COLLOQUIUM MATHEMATICUM

XIV WARSZAWA—WROCŁAW 1966

A SEMIGROUP REPRESENTATION OF VARIETIES OF ALGEBRAS*

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It is well known that the class of groups can be defined in a number of different ways: in terms of a binary multiplication, or in terms of a binary multiplication and a unary inversion, or in terms of a binary right division, and so on. How are these different definitions interrelated? In order to study this problem, not just for the class of groups, but more generally for classes of algebras, a semigroup will be defined in connection with the class; and if the class is a variety, that is to say, equationally defined, then the semigroup will in a sense specify it completely.

When dealing with several operations simultaneously, it is convenient to adopt a bracket-free notation like that of Łukasiewicz. Thus, to return to the example of groups, we might denote multiplication, inversion, right division by μ, η, ϱ, respectively, so that $xy\mu$ stands for the product of the elements x, y, and $x\eta$ for the inverse of x, and so on. The associative law of multiplication then becomes

$$xy\mu z\mu = xyz\mu\mu, \tag{1}$$

interpreted as a law, that is to say, as if the element variables x, y, z were bound by universal quantifiers, over the appropriate domain.

Right division ϱ can be expressed in terms of multiplication μ and inversion η by

$$xy\varrho = xy\eta\mu; \tag{2}$$

and multiplication and inversion can also be expressed in terms of right division by

$$x\eta = xx\varrho x\varrho, \qquad xy\mu = xyy\varrho y\varrho\varrho. \tag{3}$$

* This paper was read by the first author at the Conference on General Algebra, Warsaw, September 7-11, 1964. It summarizes parts of the unpublished Ph. D. thesis [2] of the second author, submitted under her maiden name.

The group laws can be correspondingly translated from their multiplication-and-inversion form to their right-division form, and conversely: this is the reason why we think of the class of groups as one and the same thing, whether defined one way or the other.

Equation (2) can be abbreviated to

$$\varrho = \eta\mu,$$

and this form conveys the essence without reference to the element variables; but equations (1) or (3) can not, as they stand, be so relieved of inessentials.

To make it possible to write all expressions that contain latin letters (which in our convention stand for element variables) and greek letters (which stand for operators) in the form of a row of latin letters in some standard order followed by a row of greek letters, say

$$\ldots x_3 x_2 x_1 \omega_1 \omega_2 \ldots \omega_n,$$

we introduce new operators, which we call *standard* operators, namely the *identity* operator ι, the *repetition* operator φ, the *deletion* operator ψ, and the *transposition* operators τ_i, one for each $i = 1, 2, 3, \ldots$; they are defined by

$$\ldots x_3 x_2 x_1 \iota = \ldots x_3 x_2 x_1,$$
$$\ldots x_3 x_2 x_1 \varphi = \ldots x_3 x_2 x_1 x_1,$$
$$\ldots x_3 x_2 x_1 \psi = \ldots x_3 x_2,$$
$$\ldots x_{i+2} x_{i+1} x_i x_{i-1} \ldots x_1 \tau_i = \ldots x_{i+2} x_i x_{i+1} x_{i-1} \ldots x_1.$$

Using these operators jointly with the *variety* operators such as μ, η, ϱ, we can rewrite equations (1), (3) in the form

$$\ldots x_3 x_2 x_1 \tau_1 \tau_2 \mu \tau_1 \mu = \ldots x_3 x_2 x_1 \mu\mu,$$
$$\ldots x_3 x_2 x_1 \eta = \ldots x_3 x_2 x_1 \varphi\varphi\varrho\tau_1\varrho,$$
$$\ldots x_3 x_2 x_1 \mu = \ldots x_3 x_2 x_1 \varphi\varphi\varrho\tau_1\varrho\varrho.$$

Here we have made the convention that the string of latin letters (element variables) is infinite; and now, equating strings of greek letters (operators) if they have the same effect on $\ldots x_3 x_2 x_1$, we can further abbreviate the associative law of multiplication to

$$\tau_1 \tau_2 \mu \tau_1 \mu = \mu^2, \tag{4}$$

and the definitions of η and μ in terms of ϱ to

$$\eta = \varphi^2 \varrho \tau_1 \varrho, \qquad \mu = \varphi^2 \varrho \tau_1 \varrho^2.$$

We now consider the semigroup Γ generated by all standard operators $(\iota, \varphi, \psi, \tau_1, \tau_2, \ldots)$ and by the variety operators (such as μ, η, or ϱ) appropriate to the variety we are interested in. The standard operators generate a subsemigroup Σ of Γ, and Σ is fixed once and for all. The relations between the generators of Σ are imposed by their meaning, and exemplified by the following:

$$\varphi\psi = \iota, \quad \tau_i^2 = \iota, \quad \varphi\tau_2\tau_1\psi = \tau_1\psi\varphi. \tag{5}$$

These relations have been completely determined; and, as usual, the proof of the completeness of a set of defining relations is intricate and long. Here we shall not even give a complete list of the defining relations of Σ, but content ourselves with (5) as typical examples.

The standard operators and the variety operators are connected by two kinds of relations. The first kind, exemplified by

$$\eta\psi = \psi, \quad \eta\tau_2 = \tau_2\eta \tag{6}$$

and

$$\mu\psi = \psi^2, \quad \mu\tau_3 = \tau_3\mu, \tag{7}$$

stems from the fact that η is unary and μ binary; we call such relations *denomination relations*. Finally there are the relations, exemplified by (4), which express the laws that distinguish the variety. The denomination relations can again be completely listed, and again the completeness proof is laborious. The relations that express laws have to be thought of as additionally given.

Thus we connect a certain operator semigroup Γ_1 with the variety of groups expressed in terms of multiplication μ and inversion η; and another semigroup Γ_2 with the variety of groups expressed in terms of right division ϱ. What is the common feature of Γ_1 and Γ_2 that expresses the fact that groups are groups? Both Γ_1 and Γ_2 contain Σ, as do all operator semigroups Γ; but, moreover, Γ_1 and Γ_2 are isomorphic under an isomorphism mapping Σ identically and mapping $\eta\mu \in \Gamma_1$ on $\varrho \in \Gamma_2$, or, inversely, $\varphi^2\varrho\tau_1\varrho \in \Gamma_2$ on $\eta \in \Gamma_1$ and $\varphi^2\varrho\tau_1\varrho^2 \in \Gamma_2$ on $\mu \in \Gamma_1$. This is typical of the situation, and it answers our main question: The semigroups Γ_i that belong to varieties $\mathfrak{V}_i$ are isomorphic, under an isomorphism whose restriction to Σ is the identity isomorphism, if, and only if, the varieties $\mathfrak{V}_i$ are the same, that is to say, consist of the same algebras in terms of possibly different operations. This can be made precise, but the above example of the class of groups will sufficiently illustrate what is meant.

Finally we mention, without going into any details, that the principal tool in the discussion of Γ, and in particular of the defining relations of Γ, was a representation of the elements of Γ by braids similar in kind to those introduced by Artin [1].

REFERENCES

[1] E. Artin, *Theorie der Zöpfe*, Abhandlungen aus dem Mathematischen Seminar der Universität zu Hamburg 4 (1926), p. 47-72.

[2] E. C. Dale, *Semigroup and braid representations of varieties of algebras*, Ph. D. thesis, University of Manchester, 1956.

Reçu par la Rédaction le 29. 11. 1964