COLLOQUIUM MATHEMATICUM

VOL. XXVII 1973 FASC. 2

DIRECT LIMITS OF CONGRUENCE SYSTEMS

BY

MANFRED ARMBRUST (KÖLN)

This is a contribution to Problem 1.2 of Grätzer [3]:

Given a system C of equivalence relations on a set A, how can one decide whether there is a full finitary algebraic structure on A whose congruence lattice is C?

As I had conjectured in my paper [1], the existence of a reasonable answer is somewhat unlikely. An effort to give more substantiation for this conjecture produced some ideas and results which — though still far from being conclusive — shed some new light on the subject and are most certainly capable of further development.

A suitable framework is devised in Section 1: the category **CON** of congruence systems, i.e., sets together with congruence lattices of finitary partial algebras, with an appropriate notion of morphism. Among other things, this category has direct limits, some preservation problems of which are discussed in Section 2. In particular, an example shows that the property of being the congruence lattice of a full two-valued algebra (cf. [1], Theorem 3) is not preserved under direct limits in **CON**. Finally, in Section 3, a fairly powerful higher-order formal language $\mathscr{L}$ is developed whose statements are preserved under certain direct limits in **CON**.

It follows that one cannot characterize in $\mathscr{L}$ (within **CON**) congruence lattices of full two-valued algebras. Although it seems clear upon inspection that the property of being the congruence lattice of an arbitrary full algebra is not preserved under direct limits in **CON** either, I failed to construct a counterexample. If such a counterexample were found, we would at least have the modest result that Grätzer's problem is not solvable in $\mathscr{L}$.

1. The category of congruence systems. It is well known that a system C of equivalence relations on a set A is the congruence lattice of a finitary partial algebraic structure on A if and only if C contains ι_A (the identity relation on A) and is closed with respect to arbitrary intersections and under unions of directed subsystems ("inductivity"). This prepares the stage for the present discussion.

Definition 1. A *congruence system* is a pair (A, C), where A is a set, and C a system of equivalence relations on A containing ι_A and closed with respect to arbitrary intersections and under unions of directed subsystems. The elements of C are called *congruences*.

The objects given, we only need to define suitable morphisms in order to establish the category **CON** of congruence systems. (For notions and results from category theory used in the sequel, see the first two chapters of Pareigis [4].) Considering that it would not make much sense to recur to some type of algebraic structure, we have to concentrate on those properties of ordinary homomorphisms of algebras which can be formulated in terms of congruence relations. The following seems to be the most appropriate definition:

Definition 2. A *morphism* from the congruence system (A, C) to the congruence system (B, D) is a mapping $\varphi\colon A \to B$ such that the counterimage of any $\varrho \in D$ under φ belongs to C.

For a congruence system (A, C), let $P(A, C)$ be the set of all finitary partial operations f on A that are compatible with C in the usual sense: If $\varrho \in C$, if $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \operatorname{dom} f$, and if $(x_i, y_i) \in \varrho$ for each $i = 1, \ldots, n$, then $\big(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)\big) \in \varrho$. Then we have the following useful characterization of morphisms (cf. the r-morphisms of Goetz [2]):

THEOREM 1. *Let Q be any set of partial operations on A such that C is the congruence lattice of the partial algebra (A, Q). Then the mapping $\varphi\colon A \to B$ is a morphism from (A, C) to (B, D) in* **CON** *if and only if, for each partial operation $f \in Q$, there exists a partial operation $g \in P(B, D)$ such that, for any $(x_1, \ldots, x_n) \in \operatorname{dom} f$, we have $(\varphi x_1, \ldots, \varphi x_n) \in \operatorname{dom} g$ and $\varphi f(x_1, \ldots, x_n) = g(\varphi x_1, \ldots, \varphi x_n)$.*

Proof. Let $\varphi\colon A \to B$ be a morphism and suppose $f \in P(A, C) \supseteq Q$. We claim that, for $(x_1, \ldots, x_n) \in \operatorname{dom} f$,

$$g(\varphi x_1, \ldots, \varphi x_n) := \varphi f(x_1, \ldots, x_n)$$

defines a partial operation $g \in P(B, D)$. First, assume that $\varphi x_i = \varphi y_i$ for $i = 1, \ldots, n$, and $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \operatorname{dom} f$. Then $(x_i, y_i) \in \varphi^{-1}(\iota_B)$, and since $\varphi^{-1}(\iota_B) \in C$, it follows that $\big(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)\big) \in \varphi^{-1}(\iota_B)$, i.e., $\varphi f(x_1, \ldots, x_n) = \varphi f(y_1, \ldots, y_n)$. In the same way, we infer that $g \in P(B, D)$; we just have to substitute an arbitrary $\varrho \in D$ for ι_B in the argument.

For the converse, it suffices to show that, for $\varrho \in D$, $\varphi^{-1}(\varrho)$ is compatible with Q, which is a routine argument.

Observe that, in Theorem 1, we can always choose $Q = P(A, C)$.

It is easy to see that monomorphisms are injective and epimorphisms surjective mappings. An isomorphism is a bijective morphism whose inverse is also a morphism.

Accordingly, the subobjects of $(A, \boldsymbol{C})$ are represented by congruence systems $(A', \boldsymbol{C}')$ such that $A' \subseteq A$ and $\boldsymbol{C}' \supseteq \boldsymbol{C} \mid A'$ (the restrictions of the congruences in $\boldsymbol{C}$ to A'). Note that, in particular, $(A', \boldsymbol{C} \mid A')$ is a congruence system; this is what we mean by a ***subsystem*** of $(A, \boldsymbol{C})$.

The quotient objects are given by the congruence systems $(A/\varrho, \boldsymbol{D})$, where $\varrho \in \boldsymbol{C}$ and $\boldsymbol{D} \subseteq \boldsymbol{C}/\varrho$ (A/ϱ is the partition of A modulo ϱ, and $\boldsymbol{C}/\varrho$ consists of the equivalence relations on A/ϱ induced by the congruences $\sigma \in \boldsymbol{C}$ with $\sigma \supseteq \varrho$). In particular, $(A/\varrho, \boldsymbol{C}/\varrho)$ is a congruence system; this is what we mean by a *quotient system* of $(A, \boldsymbol{C})$.

THEOREM 2. *The category* **CON** *has difference kernels and difference cokernels.*

Proof. If φ, ψ: $(A, \boldsymbol{C}) \to (B, \boldsymbol{D})$ are morphisms, then it is obvious that the inclusion mapping i: $A' \to A$ of the subsystem $A' = \{a \in A \mid \varphi a = \psi a\}$ of $(A, \boldsymbol{C})$ constitutes a difference kernel for φ, ψ.

Dually, let φ, ψ: $(A, \boldsymbol{C}) \to (B, \boldsymbol{D})$ be morphisms, and let ϱ be the smallest congruence in $\boldsymbol{D}$ containing $(\varphi a, \psi a)$ for each $a \in A$. Then the canonical projection of B onto the quotient system B/ϱ of $(B, \boldsymbol{D})$ yields a difference cokernel for φ, ψ.

THEOREM 3. *The category* **CON** *has products and coproducts.*

Proof. For a family of objects $(A_t, \boldsymbol{C}_t)$, $t \in T$, we obtain a product as follows (cf. the ϱ-product of Goetz [2]):

Let $A = \prod_{t \in T} A_t$ (the cartesian product) and define, for every positive integer n and each family of n-ary partial operations $f_t \in P(A_t, \boldsymbol{C}_t)$, an n-ary partial operation f on A by

$$f\big((x_{1t})_{t \in T}, \ldots, (x_{nt})_{t \in T}\big) = \big(f_t(x_{1t}, \ldots, x_{nt})\big)_{t \in T}$$

whenever $(x_{1t}, \ldots, x_{nt}) \in \operatorname{dom} f_t$ for each $t \in T$. Let Q be the set of all partial operations on A obtained this way, and let $\boldsymbol{C}$ be the congruence lattice of the partial algebra (A, Q). Then the congruence system $(A, \boldsymbol{C})$ together with the cartesian projections π_t: $A \to A_t$ is the desired product. In fact, the π_t's are morphisms by Theorem 1. If φ_t: $(B, \boldsymbol{D}) \to (A_t, \boldsymbol{C}_t)$ is a family of morphisms, then the unique mapping φ: $B \to A$ with $\pi_t \varphi = \varphi_t$, $t \in T$, is easily seen to be a morphism, again by Theorem 1.

The coproduct can be defined without recurrence to partial operations. Let A be the disjoint union of the A_t's, and let $\boldsymbol{C}$ consist of all equivalence relations ϱ on A such that $\varrho \mid A_t \in \boldsymbol{C}_t$ for all $t \in T$. Then $(A, \boldsymbol{C})$ is a congruence system, the inclusion mappings i_t: $A_t \to A$ are morphisms and, for a family of morphisms φ_t: $(A_t, \boldsymbol{C}_t) \to (B, \boldsymbol{D})$, the unique mapping φ: $A \to B$ with $\varphi i_t = \varphi_t$, $t \in T$, is a morphism.

Remark. The coproduct can also be described by means of partial operations. If C_t is the congruence lattice of the partial algebra (A_t, Q_t), $t \in T$, and if (A, C) is the coproduct of the (A_t, C_t) as constructed above, then it is clear that C is the congruence lattice of the partial algebra (A, Q), where Q is the disjoint union of the Q_t's.

COROLLARY. *The category* **CON** *is complete and cocomplete.*

2. Direct limits in CON. We will study the behavior of some distinguished properties of congruence systems with respect to direct limits. A congruence system (A, C) is called *unary* if C is the congruence lattice of a unary partial algebraic structure on A (cf. [1], Theorem 2). Suppose that (A_t, C_t) is a family of unary congruence systems; then it is clear from the remark following the proof of Theorem 3 that the coproduct of the family (A_t, C_t) is unary. Also, a quotient $(A/\varrho, C/\varrho)$ of a unary system (A, C) is trivially unary. So we have

THEOREM 4. *The limit of a direct system of unary congruence systems is unary.*

Now consider the property of permutability of the congruences. This one is not preservedu nder coproducts. However, it is clearly preserved under the formation of quotients and also under the following special kind of direct limit:

Let T be a directed set, and let (A_t, C_t) be a family of congruence systems such that, for $t_1 \leqslant t_2$, (A_{t_1}, C_{t_1}) is a subobject of (A_{t_2}, C_{t_2}), i.e., $A_{t_1} \subseteq A_{t_2}$ and $C_{t_2} \mid A_{t_1} \subseteq C_{t_1}$ (such a family will be called an *injective direct system*). The direct limit of this family is the congruence system (A, C), where A is the union of the A_t's and C is the set of all equivalence relations ϱ on A such that $\varrho \mid A_t \in C_t$ for all $t \in T$. Thus we have proved

THEOREM 5. *The limit of a direct system of congruence systems with permutable congruences has permutable congruences.*

Another important property is the closure under equivalence-theoretic joins. Trivial examples show that this property is not preserved under coproducts. Again, it is easy to see that it is preserved under the formation of quotients and under limits of injective direct systems. Let (A_t, C_t), $t \in T$, be an injective direct system, let (A, C) be its limit as described above, and assume that all C_t's are closed with respect to equivalence-theoretic joins. If $\varrho, \sigma \in C$, we have to show that $(\varrho \vee \sigma) \mid A_t \in C_t$ for each $t \in T$, which follows from the inductivity of the C_t's and the fact that $(\varrho \vee \sigma) \mid A_t$ is the union of the relations $\big((\varrho \mid A_s) \vee (\sigma \mid A_s)\big) \mid A_t$ for $s \geqslant t$.

THEOREM 6. *The limit of a direct system of congruence systems that are closed with respect to equivalence-theoretic joins is closed under equivalence-theoretic joins.*

Let us call a congruence system (A, C) *full two-valued* if C is the congruence lattice of a full two-valued structure on A (i.e., every operation used is supposed to have exactly two values, which may depend on the operation; note that the operations can always be chosen unary). Clearly, a quotient of a full two-valued system is full two-valued. However, this property is not preserved under direct limits. This fact appears trivial, but the counterexample is rather sophisticated and needs some preparation.

Let (A, C) be an arbitrary congruence system. We denote by $F_2(A, C)$ the set of all full unary two-valued operations on A that are compatible with C; for $h \in F_2(A, C)$, ε_h denotes the equivalence relation on A induced by h, i.e., $(x, y) \in \varepsilon_h$ iff $hx = hy$. The proof of Lemma 1 is obvious.

LEMMA 1. *Let $h \in F_2(A, C)$, $\operatorname{im} h = \{u, v\}$; let $\varrho \in C$ have exactly two equivalence classes, and suppose $(u, v) \notin \varrho$. Then $\varrho = \varepsilon_h$.*

For the moment, we will call a congruence system (A, C) *regular* if it is full two-valued and satisfies the following conditions for any $h_1, h_2 \in F_2(A, C)$:

If $\operatorname{im} h_1 = \operatorname{im} h_2 = \{u, v\}$, *then* $\varepsilon_{h_1} = \varepsilon_{h_2}$ *and* $h_i u \neq h_i v$ $(i = 1, 2)$; *if* $\operatorname{im} h_1 \neq \operatorname{im} h_2$, $\operatorname{im} h_i = \{u_i, v_i\}$ $(i = 1, 2)$, *then* $h_1 u_2 = h_1 v_2$, $h_2 u_1 = h_2 v_1$.

Observe that if (A, C) is regular and $h \in F_2(A, C)$, then $\varepsilon_h \in C$.

LEMMA 2. *Let (A, C) be regular, $f \in F_2(A, C)$, $\operatorname{im} f = \{a, b\}$; let ∞ be an element not in A, and suppose $B = A \cup \{\infty\}$. Then (A, C) is a subsystem* [1] *of a regular system (B, D) such that, for every $h \in F_2(B, D)$, $\operatorname{im} h \neq \{a, b\}$.*

Proof. We define the following two-valued operations on B:

$$f_1 x = \begin{cases} a & \text{if } x \in A \text{ and } fx = fa, \\ \infty & \text{otherwise;} \end{cases}$$

$$f_2 x = \begin{cases} b & \text{if } x \in A \text{ and } fx = fb, \\ \infty & \text{otherwise;} \end{cases}$$

$$\bar{g} x = \begin{cases} gx & \text{if } x \in A, \\ ga \ (= gb) & \text{if } x = \infty, \end{cases}$$
$$\text{for each } g \in F_2(A, C) \text{ with } \operatorname{im} g \neq \{a, b\}.$$

(In this proof, the letter g will always mean an operation of this kind.) Let D be the congruence lattice of the algebra on B given by f_1, f_2 and the $\bar{g}$'s.

If ϱ is an equivalence relation on A and $c \in A$, we denote by ϱ^c the smallest equivalence relation on B containing ϱ and (c, ∞). Whenever $\varrho \in C$, we have $\varrho^a, \varrho^b \in D$. Trivially, ϱ^a is compatible with f_1. For $(x, y) \in \varrho$, we have $(\bar{g}x, \bar{g}y) = (gx, gy) \in \varrho \subseteq \varrho^a$, and $(fx, fy) \in \varrho$. If $fx = fy$, then also

[1] Subsystem means $D | A = C$.

$f_2x = f_2y$; if $fx = a$ and $fy = b$, then $(a, b) \in \varrho$, hence $(b, \infty) \in \varrho^a$. For $(x, \infty) \in \varrho^a$, $x \in A$, we have $(x, a) \in \varrho$, so $(fx, fa) \in \varrho$. If $fx = fa$, then $fx \neq fb$, hence $f_2x = \infty = f_2\infty$; if $fx \neq fa$, then $(b, \infty) \in \varrho^a$ as above. Finally, $(\bar{g}x, \bar{g}\infty) = (gx, ga) \in \varrho \subseteq \varrho^a$.

In particular, it follows that $\boldsymbol{D} \mid A \supseteq \boldsymbol{C}$. Conversely, suppose that $\sigma \in \boldsymbol{D}$, and let $\varrho = \sigma \mid A$. Clearly, ϱ is compatible with each g since σ is compatible with $\bar{g}$. It remains to show that ϱ is compatible with f. Let $(x, y) \in \varrho$, $fx = a$, $fy = b$. If $fa = a$, we have $(a, \infty) = (f_1x, f_1y) \in \sigma$ and $(\infty, b) = (f_2x, f_2y) \in \sigma$, hence $(a, b) \in \sigma$; if $fa = b$, we have $(b, \infty) = (f_2x, f_2y) \in \sigma$ and $(\infty, a) = (f_1x, f_1y) \in \sigma$, hence $(a, b) \in \sigma$.

It is easy to see that $\varepsilon_{f_1} = (\varepsilon_f)^b$, $\varepsilon_{f_2} = (\varepsilon_f)^a$, and $\varepsilon_{\bar{g}} = (\varepsilon_g)^a = (\varepsilon_g)^b$.

Now let $h \in F_2(B, \boldsymbol{D})$. Then $\operatorname{im} h \neq \{a, b\}$, for otherwise we would have, by Lemma 1, $\varepsilon_h = (\varepsilon_f)^b$ and $\varepsilon_h = (\varepsilon_f)^a$, a contradiction. If $\operatorname{im} h \subseteq A$, then $g := h \mid A \in F_2(A, \boldsymbol{C})$, so $\varepsilon_h = (\varepsilon_g)^a$ by Lemma 1, hence $h\infty = ha = ga = \bar{g}\infty$ and $h = \bar{g}$. If $\operatorname{im} h = \{c, \infty\}$ with $c \in A$, then $c = a$ or $c = b$. For suppose, without loss of generality, that $(a, c) \in \varepsilon_f \subseteq (\varepsilon_f)^b \in \boldsymbol{D}$; then $ha = hc \neq hb = h\infty$, and this is only possible if $c = a$, in view of $(\iota_A)^a \in \boldsymbol{D}$. If $\operatorname{im} h = \{a, \infty\} = \operatorname{im} f_1$, then $\varepsilon_h = (\varepsilon_f)^b = \varepsilon_{f_1}$; if $\operatorname{im} h = \{b, \infty\} = \operatorname{im} f_2$, then $\varepsilon_h = (\varepsilon_f)^a = \varepsilon_{f_2}$. It is now straightforward that $(B, \boldsymbol{D})$ is regular.

Finitely, many successive applications of Lemma 2 yield the following construction:

THEOREM 7. *Let $(A, \boldsymbol{C})$ be a finite regular congruence system. Then there exists a finite regular congruence system $(B, \boldsymbol{D})$ such that $(A, \boldsymbol{C})$ is a subsystem of $(B, \boldsymbol{D})$ (i.e., $A \subseteq B$ and $\boldsymbol{C} = \boldsymbol{D} \mid A$) and, for every $h \in F_2(B, \boldsymbol{D})$, $\operatorname{im} h \nsubseteq A$.*

Now, we are in a position to give an example of an ascending chain $(A_n, \boldsymbol{C}_n)$, $n \geqslant 0$, of full two-valued congruence systems whose direct limit is not of this kind. We choose $A_0 = \{0, 1, 2\}$ and $\boldsymbol{C}_0 = \{\iota_{A_0}, \alpha, \beta, A_0 \times A_0\}$, where $A_0/\alpha = \{\{0, 1\}, \{2\}\}$, $A_0/\beta = \{\{0, 2\}, \{1\}\}$. This is easily seen to be a regular system. Inductively, we proceed as follows:

Suppose $(A_n, \boldsymbol{C}_n)$ is a finite regular congruence system. Applying Theorem 7, we obtain a finite regular system $(A_{n+1}, \boldsymbol{C}_{n+1})$ such that $A_n \subseteq A_{n+1}$, $\boldsymbol{C}_{n+1} \mid A_n = \boldsymbol{C}_n$ and, for every $h \in F_2(A_{n+1}, \boldsymbol{C}_{n+1})$, $\operatorname{im} h \nsubseteq A_n$. In the direct limit $(A, \boldsymbol{C})$ of this sequence, A is the union of the A_n's, and $\boldsymbol{C} \mid A_n = \boldsymbol{C}_n$ for every n. So $\boldsymbol{C}$ does not contain all equivalence relations on A, but there is not a single full two-valued operation on A compatible with $\boldsymbol{C}$. Suppose $h \in F_2(A, \boldsymbol{C})$, $\operatorname{im} h \subseteq A_n$. Then $h \mid A_{n+1} \in F_2(A_{n+1}, \boldsymbol{C}_{n+1})$, a contradiction. Therefore, $\boldsymbol{C}$ cannot be the congruence lattice of a full two-valued algebra.

3. The inductive language $\mathscr{L}$. The following symbols will be used besides the usual logical constants: predicates $=$, $\subseteq$, K and H, individual variables $x, y, \ldots$ and variables for equivalence relations $\varrho, \sigma, \ldots$

The interpretation of the predicates in a congruence system (A, C) is as follows.

Equality and inclusion in the usual set-theoretic sense; if α is an equivalence relation on A, and $a, b, c, d \in A$, then $K(\alpha)$ means $\alpha \in C$, and $H(a, b, c, d)$ means that (c, d) is not in the C-hull of (a, b), i.e., $(c, d) \notin \beta$ for some $\beta \in C$ with $(a, b) \in \beta$; moreover, $\alpha(a, b)$ means $(a, b) \in \alpha$. Accordingly, our atomic formulas are of the forms $x = y$, $\varrho = \sigma$, $\varrho \subseteq \sigma$, $K(\varrho)$, $H(u, v, x, y)$ and $\varrho(x, y)$.

Now let (A_t, C_t), $t \in T$, be an injective direct system of congruence systems and (A, C) its limit. If α and β are equivalence relations on A and $a, b, c, d \in A$, and if $S(\alpha, \beta, a, b, c, d)$ is an atomic statement or negation of such, then $S(\alpha, \beta, a, b, c, d)$ holds in (A, C) if and only if there is a $t_0 \in T$ such that $S(\alpha \mid A_t, \beta \mid A_t, a, b, c, d)$ holds in (A_t, C_t) for each $t \geqslant t_0$ (*S holds in almost every (A_t, C_t)*). This is obvious except for $H(a, b, c, d)$; here we use the inductivity of the congruence systems to verify that the C-hull of (a, b) in A is the set-theoretic union of the C_t-hulls of (a, b) in A_t (provided $a, b \in A_t$). Moreover, if one of the statements $a = b$, $a \neq b$, $\alpha \mid A_{t_0} \neq \beta \mid A_{t_0}$, $\alpha \mid A_{t_0} \nsubseteq \beta \mid A_{t_0}$, $\neg K(\alpha \mid A_{t_0})$, $\neg H(a, b, c, d)$, $(\alpha \mid A_{t_0})(a, b)$, $\neg(\alpha \mid A_{t_0})(a, b)$ holds in (A_{t_0}, C_{t_0}), then the corresponding statement holds in (A_t, C_t) for each $t \geqslant t_0$. This is not the case for the atomic statements $\alpha = \beta$, $\alpha \subseteq \beta$, $K(\alpha)$ and $H(a, b, c, d)$. We call these four types of atomic formulas *critical*.

Suppose that $\alpha_1, \ldots, \alpha_m$ are equivalence relations on A, $a_1, \ldots, a_n \in A$, and $S(\alpha_1, \ldots, \alpha_m, a_1, \ldots, a_n)$ is a statement without quantifiers. Observing that the truth values of atomic statements are ultimately constant, we conclude that $S(\alpha_1, \ldots, \alpha_m, a_1, \ldots, a_n)$ holds in (A, C) if and only if $S(\alpha_1 \mid A_t, \ldots, \alpha_m \mid A_t, a_1, \ldots, a_n)$ holds in almost every (A_t, C_t). A statement of the form

$$\exists y_1 \ldots \exists y_q S(\alpha_1, \ldots, \alpha_m, a_1, \ldots, a_n, y_1, \ldots, y_q),$$

where S is quantifier-free, still holds in (A, C) if and only if

$$\exists y_1 \ldots \exists y_q S(\alpha_1 \mid A_t, \ldots, \alpha_m \mid A_t, a_1, \ldots, a_n, y_1, \ldots, y_q)$$

holds in almost every (A_t, C_t), provided that none of the variables $y_1, \ldots, y_q$ appears as an argument in any positive occurrence of the predicate H in the disjunctive normal form of S. More care must be taken with statements

$$\exists \sigma_1 \ldots \exists \sigma_p \exists y_1 \ldots \exists y_q S(\alpha_1, \ldots, \alpha_m, a_1, \ldots, a_n, \sigma_1, \ldots, \sigma_p, y_1, \ldots, y_q).$$

If such a statement holds in (A, C), then it is clear that the same statement with the α_i's properly restricted holds in almost every (A_t, C_t). The converse is true if we require that no bound individual or relational variable occur as an argument in any positive instance of a critical atomic formula in the disjunctive normal form of S. In fact, we can find a t_0 such that, for each positive occurrence of a critical formula, the truth value is constant for $t \geqslant t_0$ (with the proper restrictions of the a_i's), and

$$S(a_1 \mid A_{t_0}, \ldots, a_m \mid A_{t_0}, a_1, \ldots, a_n, \beta_1, \ldots, \beta_p, b_1, \ldots, b_q)$$

holds in (A_{t_0}, C_{t_0}) for some $b_1, \ldots, b_q \in A_{t_0}$ and some equivalence relations $\beta_1, \ldots, \beta_p$ on A_{t_0}. Choosing equivalence relations $\bar{\beta}_1, \ldots, \bar{\beta}_p$ on A such that $\bar{\beta}_i \mid A_{t_0} = \beta_i$ $(i = 1, \ldots, p)$, we infer that

$$S(a_1, \ldots, a_m, a_1, \ldots, a_n, \bar{\beta}_1, \ldots, \bar{\beta}_p, b_1, \ldots, b_q)$$

holds in (A, C).

This discussion justifies the following definition:

The language $\mathcal{L}$ consists of all statements of the form

$$\forall \varrho_1 \ldots \forall \varrho_m \forall x_1 \ldots \forall x_n \exists \sigma_1 \ldots \exists \sigma_p \exists y_1 \ldots \exists y_q S(\varrho_1, \ldots, y_q),$$

where S is quantifier-free and so existentially bound individual or relational variable occurs in a positive instance of a critical atomic formula in the disjunctive normal form of S. We have proved

THEOREM 8. *Let (A_t, C_t), $t \in T$, be an injective direct system in* **CON** *and (A, C) its limit. Then a statement of $\mathcal{L}$ holds in (A, C) whenever it holds in almost every (A_t, C_t).*

As examples, we formalize the properties discussed in Section 2.

(i) Unary systems (Theorem 4):

$$\forall \varrho \exists u \exists v \exists x \exists y \Big(\big(\varrho(u, v) \mathbin{\&} \neg H(u, v, x, y) \Rightarrow \varrho(x, y)\big) \Rightarrow K(\varrho)\Big).$$

(ii) Permutability of the congruences (Theorem 5):

$$\forall \varrho \forall \sigma \forall x \forall y \forall z \exists u \big(K(\varrho) \mathbin{\&} K(\sigma) \mathbin{\&} \varrho(x, y) \mathbin{\&} \sigma(y, z) \Rightarrow \varrho(z, u) \mathbin{\&} \sigma(u, x)\big)$$

or

$$\forall u \forall v \forall x \exists y \big(\neg H(u, v, x, y) \mathbin{\&} \neg H(v, x, u, y)\big).$$

(iii) Closure with respect to equivalence-theoretic joins (Theorem 6). Note that, in this case, (i) also holds (see [1], Corollary of Theorem 2). So we can use statement (i) along with infinitely many statements of the form

$$\forall \varrho \forall \sigma \forall \tau \forall z_1 \ldots \forall z_n \forall u \forall v \forall x \forall y \big(K(\varrho) \mathbin{\&} K(\sigma) \mathbin{\&} \varrho(u, z_1) \mathbin{\&} \sigma(z_1, z_2) \mathbin{\&} \ldots$$
$$\ldots \mathbin{\&} \varrho(z_{n-1}, z_n) \mathbin{\&} \sigma(z_n, v) \mathbin{\&} \varrho \subseteq \tau \mathbin{\&} \sigma \subseteq \tau \mathbin{\&} \neg H(u, v, x, y) \Rightarrow \tau(x, y)\big),$$

one for each odd $n \geqslant 1$.

The fact that these properties can be formulated in $\mathcal{L}$ indicates that our language is not too trivial. However, it is impossible to formulate in $\mathcal{L}$ (within **CON**) the property of being the congruence lattice of a full two-valued algebra, and it is my conjecture that this is also true for congruence lattices of arbitrary full algebras.

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Reçu par la Rédaction le 26. 10. 1971