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ORDER IN ABSOLUTELY FREE AND RELATED ALGEBRAS*

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Introduction. The general notion of an *absolutely free algebra* implicitly occured first in Mathematical Logic as a device for investigating formal languages: The class of all well-formed formulas of a formal language together with the operations corresponding to the logical connectives forms an absolutely free algebra; moreover, the recent study of formal languages with infinitely long expressions requires the admission of *infinitary operations*. On the other hand, absolutely free algebras, being special cases of free algebras, have been investigated in General Algebra both for their own interest and as a tool to define such notions as "equation" etc. without using metamathematical concepts (see e. g. Löwig [6], [7], Słomiński [11]). It is well known that absolutely free algebras can be axiomatized by an axiom system which is a direct generalization of Peano's axiom system for the natural numbers. More precisely: In case of one unary operation (the successors operation "$'$") and a one-element generating set $\{0\}$ this axiom system reduces to the classical Peano axiom system as formulated first in a more algebraic way by Dedekind ([3], p. 16). This fact suggests the generalization of well known methods, used in the theory of natural numbers, to the general case, a suggestion that very often seems to be neglected.

The main purpose of this paper is to show how the Dedekind *theory of order* can be generalized to absolutely free and related algebras with infinitary operations. In this development we make use only of logic and the most elementary parts of the theory of sets. We do not use in fact ordinals and transfinite induction on *rank numbers*. Since indeed ordinals, or even natural numbers, do not enjoy privileges in General Algebra, their use is in this connection an artificial device, not directly given by

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the algebraic structure (1). But there is another reason for avoiding the use of natural numbers. Because the natural number system itself is a very special absolutely free algebra, we want to derive its properties by applying general theorems, and not — vice versa — to prove the general theorems by using special propositions on natural numbers. This is the point of view of General Algebra.

Because we do not make any restrictions on the number of operations or their variables, one can't fairly expect that the order we introduce is a linear ordering. But as a fundamental property even in case of infinitary operations, the *minimal condition* is satisfied. Thus one can make use of the efficient *principles of proof* and *definition by "course-of-values induction"*; these principles are very well suited to replace induction on ordinals.

After an introductory part 0 fixing the notation and terminology, in part 1 we define the ordering relation, state its fundamental properties, and give characterizations of special classes of absolutely free algebras by order-theoretic properties. In part 2 we apply the notion of order to derive easily theorems on absolutely free algebras, some of them known, but only with unnecessarily complicated proofs.

As starting point we do not use the standard definition of an absolutely free algebra (that is as a free algebra in the class of all algebras of a particular similarity type), but the *generalized Peano axiom system*. To make this clear, we may use the term "*generalized Peano algebras*" for the models of this axiom system (2).

0. Notation, terminology. M_1 and M_2 being arbitrary sets, the set of all *functions* on M_1 to M_2 (i. e. the set of all single-valued mappings from M_1 into M_2) is denoted by $M_2^{M_1}$. For any set M, $|M|$ means the cardinality of M.

Given a function $\mathfrak{a}$ assigning to every element k of a set K exactly one element a_k of the set A, it is sometimes convenient to write also $\mathfrak{a} = (a_k)_{k \in K}$ or $\mathfrak{a} = (a_k \mid k \in K)$, and to call $\mathfrak{a}$ a *K-sequence in A* or a *family of A-elements with index set K*. An *operation f of type K in A* is a function on A^K to A, thus f assigns to every K-sequence $\mathfrak{a}$ in A exactly one element $f(\mathfrak{a}) = f(a_k \mid k \in K)$ of A. If K is an *empty* or, resp., *a one-element, finite,* or *infinite set*, then the operation f is said to be *nullary* or, resp., *unary, finitary,* or *infinitary*. It is customary to identify a nullary operation with the unique element of its range.

(1) Moreover, many proofs are even essentially simpler if ordinals and transfinite induction are not used.

(2) The equivalence of the two notions can easily be proved, but the proof presupposes a somewhat efficient set theory. As we do not want to assume a special set theory explicitly, we prefer the above-mentioned approach. So, in fact, our investigations are concerned with models of the generalized Peano axiom system.

Let A and I be arbitrary sets, and $\tau = (K_i)_{i\in I}$ any I-sequence of sets. The ordered pair $\boldsymbol{A} = \langle A, (f_i)_{i\in I}\rangle$ is an *algebra of type* τ (or τ-*algebra*) if $(f_i)_{i\in I}$ is an I-sequence of operations in A, every f_i being of type K_i. We presuppose the basic notions of General Algebra as those of a *homomorphism*, a *subalgebra*, a *congruence relation*, the *direct product* of a family of τ-algebras etc. (3).

Let $\boldsymbol{A} = \langle A, (f_i)_{i\in I}\rangle$ be any algebra of type $\tau = (K_i)_{i\in I}$. As usual, we identify the algebra $\boldsymbol{A}$ with its fundamental set A if no confusion is possible. $W_{\boldsymbol{A}i}$ denotes the range of the operation f_i, $W_{\boldsymbol{A}}$ the union of all ranges $W_{\boldsymbol{A}i}$, $i\in I$. Complementation with respect to the fundamental set is indicated by a *tilde*, thus $\tilde{W}_{\boldsymbol{A}}$ means the set of all elements which are not values of any of the operations f_i. In these and all following cases we omit the indices denoting the algebra if no confusion can arise.

The operator $\boldsymbol{C}_{\boldsymbol{A}}$, assigning to every subset M of the algebra $\boldsymbol{A}$ the subalgebra $B = \boldsymbol{C}_{\boldsymbol{A}}M$ generated by M, is a *closure operator* (see e. g. Birkhoff [2]). From this very property the simple *principle of algebraic induction* follows: In order to prove a proposition $\boldsymbol{H}$ for all elements of $B = \boldsymbol{C}_{\boldsymbol{A}}M$, it is sufficient to prove that the set of all elements for which $\boldsymbol{H}$ holds (1) includes the generating set M and (2) is closed under the operations f_i.

If R is a binary relation in a set M, R^{-1} denotes the *inverse relation*. Given any two elements $a, b\in M$, we say that b is an *R-predecessor* of a if bRa holds. Subsets which contain with any element a also all R-predecessors of a are called *initial R-sections*. The intersection of initial R-sections is obviously an initial R-sections. Hence for any subset M' of M there exists the *smallest* initial R-section of M containing M'. Initial R^{-1}-sections are also called *final R-sections*.

In the sequel we use the following logical symbols: "$\sim$" for negation, "$\wedge$" for conjunction, "$\vee$" for disjunction, "$\Rightarrow$" for implication, "$\Leftrightarrow$" for equivalence, "$\bigvee\limits_{x}$" for the existential quantifier, and "$\bigwedge\limits_{x}$" for the universal quantifier.

1. Generalized Peano algebras, order, particular classes of generalized Peano algebras. An algebra $\boldsymbol{A} = \langle A, (f_i)_{i\in I}\rangle$ of type $\tau = (K_i)_{i\in I}$ will be called a *generalized Peano algebra* if it satisfies the following three conditions:

P1. For all $i, j\in I$, $\mathfrak{a}\in A^{K_i}$, and $\mathfrak{b}\in A^{K_j}$, if $f_i(\mathfrak{a}) = f_j(\mathfrak{b})$, then $i = j$, i. e. the ranges W_i, $i\in I$, of the operations f_i form a disjoint family of sets.

P2. For all $i\in I$ and $\mathfrak{a}, \mathfrak{b}\in A^{K_i}$, if $f_i(\mathfrak{a}) = f_i(\mathfrak{b})$, then $\mathfrak{a} = \mathfrak{b}$, i. e. the operations f_i are one-one mappings from A^{K_i} into A.

(3) For a detailed exposition see e. g. Schmidt [10].

P3. $C\tilde{W} = A$ (axiom of induction) (4).

The axiom P3 implies the principle of algebraic induction with $\tilde{W}$ as generating set. In case of the natural number system, i. e. the Peano algebra with one unary operation and a one-element generating set $\tilde{W} = \{0\}$, this principle coincides with *complete induction.*

For the present, let $\mathbf{A} = \langle A, (f_i)_{i \in I}\rangle$ be an arbitrary algebra of type $\tau = (K_i)_{i\in I}$. Generalizing the successor operation for the *natural number system*, we define an *algebraic successor relation* S in A: Given any $a, b \in A$, aSb shall hold if and only if there exist $i \in I$, $\mathfrak{a} \in A^{K_i}$, $k_0 \in K_i$ such that $\mathfrak{a}(k_0) = a$ and $f_i(\mathfrak{a}) = b$. If aSb holds, we call b an *algebraic successor* of a, or a an *algebraic predecessor* of b. The ordered pair $\langle A, S\rangle$ is said to be the *diagram* of the algebra $\mathbf{A}$. Even in a case of generalized Peano algebras, neither S nor the inverse relation S^{-1} is necessarily single-valued, of course (5).

Finally, using the algebraic successor relation S, we define our relation $<$ in A: Given any two elements $a, b \in A$, $a < b$ shall hold if and only if there exists an algebraic predecessor b' of b such that a is contained in the smallest initial S-section that contains b'.

From the definition follows at once that $<$ is transitive (6). In general, nothing more can be said about the relation $<$ which may be called the "*natural*" *less-than relation* of the algebra $\mathbf{A}$. But for Peano algebras we have

PROPOSITION 1. *The natural less-than relation of any generalized Peano algebra is irreflexive, hence an irreflexive ordering* (7).

Proof by algebraic induction.

COROLLARY. *For any two elements a and b of a generalized Peano algebra,*

(1.1) $$a < b \Leftrightarrow a \leqslant b \wedge a \neq b.$$

Remark. In an arbitrary algebra $\mathbf{A}$ (1.1) obviously holds if and only if the natural less-than relation is irreflexive. But this may be false even if the relation $\leqslant$ is a (reflexive) ordering. On the other hand, if $<$ is irreflexive, then $\leqslant$ is anti-symmetric, hence a reflexive ordering. In this case we may simply speak of the *natural ordering* of $\mathbf{A}$.

(4) One can treat the set $\tilde{W}$ as a primitive notion, too, as is usually done in the case of the natural number system; see e. g. Henkin [5].

(5) See proposition 4 and proposition 5.

(6) As a matter of fact, the relation $<$ is the transitive hull of S.

(7) An *irreflexive ordering* is an irreflexive and transitive relation; a *reflexive ordering* is a reflexive, anti-symmetric, and transitive relation. Let R be any transitive relation, then the union of R with the identity is called the *quasi-ordering corresponding to R*. Throughout this paper the symbol "$\leqslant$" denotes the quasi-ordering corresponding to the relation $<$.

For the sequel the following proposition — in a case of infinitary operations perhaps somewhat unexpected — is fundamental.

PROPOSITION 2. *The natural ordering of any generalized Peano algebra satisfies the minimal condition.*

It is well known that, for any ordered set $(M, <)$, the minimal condition is equivalent with the

PRINCIPLE OF ORDER-THEORETIC INDUCTION. *In order to prove a proposition* $\boldsymbol{H}$ *for all elements of* M, *it is sufficient to prove that, given any element* $a \in M$, *the assumption that* $\boldsymbol{H}$ *holds for all elements* $x < a$ *implies that* $\boldsymbol{H}$ *holds for* a.

In particular, this principle may be applied in every algebra whose natural less-than relation is an ordering with minimal condition. Then we call it *course-of-values induction.*

Remark. In any ordered set $(M, <)$ the minimal condition also implies the principle of *definition by order-theoretic induction* (see Schmidt [8]). This principle is another very useful tool in the theory of generalized Peano algebras, but since we will not apply it in this paper, we omit its exact formulation. We only mention that it implies as a special case the following

PROPOSITION 3. *If* $\boldsymbol{A}$ *is any generalized Peano algebra, then every single-valued mapping from* $\tilde{W}_{\boldsymbol{A}}$ *into an arbitrary algebra* $\boldsymbol{B}$ *of the same type can be extended to a homomorphism from* $\boldsymbol{A}$ *into* $\boldsymbol{B}$, *i. e.* $\tilde{W}_{\boldsymbol{A}}$ *is an absolutely free generating subset of* A.

Obviously, the following statements are true. Let $\boldsymbol{A} = \langle A, (f_i)_{i \in I} \rangle$ be any algebra, I' a subset of I, and $\boldsymbol{A}|I' = \langle A, (f_i)_{i \in I'} \rangle$ the I'-reduct of $\boldsymbol{A}$. Then

$$a <_{\boldsymbol{A}|I'} b \Rightarrow a <_{\boldsymbol{A}} b. \tag{1.2}$$

Furthermore, if for every $i \in I - I'$ the set K_i is void, then the relations $<_{\boldsymbol{A}}$ and $<_{\boldsymbol{A}|I'}$ are trivially identical: The nullary operations of $\boldsymbol{A}$ have no influence on the natural less-than relation. For any generalized Peano algebra $\boldsymbol{A}$,

$$\mathrm{Min}(\boldsymbol{A}) = \tilde{W} \cup F^{(0)} \tag{1.3}$$

($\mathrm{Min}(\boldsymbol{A})$ = set of all minimal elements of $(A, <)$; $F^{(0)}$ = set of all *constants*, i. e. nullary operations).

The Peano algebra of the natural numbers is *linearly ordered* by its natural ordering. Here the question arises which other generalized Peano algebras also have this property. Furthermore, it may be interesting to know for which generalized Peano algebras the algebraic successor relation S or its inverse relation S^{-1} are single-valued. These questions will be answered in the sequel.

To every ordering $<$ in a set M there is assigned a *covering relation* $\prec$ defined by

$$a \prec b \overset{\text{def}}{=} a < b \wedge \bigwedge_{x \in M} (a \leqslant x < b \Rightarrow a = x). \tag{1.4}$$

Obviously, for any algebra $\boldsymbol{A}$ with irreflexive natural ordering $<_{\boldsymbol{A}}$ the corresponding covering relation $\prec_{\boldsymbol{A}}$ is contained in $S_{\boldsymbol{A}}$. Even for Peano algebras the converse (and hence equality) holds only in very special cases.

PROPOSITION 4. *For any non-void generalized Peano algebra $\boldsymbol{A}$ the following conditions are equivalent:*

1) $\bigwedge_{i \in I} |K_i| \leqslant 1$, *i. e. there are only nullary and unary operations;*

2) $S_{\boldsymbol{A}}^{-1}$ *is single-valued;*

3) $S_{\boldsymbol{A}} = \prec_{\boldsymbol{A}}$, *i. e. $S_{\boldsymbol{A}}$ is consecutive* (8).

PROPOSITION 5. *For any non-void generalized Peano algebra $\boldsymbol{A}$ the following conditions are equivalent:*

1) $S_{\boldsymbol{A}}$ *is single-valued;*

2) *Besides constants there exists at most one unary operation.*

Peano algebras with natural linear ordering are characterized by

PROPOSITION 6. *For any non-void generalized Peano algebra $\boldsymbol{A}$ the following conditions are equivalent:*

1) $<_{\boldsymbol{A}}$ *is a linear ordering, hence a well-ordering;*

2) α) $|\tilde{W} \cup F^{(0)}| = 1$,

β) *Besides constants there exists at most one unary operation.*

Checking the condition 2, one obtains the following

COROLLARY. *There exist exactly four essentially different* (9) *(non-void) linearly ordered Peano algebras.*

In order theory ordering relations which have the property that they are equal to the transitive hulls of their corresponding covering relations are of importance. They are said to be *jump-orderings* (see Schmidt [9]).

PROPOSITION 7. *For any non-void generalized Peano algebra $\boldsymbol{A}$ the following conditions are equivalent:*

1) $\bigwedge_{i \in I} K_i$ *finite: all operations are finitary.*

2) *The natural ordering $<_{\boldsymbol{A}}$ is a jump-ordering.*

(8) Consecutive relations have been investigated e. g. by Schmidt [9].

(9) Here "essentially different" means "non-polyisomorphic" in the sense of Birkhoff [2], p. 167.

Let $\boldsymbol{A}$ be an arbitrary algebra, and $\boldsymbol{B}$ a subalgebra of $\boldsymbol{A}$. Then obviously

(1.5) $$\bigwedge_{a \in B} \bigwedge_{b \in B} a <_{\boldsymbol{B}} b \Rightarrow a <_{\boldsymbol{A}} b$$

holds, i. e. the natural less-than relation of the subalgebra $\boldsymbol{B}$ is contained in the restriction of $<_{\boldsymbol{A}}$ on the subset B of A. The question in which Peano algebras the converse also is true leads us back to algebras with at most unary operations.

PROPOSITION 8. *For any non-void generalized Peano algebra* $\boldsymbol{A}$ *the following conditions are equivalent:*

1) α) $\bigwedge_{i \in I} |K_i| \leqslant 1$, *or* β) $\tilde{W}_{\boldsymbol{A}} = \emptyset$.

2) *For every subalgebra* $\boldsymbol{B}$ *of* $\boldsymbol{A}$,

$$\bigwedge_{a \in B} \bigwedge_{b \in B} a <_{\boldsymbol{B}} b \Leftrightarrow a <_{\boldsymbol{A}} b .$$

3) *Every subalgebra of* $\boldsymbol{A}$ *is a final* $<_{\boldsymbol{A}}$*-section.*

2. The class of generalized Peano algebras. Using the notion of order, we can easily derive some interesting properties of generalized Peano algebras. Except for the propositions 1 and 2, and the simple statement (1.5), we do not apply results of the preceding part.

We say that an *algebra* $\boldsymbol{A}$ *satisfies axiom* P3* if the natural less-than relation is an irreflexive ordering with minimal condition. Simple, but fundamental for what follows is

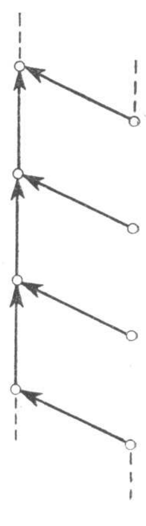
PROPOSITION 9. *For any algebra* $\boldsymbol{A}$, *the axiom* P3* *implies* P3.

This proposition is not true in general, if instead of P3* it is only required that the natural quasi-ordering $\leqslant_{\boldsymbol{A}}$ is a (reflexive) ordering with minimal condition. This is shown by the following

Example 1. Let A be the set of all natural numbers, f the successor operation, and g the identity on A. The natural quasi-ordering of $\boldsymbol{A} = \langle A, f, g \rangle$ is a reflexive ordering with minimal condition, but P3 is not satisfied.

The axiom P3* is really stronger than P3:

Example 2. Let $\boldsymbol{A} = \langle A, f \rangle$ be the algebra with one unary operation which is given by the adjoining diagram. $\boldsymbol{A}$ satisfies P3, but not P3*.



There are some algebraic standard processes forming new algebras from given ones, as e. g. *formation of subalgebras, homomorphic images, direct products.* Axiom P3*, compared with P3, has the advantage that it is preserved in some of these cases whereas P3 is not.

While simple counter-examples show that axiom P3 is not *hereditary*, i. e. is not preserved under formation of subalgebras, (1.5) implies

(2.1) *If an algebra A satisfies* P3*, *so does every subalgebra of A.*

Trivially, the axioms P1 and P2 are hereditary, thus we obviously have

PROPOSITION 10. *Every subalgebra of a generalized Peano algebra is a generalized Peano algebra* ([10]).

In the sequel, if we speak of the direct product of a family $(A_t)_{t\in T}$ of algebras, we will always exclude the trivial case $T = \emptyset$.

PROPOSITION 11. *The direct product of a family $(A_t)_{t\in T}$ of non-void algebras satisfies* P2 *if and only if every factor A_t does.*

The proof is routine.

A corresponding proposition holds for axiom P1, but one can even prove the much stronger

PROPOSITION 12. *If the algebra A satisfies* P1, *so does every homomorphic inverse image of A.*

COROLLARY. *If at least one factor A_t of the family $(A_t)_{t\in T}$ satisfies* P1, *so does the direct product.*

A corresponding proposition holds for the order-theoretic property P3*, but is false for P3.

PROPOSITION 13. *If the algebra A satisfies* P3*, *so does every homomorphic inverse image of A.*

COROLLARY. *If at least one factor A_t of the family $(A_t)_{t\in T}$ satisfies* P3*, *so does the direct product.*

From the preceding propositions, resp. corollaries, we get

PROPOSITION 14. *The direct product of any (non-void) family of generalized Peano algebras is a generalized Peano algebra.*

This proposition has the immediate consequence that there exist many generalized Peano algebras (= absolutely free algebras) which are *decomposable* into non-trivial factors. This may be somewhat unexpected because absolutely free algebras are considered of such a simple structure that they should be indecomposable.

We will not give a complete answer to the question which generalized Peano algebras are decomposable, but mention only some particular cases:

1) Let $\tau = (K_i)_{i\in I}$ be a finitary type, i. e. all K_i be finite. If I is *at most countable*, then the generalized Peano algebra A of type τ with a *countable* set $\tilde{W}_A$ is *decomposable* into a direct product of two non-trivial generalized Peano algebras.

([10]) A proof by transfinite induction in Słomiński [11], p. 24.

In a case of a *finite* set $\tilde{W}_A$ the situation is different. But even then some generalized Peano algebras are decomposable.

2) Let τ be a type with only one operation f of type K.

α) If f is *at least binary*, i. e. $|K| \geqslant 2$, then any generalized Peano algebra $\boldsymbol{A}$ with a *finite* set $\tilde{W}_A$ is *indecomposable*;

β) If f is *unary*, then no generalized Peano algebra $\boldsymbol{A}$ with a finite set $\tilde{W}_A$ can be decomposed into a direct product of *generalized Peano algebras*. But using the full content of the preceding propositions and corollaries, one can prove that any generalized Peano algebra $\boldsymbol{A}$ with a finite set $\tilde{W}_A$ containing at least two elements is decomposable into a direct product of two algebras, *one of them being a generalized Peano algebra*.

Furthermore, it follows from the foregoing that the direct product of a family of non-void algebras, none of which is a generalized Peano algebra, may be a generalized Peano algebra:

Example 3. Let $\boldsymbol{A} = \langle A, (f, g)\rangle$ be the Peano algebra with two unary operations f and g, and a one-element set $\tilde{W}_A = \{x_0\}$. We define two binary relations R_0 and T_0, each consisting of just one ordered pair, $\langle x_0, f(f(x_0))\rangle$ and $\langle f(g(x_0)), g(g(x_0))\rangle$, respectively. Let R and T be the smallest congruence relations in A containing R_0 resp. T_0. The quotients $\boldsymbol{A}/R$ and $\boldsymbol{A}/T$ both satisfy P2. Moreover, $\boldsymbol{A}/R$ satisfies P1, but not P3, and $\boldsymbol{A}/T$ satisfies P3*, but not P1. Thus the direct product is a generalized Peano algebra, but neither factor is.

It may be of some interest to give characterizations of those algebras which have *absolutely free quotients*. We know already that the properties P1 and P3* are *necessary* conditions. We conclude this paper with the remark that these conditions are *not* sufficient.

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