in A} G_\alpha$$

such that for any H in Γ and any system of homomorphisms $h_\alpha\colon G_\alpha\to H$ there exists a unique homorphism $h\colon \mathop{\bar{*}}\limits_{\alpha\in A} G_\alpha \to H$ such that $h_\alpha = h\iota_\alpha$.

THEOREM (Golema). *For every family* $\{G_\alpha\}_{\alpha\in A}$, $G_\alpha\in\Gamma$, *the free product* $\mathop{\bar{*}}\limits_{\alpha\in A} G_\alpha$ *exists.*

Postponing the proof of Proposition 1 to the next section, we consider now some of its consequences. Let φ_α be the restriction of φ to G_α.

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(1) The Bohr compactification of a group G is understood to be a compact group H in which G is continuously *isomorphically* embedded and such that any continuous almost periodic function on G extends to H.

We see that the group G generated by the subgroups $\iota_a(G_a)$, $a \in A$, in $\mathop{\bar{*}}\limits_{a\in A} G_a$ is mapped isomorphically by h onto $\mathop{*}\limits_{a\in A} G_a$. The uniquness of $\mathop{\bar{*}}\limits_{a\in A} G_a$ shows that G is dense in $\mathop{\bar{*}}\limits_{a\in A} G_a$. This proves

PROPOSITION 1.2. *The free product in the category of compact groups of a system of groups $\{G_a\}_{a\in A}$ is the closure of the algebraic free product $\mathop{*}\limits_{a\in A} G_a$ (the monomorphisms ι_a, $a \in A$, being the ordinary mappings of the groups G_a onto "one letter words" subgroups in $\mathop{*}\limits_{a\in A} G_a$).*

The condition of continuity of Φ_a can be formulated in a slightly more convenient way by introducing a suitable topology τ in $\mathop{*}\limits_{a\in A} G_a$ such that the isomorphisms ι_a are continuous and any homomorphism h of $\mathop{*}\limits_{a\in A} G_a$ into a topological group H is continuous in τ if and only if $\iota_a h$ is continuous, for all $a \in A$.

An easy way of defining the topology τ is by exhibiting a system of pseudo-norms d each of which is given by a family of pseudonorms $\{d_a\}_{a\in A}$, where d_a is a pseudo-norm in G_a. If $\dot{g}_{a_1} \dots \dot{g}_{a_n} \in \mathop{*}\limits_{a\in A} G_a$, where $\dot{g}_a = \iota_a(g_a)$, $g_a \in G_a$, and $a_i \neq a_{i+1}$ for all $i = 1, \dots, n-1$, then

$$d(\dot{g}_{a_1} \dots \dot{g}_{a_n}) = \sum_{i=1}^{n} d_{a_i}(g_{a_i}).$$

It is easy to see that d is a pseudo-norm and that if for an element $g \in \mathop{*}\limits_{a\in A} G_a$ we have $d(g) = 0$ for all pseudonorms d, then $g = e$. Moreover, if τ is the topology defined by the pseudonorms d, then ι_a are continuous mappings from G_a into $\mathop{*}\limits_{a\in A} G_a$, and, if for a homomorphism h of $\mathop{*}\limits_{a\in A} G_a$ into a topological group H the mappings $h\iota_a$ are continuous from G_a to H, then h is continuous in τ.

Remark. It would seem desirable to find an easy way of describing the topology τ_0 in $\mathop{*}\limits_{a\in A} G_a$ which is the weakest with respect to the property that every homomorphism h from $\mathop{*}\limits_{a\in A} G_a$ into a compact group H is continuous if and only if $h\iota_a$ is a continuous mapping for each $a \in A$. The theorem of K. Golema and Proposition 1 imply that such a topology exists: it is simply the topology induced from $\mathop{\bar{*}}\limits_{a\in A} G_a$ onto $\mathop{*}\limits_{a\in A} G_a$ after the embedding. This, however, is a highly ineffective procedure and, may be, one cannot hope to find a description of the topology τ_0 as simple as that of τ.

The Bohr compactification of a topological group G is a compact group $\bar{G}$ together with a continuous isomorphism φ of G into $\bar{G}$ such that for any compact group H and any homomorphism h of G into H there is a unique homomorphism $\bar{h}$ such that $\bar{h}\varphi = h$.

An equivalent definition of $\bar{G}$ is that for any continuous almost periodic function f on G there exists a unique continuous function $\bar{f}$ on $\bar{G}$ such that $f(g) = \bar{f}(\varphi(g))$ for all $g \in G$.

In virtue of what we said above, we have

PROPOSITION 1.3. *The free product* $\bar{*}_{a\in A} G_a$ *in the category of compact groups of a system of groups* $\{G_a\}_{a\in A}$ *is the Bohr compactification of the algebraic free product* $*_{a\in A} G_a$ *equipped with the topology* τ.

2. Unitary representations of free products of compact groups. In this section we prove a theorem on finite-dimensional unitary representations of compact groups of which Proposition 1 is an immediate consequence. Before we formulate the theorem we recall some definitions and notions.

All representations considered here are homomorphisms into the group of unitary operators of a finite-dimensional complex linear space. If the represented group is a topological group, we assume that the representation is continuous. For a given representation ϱ of a group G, we denote by $L(\varrho)$ the linear space on which the unitary operators $\varrho(g)$, $g \in G$, act. For a given finite family of representations ϱ_i, $i = 1, \ldots, n$, of a group G the direct sum of the representations ϱ_i is defined as the representation ϱ of G such that

$$L(\varrho) = \oplus \sum_{i=1}^{n} L(\varrho_i) \quad \text{and} \quad \varrho(g)\sum_{i=1}^{n} \xi_i = \sum_{i=1}^{n} \varrho_i(g)\,\xi_i$$

for all $\xi_i \in L(\varrho_i)$, $i = 1, \ldots, n$.

THEOREM. *Let* $G_1, \ldots, G_k$ *be a finite system of compact groups and let* $e \neq g \in *_{s=1}^{k} G_s$. *Then there exists a representation* ϱ *of* $*_{s=1}^{k} G_s$ *continuous on each of the subgroups* G_s, $s = 1, \ldots, k$, *such that*

$$\varrho(g) \neq I,$$

where I *is the identity operator.*

The proof is based on two lemmas the first of which is a well-known, and easy to prove directly, consequence of the Frobenius reciprocity theorem (cf. [2] and [3]).

LEMMA 2.1. *Let* G *be a compact group and* A *a closed Abelian subgroup of* G. *Then for any character* χ *of* A *there exists a representation* ϱ *of* G *and a vector* $\xi \in L(\varrho)$ *such that* $\xi \neq 0$ *and*

$$\varrho(a)\,\xi = \chi(a)\,\xi \quad \text{for all } a \in A.$$

LEMMA 2.2. *Let G be a compact group and let $e \neq a \in G$. Then there exists a representation ϱ of G and a vector $\xi \in L(\varrho)$ such that $\|\xi\| = 1$ and*

$$\big(\varrho(a)\xi, \xi\big) = 0.$$

Proof. Let A be the closure of $\{a^n \mid n = 0, \pm 1, \pm 2, \ldots\}$. Then A is an Abelian closed subgroup of G. Let χ be a character of A such that

$$\chi(a) = \exp(i2\pi w) \neq 1.$$

We consider two cases.

1. w is rational. Then there exists a positive integer n such that $\chi^n(a) = 1$. For each $k = 0, 1, \ldots, n-1$, let ϱ_k be a representation of G and ξ_k a vector in $L(\varrho_k)$ such that $\|\xi_k\| = n^{-1/2}$ and

$$\varrho_k(a)\xi_k = \chi^k(a)\xi_k. \tag{2.1}$$

Let ϱ be the direct sum of the representations ϱ_k, $k = 1, \ldots, n$, and let

$$\xi = \sum_{0 \leqslant k \leqslant n} \xi_k.$$

Then $\|\xi\| = 1$ and

$$\big(\varrho(a)\xi, \xi\big) = \sum_{0 \leqslant k, l < n} \big(\varrho_k(a)\xi_k, \xi_l\big) = \frac{1}{n} \sum_{0 \leqslant k < n} \chi^k(a) = 0.$$

2. w is irrational. Let k, l be integers such that

$$\frac{1}{2} < kw < \frac{3}{4}, \quad \frac{1}{4} < lw < \frac{1}{2} \pmod{1}. \tag{2.2}$$

Then also

$$\frac{1}{4} < (k-l)w < \frac{1}{2} \pmod{1}. \tag{2.3}$$

Let

$$a = \frac{\sin 2\pi w}{\sin 2\pi(k-l)w}, \quad \beta = \frac{-\sin 2\pi kw}{\sin 2(k-l)w}.$$

By (2.2) and (2.3), $a > 0$ and $\beta > 0$ and, moreover,

$$a \sin 2\pi kw + \beta \sin 2\pi lw = 0,$$

$$a \cos 2\pi kw + \beta \cos 2\pi lw = -1,$$

which shows that

$$1 + a\chi^k(a) + \beta\chi^l(a) = 0.$$

By Lemma 2.1 there exist representations $\varrho_0, \varrho_k, \varrho_l$ of G and vectors $\xi_i \in L(\varrho_i)$, $i = 0, k, l$, such that

$$\varrho_i(a)\,\xi_i = \chi^i(a)\,\xi_i, \qquad \|\xi_i\| = (1+\alpha+\beta)^{-1/2}, \qquad i = 0, k, l.$$

Let ϱ be the direct sum of the representations $\varrho_0, \varrho_k, \varrho_l$ and let

$$\xi = \xi_0 + \sqrt{\alpha}\,\xi_k + \sqrt{\beta}\,\xi_l.$$

Then $\|\xi\| = 1$ and

$$\big(\varrho(a)\,\xi, \xi\big) = 1 + \alpha\chi^k(a) + \beta\chi^l(a) = 0.$$

Proof of the theorem. Let $g = g_{a_1} \cdots g_{a_n}$, where $e \neq g_{a_i} \in G_{a_i}$ and $a_i \neq a_{i+1}$ for $i = 1, \ldots, n-1$. For each $i = 1, \ldots, n$, let ϱ_i be a representation of G_{a_i} with the property that there exists a vector e_i^1 in $L(\varrho_i) = \mathscr{H}_i$ such that

$$\|e_i^1\| = 1 \quad \text{and} \quad \big(\varrho_i(g_{a_i})e_i^1, e_i^1\big) = 0.$$

Let $\dim \mathscr{H}_i = m_i$. Write

$$e_i^{m_i} = \varrho_i(g_{a_i})\,e_i^1, \tag{2.4}$$

and let

$$e_i^1, e_i^2, \ldots, e_i^{m_i}$$

be an orthonormal basis of $\mathscr{H}_i$. Consider the linear space $\mathscr{H}$ whose orthonormal basis is

$$\{e_i^s \mid i = 1, \ldots, n;\ 1 \leqslant s \leqslant m_i - 1\} \cup \{e_n^{m_n}\}.$$

This basis can be written as

$$\overbrace{e_1^1, \ldots, e_1^{m_1}}^{\varrho_1(g_{a_1})} = \overbrace{e_2^1, \ldots, e_2^{m_2}}^{\varrho_2(g_{a_2})} = \ldots = \overbrace{e_n^1, \ldots, e_n^{m_n}}^{\varrho_n(g_{a_n})}$$

with identification

$$e_i^{m_i} = e_{i+1}^1, \qquad i = 1, \ldots, n-1, \tag{2.5}$$

which is another way of expressing the fact that for each $i = 1, \ldots, n$ the mapping

$$T_i e_i^s = \begin{cases} e_i^s & \text{for} \quad 1 \leqslant s \leqslant m_i - 1 \text{ and for } i = n,\ s = m_n, \\ e_{i+1}^1 & \text{for} \quad s = m_i \text{ and } 1 \leqslant i \leqslant n-1 \end{cases} \tag{2.6}$$

extends to an isometric linear mapping of $\mathscr{H}_i$ into $\mathscr{H}$. We note that if $|j-i| \geqslant 2$, $1 \leqslant i, j \leqslant n$, then the subspaces $T_i(\mathscr{H}_i)$ and $T_j(\mathscr{H}_j)$ are mutually orthogonal. For a fixed group G_s, $s = 1, \ldots, k$, let

$$\Delta_s = \{i \mid a_i = s\}.$$

We have

(2.7) *If* $i, j \in \Delta_s$ *and* $i \neq j$, *then* $|j-i| \geqslant 2$.

Consequently, $T_i(\mathscr{H}_i)$ and $T_j(\mathscr{H}_j)$ are orthogonal. We define a representation ϱ^s of the group G_s into the group of unitary operators of the space $\mathscr{H}$ as follows:

If $h \in G_s$, we put

$$\varrho^s(h)e_i^t = T_i \varrho_i(h) T_i^{-1} e_i^t \quad \text{for} \quad i \in \Delta_s, 1 \leqslant t \leqslant m_i, \tag{2.8}$$

$$\varrho^s(h)e_i^t = e_i^t \quad \text{for} \quad i \notin \Delta_s, 1 \leqslant t \leqslant m_i. \tag{2.9}$$

By (2.4), it follows that $m_i \geqslant 2$ for all $i = 1, \ldots, n$ and (2.7) shows that for any two different $i, j \in \Delta_s$ the identifications (2.5) do not hold among the vectors e_i^l and e_j^m, $1 \leqslant l \leqslant m_i$, $1 \leqslant m \leqslant m_j$. Therefore (2.8) and (2.9) are consistent and, moreover, the space

$$\mathscr{H}_s = \oplus \sum_{i \in \Delta_s} T_i(\mathscr{H}_i)$$

is contained in $\mathscr{H}$ as a ϱ^s-*invariant* subspace. The restriction of ϱ^s to $\mathscr{H}_s$ is, clearly, the direct sum of the representations $T_i \varrho_i T_i^{-1}$, $i \in \Delta_s$. (2.9) shows that the orthogonal complement of $\mathscr{H}_s$ is fixed under ϱ^s.

Let ϱ be the representation of $\mathop{*}\limits_{s=1}^{k} G_s$ defined by the representations ϱ^s, i.e., if $h = h_{a_1} \ldots h_{a_n}$, $a_i = 1, \ldots, k$, is an element of $\mathop{*}\limits_{s=1}^{k} G_s$, we put

$$\varrho(h) = \varrho^{a_n}(h_{a_n}) \ldots \varrho^{a_1}(h_{a_1}).$$

Clearly, $\varrho(h)$ is a unitary operator in $\mathscr{H}$ and the restriction of the representation ϱ to each of the subgroups G_s is continuous. All that is left to show is that $\varrho(g) \neq I$. Since, by (2.8), (2.7), (2.4) and (2.7),

$$\varrho^{a_i}(g_{a_i}) e_i^1 = T_i \varrho_i T_i^{-1} e_i^1 = T_i e_i^{m_i} = e_{i+1}^1$$

for $i = 1, \ldots, n-1$, we have

$$\begin{aligned} \varrho(g)e_1^1 &= \varrho^{a_n}(g_{a_n}) \ldots \varrho^{a_1}(g_{a_1}) e_1^1 \\ &= \varrho^{a_n}(g_{a_n}) \ldots \varrho^{a_2}(g_{a_2}) e_2^1 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &= e_n^{m_n}. \end{aligned}$$

But $e_1^1 \neq e_n^{m_n}$, whence $\varrho(g) \neq I$.

Proof of Proposition 1. By the theorem, for each element $e = g \in \mathop{*}\limits_{a \in A} G_a$ there exists a homomorphism ϱ_g of $\mathop{*}\limits_{a \in A} G_a$ into the compact group H_g of the unitary operators of a finite-dimensional complex linear space such that $\varrho_g(g) = I$. Consequently, the map φ defined by

$$\varphi(h) = \langle \varrho_g(h) \rangle_{g \in G},$$

maps isomorphically $\underset{a \in A}{*} G_a$ into the product of the groups H_g, $g \in G$, and clearly φ is continuous on each G_a.

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INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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