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ON THE OVERLAP OF A FUNCTION WITH THE TRANSLATION OF ITS COMPLEMENT

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The following is a generalization of a problem proposed by Erdös [1]: Let the integers from 1 to n be separated into two (disjoint) sets

$$A: a_1 < a_2 < \ldots < a_k, \quad B: b_1 < b_2 < \ldots < b_{n-k}.$$

If

$$M_r = \sum_{a_i - b_j = r} 1, \quad M = M(n; k) = \min_{A,B} \max_{r} M_r,$$

find or estimate M.

In the problem originally proposed by Erdös, n was even and $k = n/2$. In this case, Erdös [2] proved $M \geqslant \cdot 25k$, which was improved by P. Scherk to

$$M > \left(1 - \frac{1}{\sqrt{2}}\right) k > \cdot 2929k$$

and later by Moser [4] to

$$M > \frac{\sqrt{2}}{4}(k-1) > \cdot 3535(k-1).$$

Moser further mentions that by combining his method with that of Scherk, one can get

$$M > \sqrt{4 - \sqrt{15}}(k-1) \doteqdot \cdot 3563(k-1).$$

Świerczkowski [6] considered the corresponding problem for point-sets: If X is a measurable point-set on $[0, 1]$ with Lebesgue measure a, and $Y = [0, 1] - X$, let $X_t = \{x + t \mid x \in X\}$ and

$$\mathscr{M}(t) = m(X_t \cap Y), \quad M = \sup \mathscr{M}(t), \quad \lambda_p = \inf_X M.$$

Świerczkowski [6] proved that

$$\lambda_p \geqslant \{2-\sqrt{4-10a(1-a)}\}/5. \tag{1}$$

Point-sets correspond to characteristic functions. We now generalize the problem to more general real-valued functions.

Let

$$f\epsilon L[0,1], \quad \int_0^1 f(x) = a, \quad 0 \leqslant f(x) \leqslant 1,$$

and let g denote the "complementary" function, i. e., $g(x) = 1-f(x)$, when $x\epsilon[0,1]$.

We define f and g to be zero outside the interval $[0,1]$.

Let

$$\mathscr{M}(t) = \mathscr{M}(f;t) = \int_0^1 f(x)g(x+t)\,dx,$$

$$M = M(f) = \sup_t \mathscr{M}(t), \quad \lambda_F = \inf_f M.$$

In this note we will prove that

$$\lambda_F \geqslant a(1-a)/\sqrt{4-5a+2a^2}$$

and, as a consequence,

$$\lambda_p \geqslant a(1-a)/\sqrt{4-5a+2a^2},$$

which is an improvement over Świerczkowski's result (1). The method used in the present note can be applied, mutatis mutandis, to the number-theoretic case to obtain

$$M \geqslant k(n-k)/\sqrt{4n^2-5nk+2k^2} \tag{2}$$

which again is sharper than the result obtained by Świerczkowski for this case, namely,

$$M \geqslant \{2n-\sqrt{4n^2-10k(n-k)}\}/5. \tag{3}$$

We now proceed to prove our result. We need some lemmas first.

LEMMA 1. *$\mathscr{M}(t)$ is a continuous, and hence integrable, function of t, and*

$$\int_{-1}^1 \mathscr{M}(t)\,dt = a(1-a).$$

Proof. We have

$$\mathscr{M}(t+h)-\mathscr{M}(t) = \int_0^1 f(x)[g(x+t+h)-g(x+t)]\,dx.$$

Hence

$$|\mathscr{M}(t+h)-\mathscr{M}(t)| \leqslant \int_0^1 |g(x+t+h)-g(x+t)|\,dx.$$

The right-hand side tends to 0 as $h \to 0$, by the Mean-continuity property.

$$\int_{-1}^1 \mathscr{M}(t)\,dt = \int_{-1}^1 dt \int_0^1 f(x)g(x+t)\,dx = \int_0^1 f(x)\,dx \int_{-1}^1 g(x+t)\,dt$$

$$= \int_0^1 f(x)\,dx \int_0^1 g(u)\,du = \alpha(1-\alpha).$$

The change in the order of integration is easily justified.

We now define

$$\bar{f} = \frac{1}{\alpha}\int_0^1 xf(x)\,dx, \qquad \bar{g} = \frac{1}{\beta}\int_0^1 xg(x)\,dx, \qquad \bar{h} = \frac{1}{\alpha\beta}\int_{-1}^1 x\mathscr{M}(x)\,dx,$$

$$V(f) = \frac{1}{\alpha}\int_0^1 (x-\bar{f})^2 f(x)\,dx,$$

$$V(g) = \frac{1}{\beta}\int_0^1 (x-\bar{g})^2 g(x)\,dx,$$

$$V(h) = \frac{1}{\alpha\beta}\int_{-1}^1 (x-\bar{h})^2 \mathscr{M}(x)\,dx,$$

where $\beta = 1-\alpha$.

Lemma 2. $\bar{h} = \bar{g}-\bar{f}$.

Lemma 3. $V(h) = V(f)+V(g)$.

These lemmas follow from the definitions of $\bar{f}$, $V(f)$ etc. by obvious manipulations of integrals.

Lemma 4. *We have*

(i) $$\int_0^1 (x-\tfrac{1}{2})^2 g(x)\,dx \geqslant \frac{\beta^3}{12},$$

(ii) $$V(h) \geqslant \frac{\alpha^2\beta^2}{12M^2}.$$

The results are intuitively obvious since the second moment about a point is minimized when the function is concentrated around that point; i. e., when

(i) $$g(x) = \begin{cases} 1, & x \in \left[\frac{1}{2} - \frac{\beta}{2}, \frac{1}{2} + \frac{\beta}{2}\right], \\ 0 & \text{elsewhere,} \end{cases}$$

and

(ii) $$\mathscr{M}(t) = \begin{cases} M, & t \in [\bar{h} - \alpha\beta/M,\ \bar{h} + \alpha\beta/M], \\ 0 & \text{elsewhere.} \end{cases}$$

THEOREM 1. *Suppose, without loss of generality, that* $\alpha \leqslant \beta$. *Then*

(4) $$\lambda_F \geqslant \alpha(1-\alpha)/\sqrt{4-5\alpha+2\alpha^2}.$$

Proof.

$$\begin{aligned} \alpha\beta\{V(f)+V(g)\} &= \beta\int_0^1 (x-\bar{f})^2 f(x)\,dx + \alpha\int_0^1 (x-\bar{g})^2 g(x)\,dx \\ &\leqslant \beta\int_0^1 (x-\tfrac{1}{2})^2 f(x)\,dx + \alpha\int_0^1 (x-\tfrac{1}{2})^2 g(x)\,dx \\ &= \beta\int_0^1 (x-\tfrac{1}{2})^2[f(x)+g(x)]\,dx - (\beta-\alpha)\int_0^1 (x-\tfrac{1}{2})^2 g(x)\,dx \\ &= \beta\int_0^1 (x-\tfrac{1}{2})^2\,dx - (\beta-\alpha)\int_0^1 (x-\tfrac{1}{2})^2 g(x)\,dx \\ &\leqslant \beta/12 - (\beta-\alpha)\beta^3/12. \end{aligned}$$

But $\alpha\beta V(h) \geqslant \alpha^3\beta^3/12M^2$. Hence, $\alpha^3\beta^3/12M^2 \leqslant \beta/12 - (\beta-\alpha)\beta^3/12$, i. e.

$$M \geqslant \alpha(1-\alpha)/\sqrt{4-5\alpha+2\alpha^2}.$$

In order to compare this result with that of Świerczkowski, we observe that $\lambda_p \geqslant \lambda_F$ and hence Theorem 1 implies

(5) $$\lambda_p \geqslant \alpha(1-\alpha)/\sqrt{4-5\alpha+2\alpha^2}.$$

Further, by a straight forward but tedious algebra, one can show that

$$\alpha(1-\alpha)/\sqrt{4-5\alpha+2\alpha^2} > \{2-\sqrt{4-10\alpha(1-\alpha)}\}/5$$

for $0 < \alpha \leqslant 1/2$. Hence the present result is an improvement over that of Świerczkowski.

To obtain an upper bound for λ_F, consider the function

$$f(x) = \begin{cases} 2a - \boldsymbol{a}, & x \in [\frac{1}{3}, \frac{2}{3}], \\ a, & x \in (\frac{1}{6}, \frac{1}{3}) \cup (\frac{2}{3}, \frac{5}{6}), \\ \boldsymbol{a}, & x \in [0, \frac{1}{6}] \cup [\frac{5}{6}, 1], \end{cases}$$

where $\boldsymbol{a}$ is given by $0 \leqslant \boldsymbol{a} \leqslant a$ and $6a\boldsymbol{a} - 3\boldsymbol{a}^2 = 4a^2 - a$. This function shows that

$$\lambda_F \leqslant (2a - \boldsymbol{a})(1 - a - \boldsymbol{a})/3 + (a + \boldsymbol{a})/6. \tag{6}$$

Formula (6) is useful only for values of a in a "small" neighbourhood of $\frac{1}{2}$. For other values, it gives a rather poor bound.

For the particular case $a = \frac{1}{2}$, Moser [5] has found a function to show that

$$\lambda_F \leqslant \cdot 1933.$$

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