COLLOQUIUM MATHEMATICUM

XVI DÉDIÉ À M. FRANCISZEK LEJA 1967

ULTRA-KRONECKERIAN SETS

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The so-called Kroneckerian sets play a significant role in the harmonic analysis. A compact set Z in a locally compact abelian group is *Kroneckerian* if every continuous function φ on Z of absolute value 1 can be approximated by single continuous characters, i.e. if for an arbitrary $\varepsilon > 0$ there is a character χ such that $|\varphi(t)-\chi(t)| < \varepsilon$ on Z. Is $Z = \{t_i\}$ $(1 \leqslant i \leqslant n)$ a finite set, then Z is Kroneckerian if and only if it is independent, i.e. iff

$$\sum_{i=1}^{n} k_i t_i = 0$$

with integer k_i's is impossible unless $k_1 = \ldots = k_n = 0$ ([10], p. 98). For the group of integers and for euclidean spaces this is a part of the classical theorem of Kronecker; for other groups it follows from an abstract form of Kronecker's theorem, as proved by Hewitt and Zuckerman ([4]; see also [1]). Obviously, a Kronecker set is always independent. It has been proved ([10], p. 100) that a compact abelian group, having an element of infinite order in every neighbourhood of the identity, contains an infinite Kroneckerian set, actually one which is homeomorphic to the Cantor ternary set. However, apparently no attention was paid up to the present to the existence of sets of Kronecker's type which are not compact but rather "dispersed" over a whole non-compact group, so that the prescribed function φ in the definition is automatically continuous. This is a special case of a more general situation as covered by the notion we intend to introduce now.

Definition. A set E in a locally compact abelian group G is called *ultra-Kroneckerian* (UK) if for every function φ of absolute value 1 and for every $\varepsilon > 0$ there is a continuous character χ of G such that

$$|\varphi(t)-\chi(t)| < \varepsilon \quad \text{for} \quad t \in E.$$

It is obvious that for a finite set to be Kroneckerian (or, equivalently, independent) and to be UK means the same and that a compact infinite set is never UK. Further,

(i) *UK-sets do not contain non-trivial convergent sequences.*

In particular, in metric compact groups there are no infinite UK-sets.

This follows from the fact that every function φ on a UK-set with $|\varphi| = 1$ is extendable to a continuous function over G, this being impossible if the set contains a sequence $\{t_i\}$ of distinct elements with $t_i \to t_0 \in G$, since one could then assume $\varphi(t_{2i+1}) = -1$, $\varphi(t_{2i}) = 1$.

A set E in a locally compact abelian group G is called an $\boldsymbol{I}_0$*-set* if every bounded function defined on E can be extended to an almost periodic function over G [2]. It is easy to see that

(ii) *every UK-set is an* $\boldsymbol{I}_0$*-set.*

In fact, in order that a set be $\boldsymbol{I}_0$ it is enough that every two-valued function on it can be extended to an almost periodic function [2]. This is just the case with a UK-set E, because for a function φ on E with $|\varphi| = 1$ we can find a sequence $\{\chi_n\}$ of characters of G with lim unif $\chi_n = \varphi$ on E, then extend the $\eta_n' s$ to (continuous) characters of the Bohr compactification $\tilde{G}$ of G, thus getting a sequence of characters uniformly convergent on the (weak) closure $\tilde{E}$ of E in $\tilde{G}$, whose limit, if extended over the whole of $\tilde{G}$, yields an almost periodic function on G, equal to φ in the points of E.

Following these lines we state also

(iii) *E is a UK-set in G iff it is an* $\boldsymbol{I}_0$*-set such that $\tilde{E}$ is Kroneckerian in $\tilde{G}$.*

In fact, if E is UK, then for a (weakly) continuous function φ on $\tilde{E}$ with $|\varphi| = 1$ we have lim unif $\chi_n = \varphi$ on $\tilde{E}$ provided the same holds on E; hence $\tilde{E}$ is Kroneckerian. Conversely, if E is $\boldsymbol{I}_0$, then we can extend every bounded φ from E to a continuous function on $\tilde{G}$, and $|\varphi| = 1$ holds on $\tilde{E}$ if it holds on E. Thus, if E is Kroneckerian, there is a sequence of characters such that lim unif $\chi_n = \varphi$ on $\tilde{E}$, and a fortiori on E, which thus appears to be UK.

So, the investigation of $\boldsymbol{I}_0$- and related sets leads in a natural way to UK-sets. Property $\boldsymbol{I}_0$ implies by no means property UK, since an $\boldsymbol{I}_0$-set is not necessarily independent; e.g., on the real line

$$\varliminf_{n} \frac{t_{n+1}}{t_n} > 1$$

is enough for the sequence $\{t_n\}$ to be an $\boldsymbol{I}_0$-set [11]. The proof of this theorem is far from simple (yet being elementary); previous examples of I_0-sets ([7] and [8]), though easier, required nevertheless some special arithmetical constructions which do not seem appropriate for the theory of almost periodic extensions, such as exposed in [2] and [3], where the methods refer rather to topology and the theory of measure, making use of arithmetics only indirectly and for the unique purpose of guaranteeing the existence of objects under consideration. In the sequel, we shall show by very modest arithmetical means that infinite UK-sets and thus infinite I_0-sets do exist on the real line and in some other groups. So we shall be able to avoid special constructions where nothing but an existence theorem is needed. This does not diminish the importance of the results in [7], [8] and some other papers; they give much more detailed information.

We still observe that if E is an I_0-set, then $\tilde{E}$ is a Helson set [6], i.e. a closed set such that every continuous function on it is a restriction of a Fourier transform, which means here an almost periodic function with absolutely convergent Fourier series. It is known that a Kronecker set is always a Helson set ([10], p. 116) and so we obtain a part of Kahane's result directly from (iii).

THEOREM 1. *A sufficient condition for an increasing seqence of positive numbers $\{t_n\}$ to be a UK-set is that t_n's be independent and that $t_n/t_{n+1} \to 0$.*

Proof. We shall denote e^{ia} by $e(a)$. If $a_1, a_2, \ldots$ are arbitrary complex numbers of absolute value 1 and ε an arbitrary positive number, we must find a real λ such that

$$(*) \qquad |e(\lambda t_n)-a_n| < \varepsilon \quad \text{for} \quad n = 1, 2, \ldots$$

Put

$$\omega_n = \left(\frac{1}{t_n} + \frac{1}{t_{n+1}} + \ldots\right) t_{n-1}.$$

It is easily seen that $\lim\limits_n \omega_n = 0$. Thus, for a given $\delta > 0 (\delta < 1)$, there is an m such that $2\pi\,\omega_n < \delta$ for $n \geqslant m$. We shall represent the required number in the form

$$\lambda = \lambda_0 + x_m + x_{m+1} + \ldots,$$

where

$$0 \leqslant x_n \leqslant \frac{2\pi}{t_n} \qquad (n \geqslant m).$$

The numbers $t_1, t_2, \ldots, t_{m-1}$ being independent, there exists by Kronecker's theorem a λ_0 such that

$$(2) \qquad |e(\lambda_0 t_n)-a_n| < \delta \quad \text{for} \quad n < m.$$

The values of $x_n (n \geqslant m)$ will be determined successively as follows: We have

$$\lambda t_n = (\lambda_0 + x_m + x_{m+1} + \ldots + x_n) t_n + r_n t_n, \quad \text{where} \quad r_n = \sum_{\nu=n+1}^{\infty} x_\nu.$$

From (1) we obtain $r_n t_n \leqslant 2\pi\omega_n < \delta$. Hence

$$(3) \qquad |e(\lambda t_n) - a_n| \leqslant |e\big((\lambda_0 + x_m + \ldots + x_n) t_n\big) - a_n| + |e(r_n t_n) - 1|.$$

If x_ν for $\nu < n$ are already fixed, we choose x_n in such a way that the first term on the right-hand side of (3) vanishes. Then

$$(4) \qquad |e(\lambda t_n) - a_n| < |e(\delta) - 1| \quad \text{for} \quad n \geqslant m.$$

On the other hand, for $n < m$ we have $\lambda t_n = \lambda_0 t_n + r_{m-1} t_n$, and, in view of (1), $0 \leqslant r_{m-1} t_n \leqslant r_{m-1} t_{m-1} \leqslant 2\pi\omega_m < \delta$. Thus, by (2) we get

$$(5) \qquad |e(\lambda t_n) - a_n| < |e(\lambda_0 t_n) - a_n| + |e(x t_n) - 1| < \delta + |e(\delta) - 1|.$$

In view of (4) and (5) we have (*) for δ sufficiently small.

The authors were unable to decide whether there exists a characterization of UK-sets on the line R as independent sequences which increase rapidly enough. If it is so, is then perhaps $t_{n+1}/t_n \to \infty$ the adequate condition? (**P 570**).

We can prove that $t_{n+1}/t_n > \delta > 1$ is not sufficient. In fact, if e.g. $t_n - 2^n \to 0$, then $\{t_n\}$ is no UK-set. Here is the reason: the (weak) cluster points of $\{t_n\}$ in $\tilde{R}$ are the same as those of $\{2^n\}$. But if x is such a point, then $2x$ is too, because if $x \in \{2^n\}$, then $2x \in \{2^n\}$, and $2x$ is a continuous function of x. Thus, the weak closure of $\{t_n\}$ is not independent, hence it is no Kroneckerian set, and $\{t_n\}$ is no UK-set, by (iii). That it is an $\boldsymbol{I}_0$-set, follows from Strzelecki's theorem in [11]. In the above reasoning the number 2 can visibly be replaced by any rational > 1. We do not know whether it can be replaced by any real > 1 (**P 571**).

We show still one partial result pertaining to these problems:

THEOREM 2. *If $\{t_n\}$ is a UK-set, then $\{t_{n+1} - t_n\}$ is a UK-set, too.*

Proof. For a given sequence $\{a_n\}$ ($|a_n| = 1$) we take β_n so that $|\beta_n| = 1$ and $a_n = \beta_{n+1}/\beta_n$. It is easily seen that $|e(\lambda t_n) - \beta_n| < \varepsilon/2$ ($n = 1, 2, \ldots$) implies $|e\big(\lambda(t_{n+1} - t_n)\big) - a_n| < \varepsilon$, which ends the proof.

From Theorem 2 it follows that for every UK-set $\{t_n\}$ one has $\lim\limits_n (t_{n+1} - t_n) = \infty$. In fact, otherwise the UK-set $\{t_{n+1} - t_n\}$ wouldconta in a convergent sequence against (i).

Now we pass to UK-sets groups different from the real line. Firstly, let us remark that if a discrete abelian group G is of infinite rank, then it contains an infinite UK-set. In fact, every independent set E in G is UK, since every function f on E with $|f| \equiv 1$ can be extended to a character. We pass to compact groups.

THEOREM 3. *If G is a compact abelian group of topological dimension $\geqslant 2^{\aleph_0}$, then it contains an infinite UK-set.*

Proof. The character group $\hat{G}$ contains $\geqslant 2^{\aleph_0}$ independent elements (see e.g. [9], p. 263); consequently, it contains the direct sum of $2^{\aleph_0}$ copies of the group of integers. Hence there exists a homomorphism h of G onto the cartesian product $\mathfrak{P}$ of $2^{\aleph_0}$ copies of the circle group. This product contains the Bohr compactification $\tilde{R}$ of the real line. Let Z denote an infinite UK-set in $\tilde{R}$ (the existence of such a set follows from Theorem 1). A UK-set in $\tilde{R}$ is also UK in $\mathfrak{P}$, since every character of R can be extended to a character of $\mathfrak{P}$. If we choose one point from every coset $h^{-1}(p)$ $(p \in Z)$, we get visibly an infinite UK-set in G.

If E is an $\boldsymbol{I}_0$-set in a locally compact abelian group G, then the weak closure $\tilde{E}$ in the Bohr compactification $\tilde{G}$ of G is of Haar measure 0 [3]. Yet we do not know whether E can generate (algebraically) the group $\tilde{G}$ (**P 572**). If not, this would be apparently a stronger result for G connected. It can be immediately proved for UK-sets, since their closures in $\tilde{G}$ are Kroneckerian, hence independent, and $\tilde{G}$ being a compact group has no basis (i.e. is not free abelian; see e.g. [5]).

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Reçu par la Rédaction le 26. 4. 1966