COLLOQUIUM MATHEMATICUM

VOL. XXV 1972 FASC. 2

REMARKS ON INFINITE PRODUCTS OF FINITELY ADDITIVE MEASURES

BY

LILIANA WAJDA (WROCŁAW)

The present paper contains some remarks about infinitely direct (i.e., satisfying formula (2)) products of measures. By a *measure* we mean here a normed (finitely) additive non-negative set function on a field of sets.

Theorem 1 of this paper has been proved by Łomnicki and Ulam (see [3], p. 256). Our proof is based on a different idea and uses the following theorem of Pettis [5] (see also Kisyński [1] and Lipecki [2]):

(i) Every strictly additive (i.e., satisfying formula (3)) non-negative set function on an additive and multiplicative family of sets can be uniquely extended to an additive non-negative set function on the ring generated by this family.

We use the following notation:

$$\underline{\mu}(Z) = \sup\{\mu(A) \mid A \subset Z, A \in \boldsymbol{K}\}, \qquad \overline{\mu}(Z) = \inf\{\mu(A) \mid A \supset Z, A \in \boldsymbol{K}\}.$$

Theorem 2 will be proved with the help of the following theorem formulated by Marczewski and Łoś in [4]:

(ii) If $\boldsymbol{K}$ is a field of sets, μ — a measure on $\boldsymbol{K}$, $Z \notin \boldsymbol{K}$, and $[\boldsymbol{K}, Z]$ denotes the field generated by $\boldsymbol{K}$ and Z, then there exists a measure ν on $[\boldsymbol{K}, Z]$ such that $\nu(Z) = \xi$, where ξ is an arbitrary value satisfying inequality $\underline{\mu}(Z) \leqslant \xi \leqslant \overline{\mu}(Z)$.

Let X_i, $\boldsymbol{K}_i$, μ_i $(i = 1, 2, \ldots)$ denote an arbitrary set, a field of subsets of this set and a measure on $\boldsymbol{K}_i$, respectively. Let $\mathscr{K}$ be the family of infinite products $(A_1 \times A_2 \times \ldots)$, where $A_i \in \boldsymbol{K}_i$, for each i and $\boldsymbol{K}$ is the field of sets generated by $\mathscr{K}$. By $\boldsymbol{K}_0$ we denote the smallest field containing all sets of the form $(A_1 \times A_2 \times \ldots)$, where $A_i = X_i$ for almost every $i = 1, 2, \ldots$ Obviously, $\boldsymbol{K}_0 \subset \boldsymbol{K}$. It will be also convenient to use the notation

$$\pi_k\colon \prod_{i=1}^{\infty} X_i \to \prod_{i=1}^{k} X_i, \qquad \text{where } \pi_k(x_1, x_2, \ldots) = (x_1, \ldots, x_k).$$

From the axiom of choice and properties of set theoretical operations we infer the following

LEMMA. *For arbitrary* $A = \bigcup_{i=1}^{m} A^i$ *and* $B = \bigcup_{j=1}^{n} B^j$, *where* $A^i, B^j \in \mathscr{K}$, *there exists* k_0 *such that*

$$\pi_k(A \cap B) = \pi_k A \cap \pi_k B \quad \textit{for } k \geqslant k_0.$$

It is well known that, for arbitrary measures μ_i on fields $\boldsymbol{K}_i$, there exists a unique measure μ_0 on $\boldsymbol{K}_0$ such that for $(A_1 \times \ldots \times A_k \times X_{k+1} \times \ldots) \in \boldsymbol{K}_0$, where $k = 1, 2, \ldots$, we have

$$(1) \qquad \mu_0(A_1 \times \ldots \times A_k \times X_{k+1} \times \ldots) = \mu_1(A_1) \ldots \mu_k(A_k).$$

A stronger theorem is also true.

THEOREM 1. *For arbitrary* (*normed finitely additive*) *measures* μ_i *on fields* $\boldsymbol{K}_i$ *there exists the unique measure* μ *on* $\boldsymbol{K}$ *such that*

$$(2) \qquad \mu(A_1 \times A_2 \times \ldots) = \mu_1(A_1)\mu_2(A_2) \ldots \quad \textit{for every } A_j \in \boldsymbol{K}_j.$$

Proof. The family of sets

$$\mathscr{K}_s = \{\bigcup_{i=1}^{m} A^i \mid A^i \in \mathscr{K},\ m = 1, 2, \ldots\}$$

is additive and multiplicative, and for $A \in \mathscr{K}_s$ the function

$$\mu(A) = \lim_{k \to \infty} \mu_0(\pi_k^{-1} \pi_k A)$$

is well defined on $\mathscr{K}_s$.

In virtue of lemma, we have

$$(3) \qquad \mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B) \quad \text{for } A, B \in \mathscr{K}_s.$$

By Pettis theorem (i) we extend μ to the unique measure on the field generated by $\mathscr{K}$. It is easy to see that μ after extension is the unique measure satisfying (2).

We will use the following notation:

$$\boldsymbol{K}_0^{\mu_0} = \Big\{A \mid A \subset \prod_{i=1}^{\infty} X_i,\ \underline{\mu}_0(A) = \overline{\mu}_0(A)\Big\}.$$

It is interesting to know whether and under what conditions measure μ_0 can be uniquely extended to a measure on $\boldsymbol{K}$ satisfying (1). The answer to this question is given by

THEOREM 2. *There exist two different measures on* $\boldsymbol{K}$ *satisfying* (1) *if and only if*

$$(4) \qquad \limsup \big(\sup\{\mu_n(Z) \mid Z \in \boldsymbol{K}_n,\ Z \neq X_n\}\big) = 1.$$

Proof. Sufficiency. If condition (4) is satisfied, then there exists a sequence $\{A_{i_k}\}$ such that $X_{i_k} \neq A_{i_k}$, $A_{i_k} \in \boldsymbol{K}_{i_k}$ and $\mu_{i_k}(A_{i_k}) \geqslant 1-1/(k+1)^2$. For i non belonging to the sequence $\{i_k\}$ we put $A_i = X_i$. Let $A = \prod_{i=1}^{\infty} A_i$. Then $\underline{\mu}_0(A) = 0$ and $\overline{\mu}_0(A) \geqslant 1/2$, and therefore $A \in \mathscr{K} \setminus \boldsymbol{K}_0^{\mu_0}$. Hence by (ii) there exist two different measures ν_1 and ν_2 on $\boldsymbol{K}_0^{\mu_0}$ which are extensions of μ_0.

Necessity. It is easy to see that if (4) is not satisfied, then $\overline{\mu}_0(A) = 0$ for every $A = A_1 \times A_2 \times \ldots$ such that $A_i \in \boldsymbol{K}_i$ and $A_i \neq X_i$ for infinitely many i. Therefore $\mathscr{K} \subseteq \boldsymbol{K}_0^{\mu_0}$ and, consequently, $\boldsymbol{K} \subseteq \boldsymbol{K}_0^{\mu_0}$. It is obvious that measure μ_0 on $\boldsymbol{K}_0$ can be uniquely extended to a measure on $\boldsymbol{K}_0^{\mu_0}$.

Condition (4) is satisfied, e.g., if

(a) measures μ_i are two-valued and $\boldsymbol{K}_i$ are not trivial,

(b) measures μ_i vanish on singletons and $\boldsymbol{K}_i$ are not trivial,

(c) $\boldsymbol{K}_i = 2^{X_i}$ and X_i are infinite.

REFERENCES

[1] J. Kisyński, *Remark on strongly additive set functions*, Fundamenta Mathematicae 63 (1968), p. 327-332.

[2] Z. Lipecki, *On strongly additive set functions*, Colloquium Mathematicum 22 (1971), p. 255-256.

[3] Z. Łomnicki et S. Ulam, *Sur la théorie de la mesure dans les espaces combinatoires et son application au calcul des probabilités*, Fundamenta Mathematicae 23 (1934), p. 237-278.

[4] J. Łoś and E. Marczewski, *Extensions of measures*, ibidem 36 (1949), p. 267-276.

[5] B. J. Pettis, *On the extension of measures*, Annals of Mathematics 54 (1951), p. 186-187.

Reçu par la Rédaction le 15. 7. 1971