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ON TOPOLOGICAL DIVISORS OF ZERO IN p-NORMED ALGEBRAS WITHOUT UNIT

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Let A be a Banach algebra with the norm $\|\cdot\|$. An element $x \in A$, $x \neq 0$, is called a *left* (*right*) *topological divisor of zero in* A if

$$\lim \|x x_n\| = 0 \quad (\lim \|x_n x\| = 0) \tag{1}$$

for some sequence (x_n) of elements of A satisfying

$$\|x_n\| = 1, \quad n = 1, 2, \ldots \tag{2}$$

Condition (2) may be replaced by

$$\|x_n\| > \delta > 0, \quad n = 1, 2, \ldots \tag{2'}$$

An element $x \in A$ is called a *topological divisor of zero* if it is both right and left topological divisor of zero. The concept of topological divisors of zero was introduced by Šilov [2], who proved that in any Banach algebra with unit element either there are topological divisors of zero, or the algebra in question is a division algebra (i.e. either the algebra of complex numbers in the case of complex scalars or any one of the three finite dimensional division algebras over reals (reals, complexes or quaternions) in the case of real scalars). Kaplansky [1] extended this result onto algebras without unit, changing, however, the concept of topological divisors of zero: he considered in formula (1) the formal sum $1+x$ instead of x, since the product $x_n(x+1)$ makes a sense. There is no mention in the literature of the subject, whether in Banach algebras without unit elements there are topological divisors of zero in the sense of formula (1).

In this paper we are going to fill this gap. We formulate and prove the results for a wider class of topological algebras, namely for locally bounded, or p-normed algebras (see [3]), i.e. for metric algebras, where the metric $\|x-y\|$ is given by means of a p-homogeneous norm, $0 < p \leqslant 1$, i.e. of a functional $\|x\|$ satisfying the following conditions:

i. $\|x\| \geqslant 0$, and $\|x\| = 0$ if and only if $x = 0$;

ii. $\|x+y\| \leqslant \|x\| + \|y\|$;

iii. $\|xy\| \leqslant \|x\| \, \|y\|$;

iv. $\|ax\| = |a|^p \|x\|$ where a is a scalar, p — a fixed real number satisfying $0 < p \leqslant 1$.

Here we do not assume the completeness, as we did in [3].

A part of the proof presented here is analogous to that of Šilov [2]. We formulate our results for algebras over real scalars, including, of course, algebras over complex scalars.

LEMMA 1. *Let A be a metric algebra with jointly continuous multiplication. If a sequence (x_n) of elements of A is bounded, while the sequence $(y_n) \subset A$ tends to zero, then both $x_n y_n$ and $y_n x_n$ tend to zero.*

Proof. It is easy to check that for a sequence $y_n \to 0$ there exists a sequence a_n of scalars, tending to infinity, such that $a_n y_n \to 0$. We have now $x_n / a_n \to 0$ by the definition of boundedness, and $x_n y_n = a_n^{-1} x_n a_n y_n$ tends to zero as well as $y_n x_n$.

LEMMA 2. *Let A be a complete metric algebra with unit e, $x_n \in A$, $x_n \to x_0$, and let x_n be invertible in A, $n = 1, 2, \ldots$ Then x_0 is invertible in A, provided the sequence (x_n^{-1}) is bounded in A.*

Proof. For any increasing sequence k_n of integers we have $x_n x_n^{-1} - x_{k_n}^{-1} x_{k_n} = x_{k_n} - x_n \to 0$. If (x_n^{-1}) is bounded in A together with any its subsequence, then by multiplying the above sequence by x_n^{-1} from the left, and by $x_{k_n}^{-1}$ from the right, and by application of Lemma 1 we obtain $x_n^{-1} - x_{k_n}^{-1} \to 0$. So (x_n^{-1}) tends to some element y, being a Cauchy sequence in a complete space. We have $x_0 y = e = \lim x_n^{-1} x_n = y x_0$, and $y = x_0^{-1}$.

LEMMA 3. *Let A be a real p-normed algebra with unit e, and A_0 — its subalgebra such that for each $x \in A_0$, $x \neq 0$, there exists an inverse $x^{-1} \in A$. Then A_0 is a division algebra.*

Proof. Let $x \in A_0$, $x \neq 0$. It is to be shown that $x^{-1} \in A_0$. Let A_x be a commutative subalgebra of A_0 containing x, and form

$$A_1 = \{zy^{-1} \in A : y, z \in A_x, y \neq 0\}.$$

It is easy to check that A_1 is a commutative division algebra containing x. Therefore A_1 is homeomorphically isomorphic either with the field of complex numbers or with the field of reals (cf. [3]). In either case x satisfies a quadratic equation $ax^2 + bx + c = 0$ with real coefficients. Since x is invertible, we may assume $c \neq 0$, otherwise we would multiply this equation by x^{-1}. This implies that the unit e of A_1 is in $A_x \subset A_0$, and by multiplying the equation by x^{-1} we have $cx^{-1} = be - ax \in A_0$.

LEMMA 4. *Let A be a p-normed algebra, and suppose that its completion $\bar{A}$ has a unit element e. Then either A is a division algebra or A has topological divisors of zero.*

Proof. Suppose that A is not a division algebra. By Lemma 3 there is an element $y \in A$ non-invertible in $\bar{A}$. Since the set of all invertible elements in $\bar{A}$ is open in $\bar{A}$ and A is dense in $\bar{A}$, there exists in A an element z which is invertible in $\bar{A}$. This implies that on the segment connecting y and z there exists a point x, non-invertible in $\bar{A}$, which is a limit point of elements x_n which are invertible in $\bar{A}$. Clearly, $x \in A$. We shall show that x is a topological divisor of zero in A. In fact, by Lemma 2 the sequence (x_n^{-1}) has no bounded subsequence, and consequently $\lim \|x_n^{-1}\| = \infty$. We set now $y_n = x_n^{-1}/\|x_n^{-1}\|$, so $\|y_n\| = 1$ and we have

$$\|xy_n\| = \|(x-x_n+x_n)y_n\| \leqslant \|x-x_n\| \, \|y_n\| + \frac{\|e\|}{\|x_n{}^{-1}\|} \to 0,$$

and, similarly, $y_n x \to 0$. Since A is dense in $\bar{A}$, we can choose elements $z_n \in A$ in such a way that $\|y_n - z_n\| < 1/2n$, $n = 1, 2, \ldots$ We have $\|z_n\| > \frac{1}{2}$, $\|xz_n\| = \|x(z_n - x_n + x_n)\| \leqslant \|x\| \, \|z_n - x_n\| + \|xx_n\| \to 0$, and, similarly, $z_n x \to 0$. Thus x is a topological divisor of zero in A.

From this our main result follows:

THEOREM. *Let A be a p-normed algebra over real scalars. Then either A has topological divisors of zero or A is isometrically homeomorphic with one of the three finite dimensional division algebras over reals* (*reals, complexes or quaternions*).

Proof. If the completion $\bar{A}$ of A has a unit element, then our theorem follows from Lemma 4, since any p-normed division algebra is homeomorphically isomorphic with one of the three finite dimensional division algebras (cf. [3]). Assume then that there is no unit in $\bar{A}$ and denote by $\tilde{A}$ the p-normed algebra obtained from $\bar{A}$ by formal adjunction of unit element e. $\tilde{A}$ is a complete p-normed algebra with the norm $\|x+\lambda e\| = \|x\| + |\lambda|^p$. We may also assume that A is commutative, since it is sufficient to show that any commutative subalgebra of A contains topological divisors of zero unless it is a field (let us remark that if every commutative subalgebra of A is a field, then A is a division algebra).

By Lemma 4 (applied to the algebra obtained from A by formal adjunction of unit) there exist elements $x_n \in A$ and real numbers λ_n, $n = 0, 1, \ldots$, such that

$$(x_0 + \lambda_0 e)(x_n + \lambda_n e) \to 0 \quad \text{and} \quad \|x_n\| + |\lambda_n|^p = 1.$$

Hence the sequence (λ_n) is bounded and some its subsequence converges, say, to μ. We may assume therefore that

$$(x_0 + \lambda_0 e)(x_n + \mu e) \to 0.$$

If $\lambda_0 = \mu = 0$, then there are topological divisors of zero in A. So assume $|\lambda_0|+|\mu| > 0$. We have $x_0 x_n + \mu x_0 + \lambda_0 x_n + \lambda_0 \mu e \to 0$. Hence one of the numbers λ_0 and μ equals zero, otherwise $e \in \bar{A}$ contrary to our assumption. Consider therefore two cases:

1° If $\lambda_0 = 0$, we have $x_0(x_n+\mu e) \to 0$. If (x_n) converges in $\bar{A}$, say to y, we have $x_0(y+\mu e) = 0$, and for some $z \in \bar{A}$ we have $z(y+\mu e) \neq 0$ (otherwise $-y/\mu$ would be a unit element in $\bar{A}$). Therefore there exists a sequence (z_n) of elements of A tending to $z(y+\mu e)$. There exists a $\delta > 0$ such that for large n we have $\|z_n\| > \delta$. Obviously $\lim x_0 z_n = 0$, so x_0 is a topological divisor of zero in A. If (x_n) diverges in $\bar{A}$, then for any increasing sequence of integers k_n we have $x_0(x_n - x_{k_n}) = x_0(x_n+\mu e) - x_0(x_{k_n}+\mu e) \to 0$, and there exists such a sequence k_n that $\|x_n - x_{k_n}\| > \delta > 0$. Thus also in this case there are topological divisors of zero.

2° If $\mu = 0$, then $(x_0+\lambda_0 e)x_n \to 0$ and $\|x_n\| \to 1$. There exists, as before, an element $y \in A$ such that $z = y(x_0+\lambda_0 e) \neq 0$. So $z \in A$ and $\lim x_n z = 0$. Thus also in this case there are topological divisors of zero and our theorem is proved.

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