COLLOQUIUM MATHEMATICUM

XIV WARSZAWA—WROCŁAW 1966

A GENERAL EXISTENCE THEOREM ON PARTIAL ALGEBRAS AND ITS SPECIAL CASES

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In this communication*, a general existence theorem is given which essentially is not new, but is meant to unify different existence theorems of General Algebra — like the existence theorem of Birkhoff [1] for free algebras in primitive classes of full algebras, like various existence theorems for "free products" of full algebras — and to extend them as far as possible *within purely algebraic patterns*. It contains as a special case the most comprehensive algebraic existence theorem given so far, the existence theorem of Słomiński [25] for the "free" (universal) homomorphism of a given *partial* algebra A into an algebra belonging to a given primitive class $\mathfrak{B}$ of *full* algebras. As Słomiński has already shown, it is useful to consider partial algebras even if one is mainly concerned with full algebras. So it seems natural to ask for a more symmetric generalization of Słomiński's theorem that admits partial algebras to class $\mathfrak{B}$ too. This will only be possible by a complete change of proof, since Słomiński's proof makes essential use of congruence relations in certain full algebras, i. e. sets of equations for full algebras. As a matter of fact, our proof is obtained by careful reexamination — from the view point of partial algebras — of the original proof of Birkhoff's existence theorem quoted above, running nearly completely along category theoretical lines.

In fact, from the view point of a more general theory of abstract structures (including partial algebras as a special case), as well as from the view point of abstract categories, further generalizations of our algebraic existence theorem are possible and well known. Nevertheless, General Algebra — the necessity of which will not be discussed at a conference on General Algebra — undeniably demands a general

* Presented to the Conference on General Algebra, held in Warsaw, September 7-11, 1964.

existence theorem within its own purely algebraic domain, interesting for its own sake, which avoids the great — not to say ugly — complications of a general notion of abstract "structures" as given by Bourbaki [3], and makes unlimited use of such algebraic notions that cannot immediately be translated into the language of a single abstract category or of functors between abstract categories.

Hence, being thorough aware of the features of more and most abstract categories, we voluntarily limit ourselves to the relatively concrete category of partial algebras of a certain type which enables us to make use of its special non-categorical features as often we want to for comfort's sake. The result, in any case, is an extremely general algebraic theorem that may be considered satisfying in itself because of the wide extend of its immediate and important special cases. A theorem of this kind ought to be contained in any future text book or monograph on General Algebra.

1. Generalities on partial algebras. Let A, K be arbitrary sets. A *sequence of type K in A* is a mapping $\mathfrak{a} = (a_\varkappa)_{\varkappa \in K} : K \to A$, i. e. an element $\mathfrak{a} \in A^K$. A *(partial) operation of type K in A* is a mapping $f : D \subseteq A^K \to A$, i. e. an element $f \in A^D$, where its *domain* D is some set of sequences of type K; applying operation f to some sequence $\mathfrak{a} = (a_\varkappa)_{\varkappa \in K} \in D$, we may write

$$f(\mathfrak{a}) = f(a_\varkappa \mid \varkappa \in K)$$

or something like that. In the special case $D = A^K$, f is a *full operation*. In the opposite extreme case $D = \emptyset$, f is the *empty operation* (*of type K*), which will be very useful in the sequel. The general case of a partial operation of type K then lies between these two extremes. In the special case of type $K = \emptyset$ however, there really are no further partial operations of this type; for one has one and only one sequence of type $K = \emptyset$, the *empty sequence* $\mathfrak{o}$, $A^\emptyset = \{\mathfrak{o}\}$, hence only $D = \{\mathfrak{o}\}$, i. e. the full operations f of type $\emptyset$, corresponding one-one to and usually identified with the elements $a = f(\mathfrak{o}) \in A$, then named *constants*, and $D = \emptyset$, i. e. the empty operation of type $\emptyset$, are possible.

Let I be another set, the *index set*, $\Delta = (K_i)_{i \in I}$ a family of sets, the *type*. A family $(f_i)_{i \in I}$ of operations f_i of types K_i in A will be called a *(partial) algebraic structure of type Δ in A*, the couple $(A, (f_i)_{i \in I})$ is a *(partial) algebra of type Δ*, with *fundamental set A*, *fundamental operations f_i*. An algebraic structure, an algebra is *full* if so are all fundamental operations. Having in mind a definite algebraic structure $(f_i)_{i \in I}$, we often speak of "algebra A" instead of "algebra $(A, (f_i)_{i \in I})$", being well aware of possible misunderstandings which occasionally have to be avoided by careful discussion.

A *homomorphism* of algebra $(A, (f_i)_{i\in I})$ into algebra $(B, (g_i)_{i\in I})$ — both of the same type, i. e. *similar* — is a mapping $\varphi : A \to B$ such that

$$\varphi\big(f_i(a_\varkappa \mid \varkappa \in K_i)\big) = g_i\big(\varphi(a_\varkappa) \mid \varkappa \in K_i\big) \tag{1}$$

for all sequences $\mathfrak{a} = (a_\varkappa)_{\varkappa\in K_i}$ in the domain of f_i, i. e. such that $f_i(a_\varkappa \mid \varkappa \in K_i)$, hence the left side of (1) exists: then the right side of (1) is meant to exist according to definition of homomorphisms.

One sometimes has to consider two similar algebraic structures $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$ in set A. Then $(f_i)_{i\in I}$ is *weaker* or *poorer* than $(g_i)_{i\in I}$, $(g_i)_{i\in I}$ *stronger* or *richer* than $(f_i)_{i\in I}$ if, for each index $i \in I, f_i : D_i \subseteq A^{K_i} \to A$ is a restriction of $g_i : E_i \subseteq A^{K_i} \to A$, i. e. if and only if the identical mapping $\mathrm{id}_A : A \to A$ is a homomorphism of algebra $(A, (f_i)_{i\in I})$ into algebra $(A, (g_i)_{i\in I})$. The "weaker-stronger"-terminology being an obvious analogue of the terminology introduced by Alexandroff and Hopf for the comparison of topologies ([1]), the "poorer-richer"-terminology reminds one of the possibility that, for instance, all operations g_i may be full, some of them coinciding with the operations f_i, while the remaining restricted operations may be empty: so in the poorer structure $(f_i)_{i\in I}$, there may be less fundamental operations "that really matter" than in the richer structure $(g_i)_{i\in I}$, yet for practical reasons, both formally are of the same type Δ.

It will be even useful to consider the extreme case of the *weakest, poorest*, or *discrete algebraic structure of type Δ in A* all fundamental operations of which are empty; hence, *any abstract set A may be considered as discrete algebra of prescribed type Δ*. If set A is of cardinal number $|A| \geqslant 2$, there is no strongest algebraic structure of type Δ in A, but the full structures are precisely those which cannot be strengthened, i. e. the maximal elements of the ordered set of algebraic structures of type Δ in A. In a one-element set A however, the strongest or richest algebraic structure of type Δ exists and is full; these *full one-element algebras of type Δ* play a special rôle in our considerations by the fact that these are precisely those partial algebras A of type Δ such that for each partial algebra B of type Δ there is one and only homomorphism $\varphi : B \to A$, i. e. in the category of all partial algebras of type Δ together with all homomorphisms, the full one-element algebras are precisely the objects called "right zero objects" by Isbell ([10], p. 25). Let us finally note that in the empty set $A = \emptyset$, the only algebraic structure of type Δ is the discrete one, which is to be considered full if and only if type Δ is *without constants*, i. e. all sets K_i are non-empty; this *empty algebra of type Δ* is the only partial algebra of type Δ such that

([1]) The topological terminology of Bourbaki, "finer" and "coarser" (moins fin), does not appear so suitable here.

for each partial algebra B of type Δ there exists at least one homomorphism $\varphi : A \to B$, which then is unique, namely the empty homomorphism, i. e. the empty algebra is the only "left zero object" (even in a very strong sense) in the category of all partial algebras of type Δ.

2. Relative algebras, direct products, partial direct sums. This comparison of algebraic structures can be used to define the standard concepts of subalgebras, direct products, etc.

First, let B be an arbitrary subset of algebra $(A, (f_i)_{i\in I})$ (i. e. of its fundamental set A). Then there exists the strongest algebraic structure $(g_i)_{i\in I}$ on B such that the inclusion mapping $i : B \to A$ is a homomorphism, g_i being the restriction

$$g_i = f_i \cap (B^{K_i} \times B). \tag{2}$$

$(B, (g_i)_{i\in I})$ is the *relative algebra of algebra* $(A, (f_i)_{i\in I})$ *associated with subset* B, which one usually identifies ("par abus de langage") with subset B itself, its *relative algebraic structure* $(g_i)_{i\in I}$ being an obvious analogue of relative topology; besides, algebra $(A, (f_i)_{i\in I})$ is often called an *extension* of relative algebra $(B, (g_i)_{i\in I})$. Beyond its definition, one has the following stronger property of this relative algebra: let $(C, (h_i)_{i\in I})$ be an arbitrary similar algebra, let φ be an arbitrary mapping of set C into subset $B \subseteq A$, then φ is a homomorphism of algebra $(C, (h_i)_{i\in I})$ into relative algebra $(B, (g_i)_{i\in I})$ if and only if $i\cdot\varphi\ (=\varphi)$ is a homomorphism from $(C, (h_i)_{i\in I})$ into algebra $(A, (f_i)_{i\in I})$.

If, in particular, B is a *closed subset*, $f_i(B^{K_i}) \subseteq B$ for each index $i\in I$, then this relative algebra on B or subset B itself is usually called a *subalgebra* [2]. It is easy to see that a partial algebra $(B, (g_i)_{i\in I})$ is full if and only if it is a subalgebra of, i. e. closed in each extension $(A, (f_i)_{i\in I})$.

For an arbitrary subset M (possibly empty) of algebra $(A, (f_i)_{i\in I})$, there exists the least subalgebra containing M, the *subalgebra generated by* M (*closed hull, closure of* M), $\mathrm{C}M = \overline{M}$. As an important clue, we state

THEOREM 1. *For any cardinal number* $\mathfrak{m}$, *there is a cardinal number* $\overline{\mathfrak{m}}$ *such that, for each partial algebra B of type Δ generated by a set M of cardinal number* $|M| \leqslant \mathfrak{m}$,

$$|B| = |\overline{M}| \leqslant \overline{\mathfrak{m}}. \tag{3}$$

It is in the proof of this theorem that a greater or less portion of pure set theory, i. e. non-algebraic arguments come into General Algebra. For the proof of Theorem 1 (which we do not intend to give here), there are two main possibilities. If one wants a more algebraic proof,

(2) In topology, "relative space" and "subspace" have the same meaning.

one has to construct a very special full algebra $(A_0, (f_i)_{i\epsilon I})$ of type Δ generated by a set M_0 of exact power $\mathfrak{m}$, in which the following *Generalized Peano Axioms* hold true:

P1. $f_i(\mathfrak{a}) \notin M_0$, for any index $i \epsilon I$ and any sequence $\mathfrak{a} \epsilon A_0^{K_i}$;

P2. $f_i(\mathfrak{a}) = f_j(\mathfrak{b})$ implies $i = j$ and $\mathfrak{a} = \mathfrak{b}$, for any $i, j \epsilon I, \mathfrak{a} \epsilon A_0^{K_i}$, $\mathfrak{b} \epsilon A_0^{K_j}$.

(As a matter of fact, together with the *Axiom of "complete"* or *Algebraic Induction*),

P3. $\overline{M}_0 = A_0$,

these axioms constitute an obvious generalization of the classical Peano axioms for natural numbers.) Thus having secured the existence of this *Peano algebra of type Δ generated by M_0* (3) one shows that M_0 is an *absolutely free* or *absolutely independent subset*, i. e. that any mapping ("valuation") β of M_0 into a full algebra B of type Δ can be — necessarily uniquely — extended to a homomorphism φ of algebra $\overline{M}_0 = A_0$ into algebra $B : \varphi$ is nothing but the subalgebra of the direct product $A_0 \times B$ generated by $\beta, \varphi = \overline{\beta}$ (4). By this latter generalization of the classical *Recursion Theorem* securing the existence of recursively defined functions of natural numbers, choosing $\overline{\mathfrak{m}} = |A_0|$, we first obtain (3) for full algebras B, then, by a simple additional argument, for arbitrary partial algebras B. This more algebraic proof of Theorem 1 has the obvious advantage of making the upper bound $\overline{\mathfrak{m}}$ in inequality (3) exact, still it does not give an arithmetical formula for this exact $\overline{\mathfrak{m}}$, On the other hand, not minding a more set-theoretical proof, one may obtain, for an arbitrary partial algebra, a deeper insight into the construction ("from below", whereas its simple definition was "from above") of subalgebra $\overline{M}$ by its usual stepwise exhaustion by the so-called *Baire* or *Borel classes*, beginning with the generating subset M itself as the class of rank zero, and running through all ordinal *rank numbers* up to a certain *ordinal dimension* number associated with type Δ; with the

(3) Intuitive descriptions of this Peano algebra have been given by Shoda [20], § 3, [21], § 4, and by Dörge [6]. For an analogous exact recursive construction, cf. e. g. Słomiński [24], chap. III (1.1). In the case of finitary fundamental operations, another recursive construction follows the lines of metamathematics, namely the usual recursive definition of well-formed formulas of a formal language without brackets (Łukasiewicz); here the proofs of P1 and P2 become more complicated, using balance criterions for well-formed formulas as given by Gerneth and Bourbaki. Just recently, the shortest possible, a very direct construction of Peano algebras — which does no longer use any recursion — has been discovered by Kerkhoff [12].

(4) This nice algebraic formulation and proof of the Recursion Theorem goes back to Lorenzen, and has been extended to arbitrary Peano algebras by Diener (this volume, p. 63 - 72) and the author. The proof given by Słomiński [24], chap. III, (1.3), is false.

help of transfinite induction on these rank numbers, one obtains a more or less exact arithmetical expression for our upper bound $\overline{\mathfrak{m}}$ (5). Let us note that, for our purpose, only the existence of upper bound $\overline{\mathfrak{m}}$ is needed, not its best possible value; this may serve to simplify the second sort of proof.

Continuing the definitions of standard concepts, we consider a family of algebras $(A_t, (f_{ti})_{i \in I})$ $(t \in T)$. Let $A = \mathbf{P} A_t$ be the usual direct product of sets A_t. Then there exists the strongest algebraic structure $(f_i)_{i \in I}$ on A such that all natural projections $p_t : A \to A_t$ — defined by $p_t(a) = a(t)$ for all $a \in A$ — are homomorphisms, *product operation* f_i being defined by

$$p_t\big(f_i(a_\varkappa \mid \varkappa \in K_i)\big) = f_{ti}\big(p_t(a_\varkappa) \mid \varkappa \in K_i\big), \tag{4}$$

where the argument on the left side exists if and only if so does the right side for all $t \in T$. From the view point of categories, this *direct product* $(A, (f_i)_{i \in I})$ *of algebras* $(A_t, (f_{ti})_{i \in I})$, an analogue of the weak product of topological spaces, has the property (generalizing a part of the definition): for each partial algebra B of type Δ and for each ("coinitial") family of homomorphisms $\varphi_t : B \to A_t$ $(t \in T)$, there is one and only one homomorphism $\varphi : B \to A$ such that $p_t \cdot \varphi = \varphi_t$ for all $t \in T$, i. e. the family of diagrams

$$\begin{tikzcd} B \arrow[rr, dotted, "\varphi"] \arrow[dr, "\varphi_t"'] & & A \arrow[dl, "p_t"] \\ & A_t & \end{tikzcd}$$

can be filled in by a unique φ (not depending on index t) to make them all commutative. This property of algebra A together with the family of homomorphisms p_t can be considered as an axiomatic definition of *abstract direct product* (which is unique up to — unique — isomorphism); the above construction of a *concrete direct product* may be considered as the proof of existence of direct products in the entire category of partial algebras of type Δ.

In the sequel, it will be comfortable to admit the *empty family* of algebras A_t, i. e. $T = \emptyset$. The direct product A of sets A_t then consists of precisely one element, the *empty choice function*, and our general definition of product algebra turns this particular one-element set A into the corresponding full algebra of type Δ; this will be the *concrete direct product of the empty family* of algebras of type Δ. More generally,

(5) Proofs of this kind have been given by Słomiński [24], chap. II, § 4, Christensen-Pierce [5], lemma 1.1, and the author [18].

the *abstract direct products of the empty family* will be all full one-element algebras of type Δ (which are the isomorphic copies of the particular full one-element algebra considered above), i. e. the right zero objects in the category of partial algebras of type Δ.

Still in this category, as is not so well known, the dual of direct products exists. Let us again consider an arbitrary family of algebras $\big(A_t, (f_{ti})_{i\in I}\big)$ $(t\in T)$. Let $A = \mathrm{S}\, A_t$ be the direct sum of sets A_t as introduced by Whitehead and Russell,

$$\underset{\epsilon T}{\mathrm{S}}\, A_t = \{(t, a) \mid t\in T,\ a\in A_t\},$$

which is used for the definition of addition of cardinal numbers by means of arbitrary representative sets A_t as the direct product $\mathbf{P}A_t$ is used for multiplication. Then there exists the weakest algebraic structure $(f_i)_{i\in I}$ on A such that all natural injections $i_t: A_t \to A$ — defined by $i_t(a) = (t, a)$ — are homomorphisms, *sum operation* f_i being defined by

$$f_i\big(i_t(a_\varkappa) \mid \varkappa\in K_i\big) = i_t\big(f_{ti}(a_\varkappa \mid \varkappa\in K_i)\big), \tag{5}$$

where, for any $t\in T$, the left side exists if and only if so does the right one, which, in particular, means that f_i only operates on such sequences all members $(t_\varkappa, a_\varkappa)$ of which belong to the same index $t_\varkappa = t\in T$, i. e. to the same equivalence class $i_t(A_t) = \{t\}\times A_t$ of direct sum A. Again, this *partial direct sum* $\big(A, (f_i)_{i\in I}\big)$ *of algebras* $\big(A_t, (f_{ti})_{i\in I}\big)$ (6) an analogue of the "topological" sum of topological spaces, has the property (generalizing a part of the definition): for each partial algebra B of type Δ and for each ("coterminal", "cofinal") family of homomorphisms φ_t: $A_t \to B$ $(t\in T)$, there is one and only one homomorphism $\varphi: A \to B$ such that $\varphi\cdot i_t = \varphi_t$ for all $t\in T$, i. e. the family of diagrams

$$\begin{tikzcd} B & & A \arrow[ll, dotted, "\varphi"'] \\ & A_t \arrow[ul, "\varphi_t"] \arrow[ur, "i_t"'] & \end{tikzcd}$$

can be filled in by a unique φ (not depending on index t) to make them all commutative. This property of algebra A together with the family of homomorphisms i_t can be considered as the axiomatic definition of *abstract direct sum* (which is unique up to — unique — isomorphism) in the category of all partial algebras of type Δ, which exists due to the above unlimited construction of a *concrete direct sum*. (Let us note that

(6) The partial direct sum has been introduced by Słomiński [25], p. 30, under the name "direct sum"; it has been discovered independently by the author.

one very often limits oneself to the special case of pairwise disjoint summands A_t, as has been done by Cantor and repeated again and again; in this case, one usually replaces the general construction of Whitehead and Russell by taking as set A the ordinary union of sets A_t, as mappings $i_t: A_t \to A$ the ordinary inclusion mappings.)

It is clear that the *partial direct sum of the empty family* of algebras is the empty algebra of type Δ, i. e. the only left zero object in the category of partial algebras of type Δ.

3. The Main Theorem. The trouble with this partial direct sum is that it is not full even if so are all summands A_t: the class of full algebras is not closed with respect to partial direct sums as it is with respect to direct products. Still, this evil will be soon remedied by our announced General Existence Theorem:

THEOREM 2. *Let A be an arbitrary partial algebra of type Δ, let $\mathfrak{B}$ be a quasi-primitive class of partial algebras of the same type Δ, i. e.*

(i) *$\mathfrak{B}$ is closed with respect to direct products,*

(ii) *$\mathfrak{B}$ is closed with respect to (closed) subalgebras,*

(iii) *$\mathfrak{B}$ is closed with respect to isomorphic images.*

Then there exists an algebra $B_0 \in \mathfrak{B}$ and a homomorphism $\varphi_0: A \to B_0$ such that, for each algebra $C \in \mathfrak{B}$ and each homomorphism $\chi: A \to C$, there is one and only one homomorphism $\psi: B_0 \to C$ such that $\chi = \psi \cdot \varphi_0$ (7).

That is: each diagram

$$\begin{tikzcd} B_0 \arrow[rr, dotted, "\psi"] & & C \\ & A \arrow[ul, "\varphi_0"] \arrow[ur, "\chi"'] & \end{tikzcd}$$

($C \in \mathfrak{B}$) can be filled in by a unique ψ to make it commutative.

Let us note that one may combine properties (i) and (iii) by saying that class $\mathfrak{B}$ shall be closed with respect to *abstract* direct products in the sense given above, i. e. with respect to isomorphic copies of the *concrete* direct product. On the other side, very frequently a condition stronger than (iii) holds, which does not admit a nice combination with (i), namely (iii') that $\mathfrak{B}$ is closed with respect to all homomorphic images,

(7) For a more general theorem on sets with arbitrary abstract structures cf. Bourbaki [3], CST 22, p. 44. Mr. Peter Freyd has kindly informed me that Theorem 2 will be contained as a special case of a simple exercise on abstract categories in his forthcoming book on categories.

which turns the quasi-primitive class $\mathfrak{B}$ into a *primitive class*: this is the hypothesis of Słomiński [25], Theorem 7.

Moreover, Słomiński assumes $\mathfrak{B}$ to be a class of full algebras. In this special case, one may construct the algebra B_0 of our theorem as a factor algebra A_0/R of the Peano algebra A_0 of type Δ (generated by a big enough set M_0) modulo a suitable congruence relation R (the set of all "A_0-equations holding in algebra A as well as in class $\mathfrak{B}$" as defined below): this is Słomiński's proof. But if $\mathfrak{B}$ is a class of arbitrary partial algebras, no longer can our algebra B_0 be expected to be full, nor a homomorphic image of a Peano algebra A_0 (which would be full): extending Słomiński's theorem for a primitive class of full algebras to a quasi-primitive class of partial algebras, we need a completely different proof.

Let us finally remark that, in any case, class $\mathfrak{B}$ contains some full algebras, since according to our convention on the empty family, conditions (i) and (iii) imply that all full one-element algebras of type Δ belong to $\mathfrak{B}$. By the latter statement, we even become sure of the existence of an algebra $B \in \mathfrak{B}$ and a homomorphism $\varphi : A \to B$; as a matter of fact, without this convention (or — if one prefers — additional hypothesis), our theorem would become false (8). So we come to the

Proof of Theorem 2. Due to Theorem 1, there is a cardinal number $\overline{\mathfrak{m}}$ such that $|B| \leqslant \overline{\mathfrak{m}}$ for all partial algebras B generated by sets M of cardinal number $|M| \leqslant |A|$. Let E be an abstract set of power $|E| \geqslant \overline{\mathfrak{m}}$. Let us first consider an algebra B such that $B \subseteq E$, and the family of algebras $B_\lambda = B$, where λ runs through all homomorphisms $\lambda : A \to B$, i. e. through set $\mathrm{Hom}\,(A, B)$; its direct product is the *direct power* $B^{\mathrm{Hom}(A,B)}$, with the natural projections $p_\lambda : B^{\mathrm{Hom}(A,B)} \to B_\lambda = B$. According to the category property of direct products, there is one and only one homomorphism $\varphi_B : A \to B^{\mathrm{Hom}(A,B)}$ such that $p_\lambda \cdot \varphi_B = \lambda$ for all $\lambda \in \mathrm{Hom}(A, B)$ (9). Next, let us consider the family of algebras $B^{\mathrm{Hom}(A,B)}$ such that $B \in \mathfrak{B}$, $B \subseteq E$, and its direct product $\mathbf{P}B^{\mathrm{Hom}(A,B)}$, with the natural projections $q_B : \mathbf{P}B^{\mathrm{Hom}(A,B)} \to B^{\mathrm{Hom}(A,B)}$. Again, there is one and only one homomorphism $\varphi : A \to \mathbf{P}B^{\mathrm{Hom}(A,B)}$ such that $q_B \cdot \varphi = \varphi_B$ for all $B \in \mathfrak{B}$, $B \subseteq E$. Let $B_0 = \overline{\varphi A}$ be the subalgebra of $\mathbf{P}B^{\mathrm{Hom}(A,B)}$ generated by the image of φ, let $i : B_0 \to \mathbf{P}B^{\mathrm{Hom}(A,B)}$ be the inclusion homomorphism; then φ may be considered as a homomorphism $\varphi_0 : A \to B_0$, more precisely: due to the general property of

(8) This is shown by the simple example given by Mr. Peter Freyd: A a non-discrete algebra, $\mathfrak{B}$ the class of all discrete algebras (of the same type as A!); as a matter of fact, this class $\mathfrak{B}$ only fulfills condition (i) with respect to direct products of *non-empty* families.

(9) φ_B is the canonical homomorphism of A into $B^{\mathrm{Hom}(A,B)}$ as considered — in the special case of full algebras — in [17], (6).

relative algebras discussed above, there is one and only one homomorphism $\varphi_0 : A \to B_0$ such that $i \cdot \varphi_0 = \varphi$.

Due to (i) and (ii), $B_0 \in \mathfrak{B}$. Let C be an arbitrary algebra belonging to class $\mathfrak{B}$, let $\chi : A \to C$ be an arbitrary homomorphism. Because of $B_0 = \overline{\varphi_0 A}$, there is at most one homomorphism $\psi : B_0 \to C$ such that $\chi = \psi \cdot \varphi_0$. We have to show the existence of ψ. This is done by "chasing the diagram":

$$\begin{tikzcd}
 & C & \\
\overline{\chi A} = C_0 \arrow[ur, "j"] \arrow[d, "\omega"'] & A \arrow[u, "\chi"'] \arrow[l, "\chi_0"'] \arrow[r, "\varphi_0"] \arrow[dl, "\lambda"'] \arrow[d, "\varphi_B"] \arrow[dr, "\varphi"] & B_0 = \overline{\varphi A} \arrow[ul, dotted, "\psi"'] \arrow[d, "i"] \\
B_\lambda = B & B^{\mathrm{Hom}(A,B)} \arrow[l, "p_\lambda"] & \mathrm{P}B^{\mathrm{Hom}(A,B)} \arrow[l, "q_B"]
\end{tikzcd}$$

Let $C_0 = \overline{\chi A}$ be the subalgebra of C generated by the image of χ, let $j : C_0 \to C$ be the inclusion homomorphism; again χ may be considered as a homomorphism $\chi_0 : A \to C_0$, i. e. $j \cdot \chi_0 = \chi$. But $|\chi A| \leqslant |A|$, hence $|C_0| \leqslant \mathfrak{m} \leqslant |E|$; there is a bijection $\omega : C_0 \to B \subseteq E$, and set B may be turned into a partial algebra of type Δ such that ω becomes an isomorphism. Due to (ii) and (iii), $B \in \mathfrak{B}$. We define the homomorphism $\lambda = \omega \cdot \chi_0 : A \to B$. Then $\psi = j \cdot \omega^{-1} \cdot p_\lambda \cdot q_B \cdot i : B_0 \to C$ is the homomorphism we want, for

$$\begin{aligned}
\psi \cdot \varphi_0 &= j \cdot \omega^{-1} \cdot p_\lambda \cdot q_B \cdot i \cdot \varphi_0 = j \cdot \omega^{-1} \cdot p_\lambda \cdot q_B \cdot \varphi = j \cdot \omega^{-1} \cdot p_\lambda \cdot \varphi_B \\
&= j \cdot \omega^{-1} \cdot \lambda = j \cdot \chi_0 = \chi,
\end{aligned}$$

q. e. d.

Let us remark that algebra B_0 together with homomorphism $\varphi_0 : A \to B_0$ is unique up to — unique — isomorphism.

From the universality property of this unique $\varphi_0 : A \to B_0$, it follows that R_{φ_0}, the equivalence relation (in fact a *congruence relation*) induced by φ_0, is the least among all equivalence relations R_χ induced by homomorphisms $\chi : A \to C \in \mathfrak{B}$,

$$R_{\varphi_0} = \bigcap_{C \in \mathfrak{B}} \ \bigcap_{\chi \in \mathrm{Hom}(A,C)} R_\chi . \tag{6}$$

Defining

$$R_C = \bigcap_{\chi \in \mathrm{Hom}(A,C)} R_\chi \tag{7}$$

as the set of all *A-equations of partial algebra* C (10), we also obtain

$$(8)\qquad R_{\varphi_0} = \bigcap_{C\in\mathfrak{B}} R_C = R_{\mathfrak{B}},$$

i. e. R_{φ_0} equals the set of all *A-equations of class* $\mathfrak{B}$. Moreover, $R_{\varphi_0} = R_{\mathfrak{B}} \subseteq R_{B_0} \subseteq R_{\varphi_0}$, hence $R_{B_0} = R_{\mathfrak{B}}$: extending the notion introduced by Tarski for full algebras, partial algebra B_0 is *A-functionally* or *A-equationally free in* $\mathfrak{B}$.

Due to (6), φ_0 is injective, $R_{\varphi_0} = \bigcap\bigcap R_\chi \subseteq \mathrm{id}_A$, if and only if $A\times A - \mathrm{id}_A \subseteq \bigcup\bigcup (A\times A - R_\chi)$, i. e. if and only if for each couple of different elements $x, y\in A$, there is an algebra $C\in\mathfrak{B}$ and a homomorphism $\chi: A\to C$ such that $\chi(x)\neq\chi(y)$ (11). Or φ_0 is injective if and only if there is an algebra $C\in\mathfrak{B}$ and an injective homomorphism $\chi: A\to C$ (12). If φ_0 is injective, it is not an isomorphism of algebra A onto relative algebra $\varphi_0 A\subseteq B_0$ in general, as is shown below by example (13).

4. Some special cases. We are going to discuss some typical special cases.

First, let algebra A be discrete, i. e. an abstract set. Then each mapping χ of set A into an arbitrary algebra C is a homomorphism. We conclude that subset $\varphi_0 A\subseteq B_0$ is $\mathfrak{B}$*-free* or $\mathfrak{B}$*-independent*, i. e. *C-free* or *C-independent* for each algebra $C\in\mathfrak{B}$, i. e. each mapping $\gamma:\varphi_0 A\to C$ can be extended to a — necessarily unique — homomorphism $\psi: B_0 = \varphi_0 A \to C$. If moreover class $\mathfrak{B}$ is *nontrivial*, i. e. contains an algebra B of cardinal number $|B|\geqslant 2$, φ_0 is injective by the criterion given above. We obtain the existence of an algebra $B_0\in\mathfrak{B}$, $\mathfrak{B}$-freely generated by a subset of prescribed cardinal number $|A|$, for any nontrivial quasi-primitive class $\mathfrak{B}$ of arbitrary partial algebras. In the special case considered by Słomiński [25], this is the famous existence theorem of Birkhoff [1] for free algebras in a non-trivial primitive class $\mathfrak{B}$ of full

(10) Tarski was the first to define an equation as a couple of algebraic operations (functions); Słomiński [24], chap. III, § 3, then defined an equation as a couple of elements of a Peano algebra A. In [16], A was admitted to be an arbitrary free full algebra, in [17], a completely arbitrary full algebra; here the congruence relations R_C are precisely the superinvariant ("überinvariante") congruence relations of A, which coincide with the fully invariant congruence relations if A is free.

(11) Cf. Bourbaki [3], CST 23, p. 45; cf. also Shoda [20], § 5, [21], § 4.

(12) E. g., this trivial injectivity criterion is used by Chevalley [4], p. 42.

(13) Słomiński [25], § 2, discusses conditions for $\varphi_0: A\to B_0$ to be an embedding. An important general embedding criterion in the case of finitary operations (all sets K_i finite) has been given by Neumann [13], theorem 4, [14], theorem 33, the latter proof making use of Steenrod's theorem on the inverse limit of compact spaces. Grätzer-Schmidt [8], chap. I, § 2, give a description of the embedding $\varphi_0: A\to B_0$ in the case $\mathfrak{B}$ is the class ("species") of *all* full algebras of finitary type.

algebras (14); as a matter of fact, the proof given above — as well as the proof of the still more general theorem of Bourbaki [3] on sets with structures — follows the lines of Birkhoff's original proof. Mind that the injective mapping φ_0 is not an isomorphism, i. e. the $\mathfrak{B}$-independent generating subset $\varphi_0 A \subseteq B_0$ is not discrete in its relative algebraic structure in general, even if $\mathfrak{B}$ is a class of full algebras (e. g. let $\mathfrak{B}$ be the class of semi-lattices); so it might become dangerous to "identify" the elements of the discrete algebra A with their images $\varphi_0(a)$, which are elements of the possibly non-discrete relative algebra $\varphi_0 A$.

Second, let us consider the special case that algebra A is full. Then its homomorphic image $\varphi_0 A$ is full, hence a closed subset of partial algebra B_0 hence $B_0 = \varphi_0 A$, φ_0 is surjective, B_0 really full. Using the homomorphism theorem for full algebras, B_0 is isomorphic to factor algebra A/R_{φ_0}; by (8) and property (iii), we obtain $A/R_{\mathfrak{B}} \in \mathfrak{B}$, a statement which — in the special case that $\mathfrak{B}$ is a primitive class of full algebras — used to play a key rôle in Birkhoff's famous theory of equationally definable classes (15).

Third, as the typical example for A to be neither discrete nor full: let $\mathfrak{B}$ be the class of groups considered as full algebras with multiplication, inversion, and unit element as fundamental operations, i. e. as full algebras of type $\Delta = (2, 1, 0)$; let A be a semi-group, considered as a partial algebra of type $(2, 1, 0)$, with multiplication as full fundamental operation of type 2, the other fundamental operations of types 1 and 0 being empty. The interested reader is asked to study the method of proof for the existence of this "universal" or "quotient" group B_0 — sometimes, in particular in the abelian case, called the Grothendieck group — of an arbitrary semi-group A as used by Chevalley (16), this method really belongs to General Algebra, for it may be extended at least to arbitrary primitive classes $\mathfrak{B}$ of full algebras, and it may be described as a reduction of the general form of Theorem 2, for completely arbitrary partial algebras A, to the classical special case that A is discrete, i. e. to Birkhoff's existence theorem for free algebras.

5. Direct sums on quasi-primitive classes. There is an easy extension of Theorem 2 to a whole family of partial algebras A_t $(t \in T)$ instead of a single algebra A:

THEOREM 3. *Let* $(A_t)_{t \in T}$ *be a family of partial algebras of type* Δ,

(14) The author has been unable to find its generalization to arbitrary partial algebras in literature.

(15) Cf. e. g. [17], corollary of theorem 25.

(16) Chevalley [4], p. 41, theorem 20. Cf. also Shoda [20], § 5, [21], § 4; Peremans [15], lemma 1.

let $\mathfrak{B}$ be a quasi-primitive class of partial algebras of type Δ. Then there is an algebra $B_0 \in \mathfrak{B}$ and a family of homomorphisms $\varphi_t : A_t \to B_0$ such that, for any algebra $C \in \mathfrak{B}$ and any ("coterminal") family of homomorphisms χ_t: $A_t \to C$, there is one and only one homomorphism $\psi : B_0 \to C$ such that $\chi_t = \psi \cdot \varphi_t$ for all $t \in T$.

I. e. the family of diagrams

$$\begin{tikzcd} B_0 \arrow[rr, dotted, "\psi"] & & C \\ & A_t \arrow[ul, "\varphi_t"] \arrow[ur, "\chi_t"'] & \end{tikzcd}$$

can be filled in by a unique ψ (not depending on index t) to make them all commutative.

For the proof, one passes from family $(A_t)_{t \in T}$ to its partial direct sum A with injections $i_t : A_t \to A$, and, using the category property of the partial direct sum, applies Theorem 2. In the special case that $\mathfrak{B}$ consists of full algebras, this method of reduction of the general Theorem 3 to its special case Theorem 2 by means of the partial direct sum has been used by Słomiński [25], Theorem (3.5) ([17]).

If all summands A_t belong to class $\mathfrak{B}$, algebra B_0 may be called the $\mathfrak{B}$-*direct sum* of (partial) algebras A_t,

$$B_0 = {}^{\mathfrak{B}}\underset{t \in T}{\mathrm{S}} A_t.$$

In the case of abelian groups, this is the ordinary direct sum, in the case of arbitrary groups, the free product. These two well-known examples show how the notion of "direct sum" depends on class $\mathfrak{B}$. In case $\mathfrak{B}$ is the whole class of partial algebras of type Δ, we get back our original partial direct sum ([18]).

Yet there is a very nice example to show that it may be important

([17]) Shoda [20], § 4, [21], § 4, starts from an arbitrary common extension A of algebras A_t, not telling the reader how to obtain it. Bourbaki [3], p. 50, ex. 3, starts from the direct sum of sets A_t without giving it an algebraic structure.

([18]) Special cases of the $\mathfrak{B}$-direct sum have been considered by many authors, most of them only admitting it if the homomorphisms $\varphi_t : A_t \to B_0$ are embeddings; cf. Shoda [20], § 4, [21], § 4; Sikorski [23], §§ 3,4 , in particular theorem (viii); Peremans [15], theorem 2; Christensen-Pierce [5], theorem 1.5; Jónsson [11], § 1; Kerkhoff [12] considers the case that $\mathfrak{B}$ is the class of full algebras of type Δ. There is a frequent interchange between the terms "product" and "sum". Hewitt [9] gives a "dual" of the theorem of Christensen-Pierce, loc. cit.

not to assume that all algebras A_t belong to class $\mathfrak{B}$. Let family $(A_t)_{t\in T}$ be reduced to a couple of algebras A and M, assume algebra A to belong to class $\mathfrak{B}$, algebra M to be discrete, i. e. an abstract set (which may be assumed to have no elements in common with set A): then B_0 is the $\mathfrak{B}$-polynomial extension of type M over A,

$$B_0 = A[M]_{\mathfrak{B}},$$

the elements of which may be called $\mathfrak{B}$-polynomials of type M over A ([19]). It is possible to generalize from classical ring theory to arbitrary quasi-primitive classes $\mathfrak{B}$ of full — if not partial — algebras the fundamental notions and propositions concerning polynomial extensions ([20]), even the connection between $\mathfrak{B}$-polynomials of type M as defined above, the "polynomial functions" of Birkhoff, i. e. the elements of his "polynomial algebra"

$$P^M(A) \subseteq A^{A^M} \text{ ([21])}$$

(the obvious generalization of "ganzrationale Funktionen"), and the M-characteristic as defined by the author [16].

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([19]) For this simple method of construction of polynomial extensions, cf. Shoda [20], § 4, [21], § 5. A much more complicated construction of polynomial extensions, first of full, then of partial algebras (even admitting many-valued operations) has been given by Dörge [6] and Dörge-Schuff [7], cf. also Schuff [19].

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Reçu par la Rédaction le 30. 11. 1964