COLLOQUIUM MATHEMATICUM

VOL. XXIX 1974 FASC. 1

IMBEDDING LOCALLY CONVEX LATTICES INTO COMPACT LATTICES

BY

ALBERT R. STRALKA (RIVERSIDE, CALIFORNIA)

Let $\mathscr{L}$ be the class of locally convex, distributive topological lattices. All those distributive topological lattices which are compact [5], or locally compact and connected [1], or discrete belong to $\mathscr{L}$. Relying on the imbedding theorem in [2], it was shown in [6] that if $L \in \mathscr{L}$ and has finite breadth n, then L can be imbedded in a product of n compact chains. The condition of local convexity thus serves to characterize sublattices of finite products of compact chains. In general, it is not even true that compact members of $\mathscr{L}$ can be imbedded in products of chains [3].

In this note* we are concerned with the question of when members of $\mathscr{L}$ — without the assumption of finite breadth — can be imbedded in compact lattices. We first show that if $L \in \mathscr{L}$ and L is locally compact and connected, then L can be so imbedded. Next, we give an example of a member of $\mathscr{L}$ — which has the discrete topology — which cannot be imbedded in a compact lattice.

Recall that a *topological lattice* is a Hausdorff space L with a pair of continuous maps $\wedge, \vee : L \times L \to L$ such that $(L, \wedge, \vee)$ is a lattice. A subset A of a lattice is *convex* if whenever $a, b \in A$ and $a \leqslant x \leqslant b$, then $x \in A$. A topological lattice is *locally convex* if its topology has a base of convex sets. As noted above, $\mathscr{L}$ will be the class of locally convex, distributive topological lattices. For $L \in \mathscr{L}$ and $a, b \in L$ with $a < b$, the interval from a to b, $[a, b]$, is $\{x \in L;\ a \leqslant x \leqslant b\}$ and $[a, b]^{\#}$ is the natural continuous homomorphism of L onto $[a, b]$, i.e., for $x \in L$,

$$[a, b]^{\#}(x) = (a \vee x) \wedge b = a \vee (x \wedge b).$$

By an *imbedding* we shall mean an open, injective, continuous homomorphism. For a set W, $\partial(W)$ will be its boundary and W^* will be its closure.

* Partially supported by N.S.F. Grant 33912.

1. Locally compact connected lattices. Throughout this section L will be an arbitrary but fixed member of $\mathscr{L}$ which is, in addition, locally compact and connected. The topology on L will have a neighborhood base of compact convex sets. $\mathscr{F}$ will denote the set of compact intervals of L. The following lemma will be used often and usually without reference.

LEMMA 1.1. *Let W be a compact, convex neighborhood of the point p of L, let $w_1, w_2 \in W$ and $q, r \in L/W$ with $w_1 < w_2$ and $q < p$. Then*

(i) $[q, p] \cap \partial(W) \neq \emptyset$;

(ii) $[w_1, w_2] \subseteq W$;

(iii) $[w_1, w_2] \in \mathscr{F}$;

(iv) *either* $p \wedge r \notin W$ *or* $p \vee r \notin W$.

Proof. (i) holds, because $[q, p]$ is connected. (ii) and (iii) result from the fact that W is both compact and convex. If both $p \wedge r$ and $p \vee r$ belong to W, then, because W is convex, r would belong to W. Hence (iv) holds.

Now endow $\times\mathscr{F}$ with coordinate-wise operations and the Tichonov topology. $\times\mathscr{F}$ becomes a compact distributive topological lattice. The parametric map $\mathscr{F}^{\#}\colon L \to \times\mathscr{F}$ defined by

$$\big(\mathscr{F}^{\#}(x)\big)_{[a,b]} = [a, b]^{\#}(x)$$

is a continuous homomorphism. From Mrówka's imbedding theorem in [4] and the discussion in section 3 of [6] we have the following

LEMMA 1.2. *If $\mathscr{F}$ separates points in L (i.e., for $x, y \in L$ with $x \neq y$, there is $[a, b] \in \mathscr{F}$ such that $[a, b]^{\#}(x) \neq [a, b]^{\#}(y)$), then $\mathscr{F}^{\#}$ is injective. $\mathscr{F}^{\#}$ is an imbedding if, in addition, given $p \in L$ and F a closed subset of L not containing p, there are $[a_1, b_1], \ldots, [a_n, b_n] \in \mathscr{F}$ and subsets $F_1, \ldots, F_n$ of L such that*

$$F = F_1 \cup \ldots \cup F_n \quad \text{and} \quad [a_i, b_i]^{\#}(p) \notin \big([a_i, b_i]^{\#}(F_i)\big)^{*}.$$

With this result we are now prepared to prove our imbedding theorem.

THEOREM 1.3. *$\mathscr{F}^{\#}\colon L \to \times\mathscr{F}$ is an imbedding. Thus every locally compact, connected distributive topological lattice can be imbedded in a compact distributive topological lattice.*

Proof. We begin by showing that $\mathscr{F}^{\#}$ is injective. Let $x, y \in L$ with $x \neq y$. Select a compact, convex neighborhood W of x which excludes y. Then either $x \vee y \notin W$ or $x \wedge y \notin W$. We assume the latter holds. Then

$$[x \wedge y, x] \cap \partial(W) \neq \emptyset.$$

Let w be any point of that set. $[w, x] \in \mathscr{F}$ and

$$[w, x]^{\#}(y) = w \neq x = [w, x]^{\#}(x).$$

Hence $\mathscr{F}^{\#}$ is injective.

Now suppose that $p \in L$ and F is a closed subset of L with $p \notin F$. There is a compact, convex neighborhood W of p which is contained in $L \setminus F$. The sets

$$B = \partial(W) \cap (p \wedge L) \quad \text{and} \quad T = \partial(W) \cap (p \vee L)$$

are compact. This implies that there are a subset $T_0 = \{t_1, \ldots, t_n\} \subseteq T$ and open sets $U(t_1), \ldots, U(t_n)$ of T such that $t_i \in U(t_i)$,

$$\bigcup_{i=1}^{n} U(t_i) = T \quad \text{and} \quad [p, t_i]^{\#}(p) \notin \big([p, t_i]^{\#}(U(t_i))\big)^* \vee [p, t_i].$$

Similarly, there are a subset $B_0 = \{b_1, \ldots, b_m\} \subseteq B$ and open sets $U(b_1), \ldots, U(b_m)$ of B such that $b_i \in U(b_i)$,

$$\bigcup_{i=1}^{m} U(b_i) = B \quad \text{and} \quad [b_i, p]^{\#}(p) \notin \big([b_i, p]^{\#}(U(b_i))\big)^* \wedge [b_i, p].$$

Next, we define a map $\gamma\colon B \cup T \to B_0 \cup T_0$ which assigns to an element x of $B \cup T$ an element $\gamma(x)$ of $B_0 \cup T_0$ with the property that $x \in U(\gamma(x))$. For $f \in F$, either $p \wedge f \notin W$ or $p \vee f \notin W$. Hence

$$([p \wedge f, p] \cup [p, p \vee f]) \cap \partial(W) \neq \emptyset.$$

Let $\delta(f)$ be any point of this set and let $F_i = \delta^{-1}(\gamma^{-1}(b_i))$ for $i = 1, \ldots, m$ and $F_{m+1} = \delta^{-1}(\gamma^{-1}(t_i))$ for $i = 1, \ldots, n$. Then $F = F_1 \cup \ldots \cup F_{m+n}$. Now, let $f \in F_i$. We may assume that $i = 1$. Then $b_1 = \gamma(\delta(f))$ and we have

$$[b_1, p]^{\#}(f) = b_1 \vee (p \wedge f) \leqslant b_1 \vee \delta(f).$$

Hence

$$[b_1, p]^{\#}(f) \in \big([b_1, p]^{\#}(U(b_1))\big)^* \wedge [b_1, p].$$

Therefore, $[b_1, p]^{\#}(p) \notin ([b_1, p]^{\#}(F_i))^*$. It then follows that $\mathscr{F}^{\#}$ is an imbedding.

Remark. The methods used in this section can be modified to show that if L is a locally compact, connected (not necessarily distributive) topological lattice such that every pair of distinct points of L can be separated by a continuous homomorphism into a compact topological lattice, then L can be imbedded in a compact lattice. In particular, if $L \in \mathscr{L}$, L is locally compact and connected and $\mathrm{Hom}(L, I)$ separates points, then L can be imbedded in a product of copies of I. (I is the real interval $[0, 1]$ with the usual operations.)

2. An example. In this section we present an example of a member of $\mathscr{L}$ which cannot be imbedded in a compact lattice. *2* will be the lattice on $\{0, 1\}$. Let Y be the cartesian product of a countable number of copies of *2* with coordinate-wise operations and the *discrete* topology. Y, obviously, belongs to $\mathscr{L}$. $\mathrm{Hom}(Y, 2)$ separates points, so Y has a continuous injective homomorphism into a compact lattice; in fact, Y has a continuous injective homomorphism *onto* a compact topological lattice. Suppose that Y is a sublattice of a compact topological lattice K. We may assume that $O_Y = O_K$ and $1_K = 1_Y$. Since $\{O_Y\}$ is an open set in Y, it cannot be a limit of a sequence of points of Y. However, define a_n to be that point of Y whose first n coordinates are zero and all subsequent coordinates are one. Then $\{a_n\}$ is an infinite decreasing sequence in K, so it must have a limit point k and $k \leqslant a_n$ for all n. Next, we define e_n to be that point of Y whose n-th coordinate is one and all other coordinates are zero. $e_n \wedge a_m = 0$ for $m \neq n$. Then, since K is a compact topological lattice, the infinite distributivity law holds, so

$$O_Y = \bigvee_{n=1}^{\infty} (e_n \wedge k) = k \wedge \Big(\bigvee_{n=1}^{\infty} e_n\Big) = k \wedge 1_Y = k.$$

This is a contradiction. Hence Y cannot be imbedded in a compact lattice. (See also [7].)

REFERENCES

[1] L. W. Anderson, *One-dimensional topological lattices*, Proceedings of the American Mathematical Society 10 (1959), p. 715-720.

[2] K. A. Baker and A. R. Stralka, *Compact, distributive lattices of finite breadth*, Pacific Journal of Mathematics 34 (1970), p. 311-320.

[3] J. D. Lawson, *Lattices with no interval homomorphisms*, ibidem 32 (1970), p. 459-466.

[4] S. Mrówka, *Further results on E-compact spaces. I*, Acta Mathematica 120 (1968), p. 161-185.

[5] L. Nachbin, *Topology and order*, Princeton, New Jersey, 1956.

[6] A. R. Stralka, *Locally convex topological lattices*, Transactions of the American Mathematical Society 151 (1970), p. 629-640.

[7] J. W. Stepp, *Semilattices which are imbeddable in a product of min intervals*, Proceedings of the American Mathematical Society 28 (1971), p. 81-86.

UNIVERSITY OF CALIFORNIA, RIVERSIDE
RIVERSIDE, CALIFORNIA

Reçu par la Rédaction le 8. 6. 1972