COLLOQUIUM MATHEMATICUM

XIV WARSZAWA—WROCŁAW 1966

FREE OBJECTS IN THE THEORY OF CATEGORIES*

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An object A is *free* with respect to a category $\mathfrak{K}$ of algebras of a fixed type if it has a set of free generators, i. e., if there exists a set M of generators of A such that for every algebra B in the category every map $\varphi: M \to B$ can be extended to a homomorphism $\varphi: A \to B$. It is known (cf. [5], p. 23) that the notion of a free algebra cannot be defined purely in categorical terms, i. e. in terms of homomorphisms and their superpositions (without using the notion of an element of an algebra). A categorical substitute of the notion of a free algebra and several examples are given in [9]. Another example is in [10], and still another example follows.

Let us consider the category of metric spaces of diameter less than or equal to 1, the morphisms being contractions (a *contraction* means a transformation which satisfies the Lipschitz condition with the constant 1). Then an isomorphism is just an isometry. A *free join* of a family $\{A_t\}_{t\in T}$ of objects in this category can be constructed as a disjoint union $A = \sum A_t$ (a free join in the category of all sets) metrized as follows: $\varrho(a, b) = \varrho_t(a, b)$ if both a and b belong to the same space A_t, and $\varrho(a, b) = 1$ if $a \in A_t$, $b \in A_u$, $t \neq u$. A *direct join* is the Cartesian product $\prod A_t$ with the uniform metric

$$\varrho(\{a_t\}, \{b_t\}) = \sup\{\varrho_t(a_t, b_t) : t \in T\}.$$

Clearly a one-point set is a basic free object in this category and hence a "free metric space" is a set A with the trivial distance $\varrho(a, b) = 1$ for $a \neq b$.

The notion of a basic free object, defined in [9], is actually very close to that of an integral object due to MacLane [7], p. 507, yet they are not equivalent. It is easy to show that if a category has both an

* This paper was presented on the Conference on General Algebra held in Warsaw, September 7-11, 1964, and is a kind of supplement to the paper [9]. The notion of a category and main definitions can be found in [6] and [8]; other terms not explained here are from [9].

integral object and a basic free object, then they are isomorphic. The same applies to cointegral and basic direct objects. In the category of compact spaces and continuous maps, however, there is no cointegral object, but the unit interval is a basic direct object.

If F is a basic free object in a category $\mathfrak{K}$, we can define the notion of a supermorphism, viz. a morphism $\alpha : A \to B$ is called a *supermorphism* if the adjoint map

$$\mathrm{Hom}(F, \alpha) : \mathrm{Hom}(F, A) \to \mathrm{Hom}(F, B)$$

is onto (1). It can be shown that every retraction is a supermorphism and every supermorphism is an epimorphism.

We can also define the notion of a projective object in a standard way using a supermorphism instead of a surjection. Then the Universality Theorem (2) can be simplified because we do not have to assume that F is projective and strict, for both the assumptions are included in the definition of a supermorphism. Another advantage of the new approach is to avoid new primary notions and use categories instead of bicategories.

In certain categories supermorphisms have nice characterizations, but even if they do not, the problem of particular validity of the Universality Theorem is reduced to the problem of characterizing the supermorphisms.

Dually we define a submorphism (using a basic direct object). If we identify submorphisms in the standard way, we get a definition of a subobject.

In [9] there is a discussion of various troubles concerning the appropriate definition of a subobject in categorical terms; Grothendieck's definition is not acceptable in certain important categories (like the category of locally compact groups and continuous homomorphisms) and in Grothendieck's definition monomorphisms should be replaced by more special morphisms (by "injections"). The definition of a subobject which we have just introduced, has several advantages and in some categories it fits better our purposes than Grothendieck's definition using any monomorphisms (3). E. g., in the category of Banach spaces and

(1) The parallel terms "submap" and "supermap" have been used by MacLane [7], p. 497, in a somewhat similar context.

(2) See [9], p. 16. The author is obliged to the reviewer for pointing out that the set T defined in the proof of Proposition 5.4 in [9], p. 15, should be assumed to be non-empty, because otherwise the statement is not true. It is enough to assume that there exist objects A and B and morphisms $\alpha: A \to B$ and $\beta: A \to B$ in the category such that $\alpha \neq \beta$. The same assumption should be added to the Universality Theorem (p. 16).

(3) Ehresmann [2] gives a definition of an injection and a surjection in a category $\mathfrak{K}$, relative to a given functor into another category $\mathfrak{K}_1$ and a subcategory of $\mathfrak{K}_1$. This, however, is not an intrinsic definition that we are looking for.

bounded linear operators, a submorphism is just an isomorphism (in Banach's sense) onto a closed subspace, and a supermorphism is just any morphism onto.

In the category of completely regular Hausdorff spaces and continuous maps, a map $\alpha : A \to B$ is a submorphism if and only if it is a homeomorphism onto a subset C of B such that every bounded real-valued continuous function on C can be extended to a continuous function on B with the same supremum and infimum. E. g., a homeomorphism onto a closed subspace of a normal space is a submorphism, and the embedding $A \to \beta A$ into the Stone-Čech compactification is also a submorphism.

In the category of locally compact Abelian groups and continuous homomorphisms, the canonical embedding into the Bohr compactification is a submorphism though it need not be a homeomorphism (it must be continuous, one-one and a homeomorphism in the weak topology determined by the dual group).

The two notions, supermorphism and submorphism, can be used as a starting point for certain homological investigations in non-abelian categories. For instance, let us consider the following definition (see Freyd [3], p. 126): An injective envelope of an object A in an abelian category is a monomorphism $\alpha : A \to B$ satisfying the two following conditions: (1) B is injective, (2) if $\beta : B \to C$ is any morphism such that $\beta\alpha$ is a monomorphism, then β is a monomorphism.

In the category of Banach spaces and linear contractions, a monomorphism is just a ono-one morphism. If we compare the definition of an injective envelope quoted above with a recent paper by Cohen [1], then we infer that this definition is not suitable in the case of Banach spaces and the word "monomorphism" should be changed in each place to "a linear isometrical embedding", i. e., to a submorphism in this category.

Finally, one may ask the question whether a basic free object (or a basic direct object) is really necessary in the definition of a supermorphism (a submorphism) given above; one may conceive that any generator (any cogenerator, respectively) will do as well. The following example shows that the definition of a submorphism would be changed to a non-equivalent one if we took another generator instead of a basic direct object. Consider the category of completely regular Hausdorff spaces and continuous maps. Let I be the closed interval $[0,1]$, let R be the real line and let $\varphi : X \to \beta X$ be the embedding into the Stone-Čech compactification. Then

$$\varphi^* = \mathrm{Hom}(\varphi, I) : \mathrm{Hom}(\beta X, I) \to \mathrm{Hom}(X, I)$$

is onto and φ^* is a submorphism, whereas the map

$$\varphi' = \mathrm{Hom}(\varphi, R) : \mathrm{Hom}(\beta X, R) \to \mathrm{Hom}(X, R)$$

is not onto (unbounded continuous functions on X need not have continuous extensions to βX).

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Reçu par la Rédaction le 14. 11. 1964;

en version modifiée le 22. 4. 1965