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COMPLETENESS AND COMPACTNESS OF LATTICES

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0. Introduction. The aim of this paper is to characterize complete lattices in terms of equational compactness or atomic compactness of lattices or related systems. These notions of compactness were given by J. Mycielski (see [3] and [5]).

In section 2, we study the general connections between completeness of lattices and atomic compactness of their orders. In section 3, the above is applied to some problems of definability of classes of compact lattices and to a problem of G. Birkhoff concerning the interval topology. Finally, in this section, we construct an example of a compact topological lattice no ultrapower of which can be embedded into an equationally compact lattice. This is a solution of a problem from [5].

The main tool of this paper is Theorem 2.3 of [5].

1. Terminology and notation. The terminology and notation of [5] will be used throughout this paper.

Let $\mathfrak{L} = \langle L, \frown, \smile \rangle$ be a lattice. By $\mathfrak{L}(\leqslant)$ we denote the ordered system $\langle L, \leqslant \rangle$, where $\leqslant$ is the ordering relation defined in the usual way by $\frown$ or $\smile$ on L; similarly, we define the systems $\mathfrak{L}(\leqslant, \frown) = \langle L, \leqslant, \frown \rangle$ and $\mathfrak{L}(\leqslant, \smile) = \langle L, \leqslant, \smile \rangle$. If $\mathfrak{L}$ has the greatest element 1 and the smallest element 0, then we put $\mathfrak{L}^* = \langle L, \frown, \smile, 0, 1 \rangle$ and call this algebra a $*$-*lattice*.

If $\mathfrak{A}$ and $\mathfrak{B}$ are ordered systems, then by $\mathfrak{A} \oplus \mathfrak{B}$ we denote their ordinal sum. Analogously, if $\mathfrak{A}_i$, $i \in I$, is a non-void family of ordered systems and the set I is ordered, then by $\bigoplus_{i \in I} \mathfrak{A}_i$ we denote their ordinal sum.

Algebraic systems will be denoted by $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ and their sets by $A, B, C, \ldots$, respectively. The cardinality of a set A will be denoted by $|A|$. Any ordinal η will be identified with the set of all smaller ordinals, thus e.g. ω is the set of all natural numbers. $\boldsymbol{K} \in EC_\Delta$ means that $\boldsymbol{K}$ is an elementary class of algebraic systems.

If $\boldsymbol{K}$ is a class of algebraic systems, then by $EC_\Delta(\boldsymbol{K})$ we denote the collection of all classes of the form $\boldsymbol{K} \frown \boldsymbol{M}$, where $\boldsymbol{M} \in EC_\Delta$. We see that $\boldsymbol{K} \in EC_\Delta$ implies $EC_\Delta(\boldsymbol{K}) \subseteq EC_\Delta$. We define $PC_\Delta(K)$, $QC_\Delta(\boldsymbol{K})$ and $UC_\Delta(\boldsymbol{K})$ in the same way (see [2] for the definitions of PC_Δ, QC_Δ and UC_Δ).

For a lattice $\mathfrak{L}$, the interval topology is the smallest topology in L, in which the sets of the form $\{x \in L: x \leqslant a\}$ and of the form $\{x \in L: a \leqslant x\}$ are topologically closed.

For a class $\boldsymbol{K}$ of lattices, $\boldsymbol{K}_{TC}$ and $\boldsymbol{K}_{RC}$ have the same meaning as in [5].

2. Compactness of lattices. First, let us write down the following simple proposition:

PROPOSITION 2.1. *If a lattice $\mathfrak{L}$ is equationally compact, then the systems $\mathfrak{L}(\leqslant, \frown)$ and $\mathfrak{L}(\leqslant, \smile)$ are atomically compact. The atomic compactness of one of the systems $\mathfrak{L}(\leqslant, \frown)$ or $\mathfrak{L}(\leqslant, \smile)$ implies that of the system $\mathfrak{L}(\leqslant)$.*

LEMMA 2.2. *If $\mathfrak{L}(\leqslant)$ is atomically compact, then $\mathfrak{L}$ is a complete lattice.*

Proof. Evidently, if $\mathfrak{L}(\leqslant)$ is atomically compact, then $\mathfrak{L}$ has both the smallest and the greatest elements 0 and 1. Let $a_t, t \in T$, be an arbitrary system of elements of L, with $T \neq \emptyset$. Let

$$B = \{b \in L: \bigvee_{t \in T} b \leqslant a_t\}$$

and consider the following set of atomic formulae, having one free variable x_0 only:

$$\Sigma = \{\text{“}x_0 \leqslant a_t\text{”}: t \in T\} \cup \{\text{“}x_0 \geqslant b\text{”}: b \in B\}.$$

It is easy to see that an element $a \in L$ satisfies Σ in $\mathfrak{L}(\leqslant)$ if and only if

$$a = \bigcap_{t \in T} a_t.$$

Let Σ' be a finite subset of Σ. Then Σ' contains only finitely many elements $a_t, t \in T$; say, $a_{t_0}, \ldots, a_{t_n}$. It is easy to verify that the element $c = a_{t_0} \frown \ldots \frown a_{t_n}$ satisfies Σ' in $\mathfrak{L}(\leqslant)$. But $\mathfrak{L}(\leqslant)$ is atomically compact, thus Σ is satisfiable in $\mathfrak{L}(\leqslant)$ and $\mathfrak{L}$ is complete.

Let $\boldsymbol{L}$ denote the class of all lattices; put $\boldsymbol{L}(\leqslant) = \{\mathfrak{L}(\leqslant): \mathfrak{L} \in \boldsymbol{L}\}$, and let $\boldsymbol{O}$ be the class of all ordered systems. We have $\boldsymbol{L}(\leqslant) \subseteq \boldsymbol{O}$.

LEMMA 2.3. *If $\mathfrak{L}$ is complete, then $\mathfrak{L}(\leqslant)$ is injective in $\boldsymbol{O}$, thus also in $\boldsymbol{L}(\leqslant)$.*

Proof. Let $\mathfrak{A} = \langle A, \leqslant\rangle$ be a subsystem of $\mathfrak{C} = \langle C, \leqslant\rangle \in \boldsymbol{O}$, and let $h: \mathfrak{A} \to \mathfrak{L}(\leqslant)$ be a homomorphism. We have to show that there is a homomorphism $h^*: \mathfrak{C} \to \mathfrak{L}(\leqslant)$ such that $h^* \mid A = h$. Let us put

$Z = \{(\mathfrak{B}, f) : \mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{C}, f : \mathfrak{B} \to \mathfrak{L}(\leqslant), f \mid A = h\}$. Z is non-void since $(A, h) \in Z$. Let $\leqslant$ be an ordering relation over Z defined as follows:

$$(\mathfrak{B}_1, f_1) \leqslant (\mathfrak{B}_2, f_2) \text{ if and only if } \mathfrak{B}_1 \subseteq \mathfrak{B}_2 \text{ and } f_2 \mid B_1 = f_1.$$

Then $\mathfrak{Z} = \langle Z, \leqslant\rangle$ is an ordered system such that each linearly ordered subset of Z has an upper bound in $\mathfrak{Z}$. Thus by Kuratowski-Zorn lemma there is a maximal element $(\mathfrak{B}_0, f_0)$ in $\mathfrak{Z}$. We will prove that $B_0 = C$. Suppose to the contrary that there is an element $c \in C \setminus B_0$. Let $X = \{b \in B_0 : b \leqslant c\}$ and $Y = \{b \in B_0 : c \leqslant b\}$. Let $u = \bigcup \{f_0(b) : b \in X\}$ and $v = \bigcup \{f_0(b) : b \in Y\}$, and if X or Y is void, then put $u = 0$ or $v = 1$, respectively (such u and v exist since $\mathfrak{L}$ is complete). Now let $a \in L$ be such that $u \leqslant a \leqslant v$. It is easy to see that the mapping $f' : B' \to L$, where $B' = B_0 \cup \{c\}$, defined by $f' \mid B_0 = f_0$ and $f'(c) = a$, is a homomorphism of $\mathfrak{B}' = \langle B', \leqslant\rangle$ into $\mathfrak{L}(\leqslant)$. But this is impossible since $(\mathfrak{B}_0, f_0)$ was chosen as a maximal element of $\mathfrak{Z}$. Hence $h^* = f_0$ has the required properties, q.e.d.

THEOREM 2.4. *For every lattice $\mathfrak{L}$, the following conditions are equivalent*:

(i) *$\mathfrak{L}(\leqslant)$ is atomically compact*;

(ii) *$\mathfrak{L}$ is complete*;

(iii) *$\mathfrak{L}(\leqslant)$ is injective in $\boldsymbol{O}$*;

(iv) *$\mathfrak{L}(\leqslant)$ is injective in $\boldsymbol{L}(\leqslant)$*;

(v) *$\mathfrak{L}(\leqslant)$ is an absolute retract in $\boldsymbol{O}$*;

(vi) *$\mathfrak{L}(\leqslant)$ is an absolute retract in $\boldsymbol{L}(\leqslant)$*.

Proof. By Lemma 2.2, (i) implies (ii), and, by Lemma 2.3, (ii) implies (iii) and (iv). Evidently (iii) implies (v) and (iv) implies (vi). Finally, by Corollary 2.5 of [5], each of the conditions (v) or (vi) implies (i).

Now we are going to prove a series of propositions concerning connections between so far introduced and related notions.

EXAMPLE 2.5. There is an atomically compact ordered system which is not an absolute retract in $\boldsymbol{O}$. It is easy to verify that such is the system $\mathfrak{A} = \langle\{a, b, c\}, \leqslant\rangle$, where a and b are incomparable, $c \leqslant a$ and $c \leqslant b$.

PROPOSITION 2.6. *If an ordered system $\mathfrak{A} = \langle A, \leqslant\rangle$ is an absolute retract in $\boldsymbol{O}$, then there is a lattice $\mathfrak{L}$ such that $\mathfrak{L}(\leqslant) = \mathfrak{A}$.*

Proof. Without loss of generality we can suppose that $\mathfrak{K}$ is a lattice such that $\mathfrak{A}$ is a subsystem of $\mathfrak{K}(\leqslant)$. By our assumptions, there is a homomorphism $h : \mathfrak{K}(\leqslant) \to \mathfrak{A}$ such that $h \mid A$ is the identity mapping. Since $\mathfrak{K}$ is a lattice for every $a, b \in A$, there is $c \in K$ which is l.u.b. of a

and b. Then we have $h(c) \geqslant h(a) = a$ and $h(c) \geqslant h(b) = b$. Let us put $X = \{d \in A : d \geqslant a \cap d \geqslant b\}$. Thus $h(c) \in X$. Let $d \in X$. We then have $d \in X \subseteq A \subseteq K$, thus $d \geqslant c$, and, consequently, $d = h(d) \geqslant h(c)$, which shows that $h(c)$ is the l.u.b. (in $\mathfrak{A}$) of a and b. The proof of the existence of g.l.b. is dual.

EXAMPLE 2.7. There is a complete lattice $\mathfrak{L}$ which is not equationally compact. Moreover, there is no equationally compact lattice $\mathfrak{K}$ such that $\mathfrak{L}$ is a sublattice of $\mathfrak{K}$.

Indeed, let S be an arbitrary infinite set and let $L = S \cup \{a, b\}$. Setting a as the greatest element, b as the smallest element of S and all elements of S as incomparable elements, we obtain a well defined complete lattice $\mathfrak{L} = \langle L, \cap, \cup \rangle$. Let $\mathfrak{K}$ be an arbitrary lattice containing $\mathfrak{L}$ as a sublattice. We will prove that $\mathfrak{K}$ is not equationally compact. Let T be a set with $|T| > |K|$ and consider the following set of equations:

$$\Sigma = \{\text{“}x_i \cap x_j = b\text{”}: i \neq j,\, i, j \in T\} \cup \{\text{“}x_i \cup x_j = a\text{”}: i \neq j,\, i, j \in T\}.$$

It is easy to see that every finite subset Σ' of Σ has a solution in L but, of course, Σ is not satisfiable in $\mathfrak{K}$.

Let us observe that the only constants in Σ are a and b, which are the greatest and the smallest elements of $\mathfrak{L}$. Thus we have immediately

EXAMPLE 2.8. There is a complete $*$-lattice $\mathfrak{L}^*$ such that no weakly equationally compact $*$-lattice contains $\mathfrak{L}^*$ as sublattice.

REMARK 2.9. Each lattice is weakly equationally compact.

Indeed, by Corollary 2.5 of [5], it is so since it contains a homomorphic image (e.g. one point image) of each lattice which contains it as a sublattice.

3. Definability. Now, we will apply the result of section 2 to some problems concerning definability. Let us denote by $\mathfrak{S}(\mathfrak{m})$ the lattice defined in Example 2.7, where $\mathfrak{m}$ is the cardinality of the set S. Let $\boldsymbol{K}$ be the class of all lattices $\mathfrak{S}(\mathfrak{m})$ with any $\mathfrak{m}$. Observe first that we have the following obvious

PROPOSITION 3.1. *$\boldsymbol{K}$ is an elementary class, i.e. $\boldsymbol{K} \in EC_\Delta$.*

Now we will use the class $\boldsymbol{K}$ to show that some problems of G. Birkhoff ([1], Problem 21b, 22, 23) are not elementarily solvable.

THEOREM 3.2. *There is no set Φ of formulae of the first order predicate calculus with identity, having one free variable only, such that an element a of a lattice $\mathfrak{L}$ satisfies Φ in $\mathfrak{L}$ if and only if a is an isolated point in the interval topology of $\mathfrak{L}$.*

Proof. Suppose the contrary. Let x be the free variable in Φ. Let $\Phi' = \Phi \cup \{\text{“} \bigwedge_y x \cap y = y \text{”}\}$. Thus Φ' is such that an element a satisfies Φ' in $\mathfrak{L}$ if and only if a is the greatest element of $\mathfrak{L}$ and is isolated in the interval topology of $\mathfrak{L}$. It is easy to see that, for each $\mathfrak{m} < \aleph_0$, Φ is satisfiable in $\mathfrak{S}(\mathfrak{m})$, but Φ cannot be satisfied in $\mathfrak{S}(2^{\aleph_0})$, which is an ultrapower of all $\mathfrak{S}(\mathfrak{m}), \mathfrak{m} < \aleph_0$. Indeed, *1* belongs to the closure (in the interval topology) of each infinite set of incomparable elements of $\mathfrak{S}(2^{\aleph_0})$. This contradiction finishes the proof.

LEMMA 3.3. *Let $\boldsymbol{M}$ be an arbitrary class of lattices. If $\mathfrak{S}(\mathfrak{m}) \in \boldsymbol{M}$ for every $\mathfrak{m} < \aleph_0$ and $\mathfrak{S}(2^{\aleph_0}) \notin \boldsymbol{M}$, then $\boldsymbol{M} \notin EC_\Delta$ and $\boldsymbol{M} \notin QC_\Delta$. If, moreover, $\mathfrak{S}(2^{\aleph_0}) \notin \mathscr{S}\boldsymbol{M}$, then $\boldsymbol{M} \notin PC_\Delta$ and $\mathscr{S}\boldsymbol{M} \notin UC_\Delta$.*

Proof. Let $\mathscr{D}$ be an arbitrary non-principal ultrafilter over ω. Then $\mathfrak{P}_{n\in\omega}\mathfrak{S}(n)/\mathscr{D}$ is isomorphic with $\mathfrak{S}(2^{\aleph_0})$. Thus by the well known properties of EC_Δ and QC_Δ, $\boldsymbol{M} \notin EC_\Delta$ and $\boldsymbol{M} \notin QC_\Delta$ (see [2], Theorem 2.3 and 2.8). By the same argument our assumption gives $\mathscr{S}\boldsymbol{M} \notin UC_\Delta$ and this already implies $\boldsymbol{M} \in PC_\Delta$ (see e.g. [4]).

THEOREM 3.4. *Neither the class $\boldsymbol{L}_{TC}$ nor $\boldsymbol{L}_{RC}$ is elementary. Moreover, $\boldsymbol{L}_{TC} \notin PC_\Delta$, $\boldsymbol{L}_{RC} \notin PC_\Delta$, $\boldsymbol{L}_{TC} \notin QC_\Delta$, $\boldsymbol{L}_{RC} \notin QC_\Delta$, $\mathscr{S}\boldsymbol{L}_{TC} \notin UC_\Delta$, and $\mathscr{S}\boldsymbol{L}_{RC} \notin UC_\Delta$.*

Proof. Indeed, for every $\mathfrak{m} < \aleph_0$, the lattice $\mathfrak{S}(\mathfrak{m})$ belongs to all considered classes but it does not belong to those classes for $\mathfrak{m} \geqslant \aleph_0$. Thus suppositions of Lemma 3.3 are satisfied and 3.4 follows.

THEOREM 3.5. *The class $\boldsymbol{H}$ of all lattices which are Hausdorff spaces in their interval topology is not elementary. Moreover, $\boldsymbol{H} \notin QC_\Delta$ and $\boldsymbol{H} \notin PC_\Delta$.*

Proof. It is easy to see that $\mathfrak{S}(\mathfrak{m}) \in \boldsymbol{H}$ for $\mathfrak{m} < \aleph_0$ and $\mathfrak{S}(\mathfrak{m}) \notin \boldsymbol{H} \notin \mathscr{S}\boldsymbol{H}$ for $\mathfrak{m} \geqslant \aleph_0$ and thus 3.5 follows by Lemma 3.3.

THEOREM 3.6. *The class $\boldsymbol{Z}$ of all complete lattices which are Hausdorff compact spaces in their interval topology is not elementary relatively to the class $\boldsymbol{C}$ of all complete lattices, i.e., $\boldsymbol{Z} \notin EC_\Delta(\boldsymbol{C})$. Moreover $\boldsymbol{Z} \notin QC_\Delta(C)$, $\boldsymbol{Z} \notin PC_\Delta(\boldsymbol{C})$ and $\mathscr{S}\boldsymbol{Z} \notin UC_\Delta(\boldsymbol{C})$.*

Proof. This follows from Lemma 3.3, since $\mathfrak{S}(\mathfrak{m}) \in \boldsymbol{Z} \cap \boldsymbol{C}$ for $\mathfrak{m} < \aleph_0$, but $\mathfrak{S}(\mathfrak{m}) \in \boldsymbol{C} \setminus \boldsymbol{Z}$ and $\mathfrak{S}(\mathfrak{m}) \in C \setminus \mathscr{S}\boldsymbol{Z}$ for $\mathfrak{m} \geqslant \aleph_0$.

EXAMPLE 3.7. The lattice

$$\mathfrak{B} = \Big(\bigoplus_{0<n<\aleph_0} \mathfrak{S}(n)\Big) \oplus \mathfrak{S}(0)$$

has the following properties:

(i) $\mathfrak{B}$ is complete;

(ii) $\mathfrak{B}$ with the interval topology is a compact Hausdorff space;

(iii) both $\cap$ and $\cup$ are continuous in the interval topology of $\mathfrak{B}$;

(iv) no non-trivial countable ultrapower of $\mathfrak{B}$ is in $\mathscr{S}\boldsymbol{L}_{RC}$.

The properties (i)-(iii) are easy to verify. Let us show (iv). It is visible that if $\mathscr{D}$ is a non-principal ultrafilter over ω, then $\mathfrak{P}_{n<\omega}\mathfrak{S}(n)/\mathscr{D}$ is isomorphic with a sublattice of $\mathfrak{B}^\omega/\mathscr{D}$. But as we know $\mathfrak{P}_{n<\omega}\mathfrak{S}(n)/\mathscr{D}$ is isomorphic with $\mathfrak{S}(2^{\aleph_0})$ and hence (iv) follows by 2.7.

From this Example we obtain at once the following result which is a solution of a problem from [5]:

THEOREM 3.8. $\boldsymbol{L}\setminus\boldsymbol{L}_{TC}\notin EC_{\Delta}$ *and* $\boldsymbol{L}\setminus\boldsymbol{L}_{RC}\notin EC_{\Delta}$.

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