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ON SUMS OF DIRECT SYSTEMS OF BOOLEAN ALGEBRAS

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0. Introduction. In [3] we described a construction of abstract algebras which we called the sum of a direct system of algebras. In this paper, we shall be concerned with algebras which can be represented as such sums of direct systems of Boolean algebras. It will turn out that they form an equational class which we shall describe in section 1. Finally, in section 2 we shall describe independent sets in such algebras. (Independence will be understood in the sense of [1].)

For the definition of the sum of a direct system of abstract algebras, as well as for the proofs of the two results (i) and (ii) that follow, the reader may consult [3].

(i) *If $\mathscr{A}$ is a direct system of algebras with the least upper bound property (i.e. such that every two elements of the set of indices of this system have the least common upper bound), containing at least two algebras, then in the sum $S(\mathscr{A})$ of this system all regular equations satisfied in all algebras of $\mathscr{A}$ are satisfied, whereas every other equation is false in $S(\mathscr{A})$. In $S(\mathscr{A})$ there are then no algebraic constants.*

Let us recall that an equation $s_1 = s_2$, where s_1 and s_2 are terms in an algebra, is called *regular* if the set of free variables occurring in s_1 is the same as the set of free variables occurring in s_2.

(ii) *Let $\mathfrak{A}$ be an algebra from an equational class $\boldsymbol{K}$, whose defining equations are all regular, and let $\boldsymbol{K}_g^*$ be the equational class defined by equations of class $\boldsymbol{K}$, to which equation $g(x, y) = x$, where $g(x, y)$ is a fixed term of $\mathfrak{A}$, has been added. Then algebra $\mathfrak{A}$ is representable as a sum of a direct system of algebras from the class $\boldsymbol{K}_g^*$ if and only if $g(x, y)$ satisfies the following equalities:*

(0.1) $$g(g(x, y), z) = g(x, g(y, z)),$$

(0.2) $$g(x, x) = x,$$

(0.3) $$g(x, g(y, z)) = g(x, g(z, y)),$$

(0.4) $$g(F(x_1, \ldots, x_n), y) = F(g(x_1, y), \ldots, g(x_n, y)),$$

$$(0.5)\qquad g\big(y, F(x_1, \dots, x_n)\big) = g\Big(y, F\big(g(y, x_1), \dots, g(y, x_n)\big)\Big),$$

$$(0.6)\qquad g\big(F(x_1, \dots, x_n), x_k\big) = F(x_1, \dots, x_n) \quad (k = 1, 2, \dots, n),$$

$$(0.7)\qquad g\big(y, F(y, \dots, y)\big) = y,$$

where in (0.4)-(0.7) $F(x_1, \dots, x_n)$ *is an arbitrary fundamental operation of* $\mathfrak{A}$ *and* n *is its arity.*

1. Sums of Boolean algebras as an equational class. We shall assume the axioms of a Boolean algebra $\mathfrak{B} = (X; +, \cdot, ')$ in the following form:

$$(1.1)\qquad x+x = x, \quad x\cdot x = x,$$

$$(1.2)\qquad x+y = y+x, \quad x\cdot y = y\cdot x,$$

$$(1.3)\qquad (x+y)+z = x+(y+z), \quad (x\cdot y)\cdot z = x(y\cdot z),$$

$$(1.4)\qquad x(y+z) = xy+xz, \quad x+yz = (x+y)(x+z),$$

$$(1.5)\qquad (x+y)' = x'\cdot y',$$

$$(1.6)\qquad (x')' = x,$$

$$(1.7)\qquad x+xx' = x,$$

$$(1.8)\qquad x\cdot x' = y\cdot y',$$

The sums of direct systems of Boolean algebras are described by the following

THEOREM I. *An algebra* $\mathfrak{A} = (X; +, \cdot, ')$ *is a sum of a direct system of Boolean algebras if and only if it satisfies equalities* (1.1)-(1.7) *and*

$$(1.8)'\qquad xx'+yy' = xx'yy'.$$

Proof. The necessity follows from (i). To prove the sufficiency consider the term $x\circ y = x+xy$. Observe first that

$$(1.9)\qquad (x\circ y)' = x'\circ y'.$$

Now we prove that the equation $g(x, y) = x\circ y$ satisfies the conditions (0.1)-(0.7). The proof of (0.1)-(0.3) and also of (0.4)-(0.7) for the fundamental operations $x+y$ and xy is similar to that given in [4] in analogous case, and so we omit it. It remains to prove that (0.4)-(0.7) hold for the fundamental operation x'.

In order to prove (0.4), remark at first that by substituting xy for y in formula (1.2) from [4], we obtain

$$(*)\qquad x+xy+xyz = x+xyz.$$

Applying in turn (1.7), (*) and (1.9), we obtain

$$\begin{aligned} x'\circ y &= x'+x'y = x'+x'(y+yy') = x'+x'y+x'yy' = x'+x'yy' \\ &= x'+x'y'+x'yy' = x'+x'(y'+yy') = x'+x'y' = x'\circ y' = (x\circ y)'. \end{aligned}$$

Formula (0.5) follows from the following chain of equalities:

$$y\circ(y\circ x)' = y\circ(y'\circ x') = y+y(y'+y'x') = y+yy'+yy'x' = y+yy'x'$$
$$= (y+yy')(y+x') = y(y+x') = y+yx' = y\circ x'.$$

Formulas (0.6) and (0.7) are consequences of (1.7).

From (ii) it follows that $\mathfrak{A}$ is a sum of a direct system of algebras $\mathfrak{A}_i$ in which conditions (1.1)-(1.7) and (1.8′), as well as

$$(1.10) \qquad x+xy = x,$$

are satisfied. It remains to prove that they are Boolean algebras, i.e. that condition (1.8) is satisfied. In fact, we have

$$xx' = xx'+xx'yy' = xx'+xx'+yy' = xx'+yy',$$

and, similarly, $yy' = xx'+yy'$, whence $xx' = yy'$.

Theorems I and 0(i) imply the following

COROLLARY. *If a direct system $\mathscr{A}$ contains precisely one Boolean algebra, then $S(\mathscr{A})$ is a Boolean algebra (in other words, we have $xx' = yy'$), and conversely. If every Boolean algebra of a direct system $\mathscr{A}$ is a one-element algebra, then $S(\mathscr{A})$ is a semilattice (in the sense that $x+y = xy$ and $x' = x$), and conversely.*

We say that a system $\mathscr{A}$ is *proper* if none of these cases occurs.

2. Algebraic operations in sums of Boolean algebras. As it is well known, the algebraic operations in Boolean algebras have a canonical representation in the form of Boolean functions. The question can be posed, whether a similar canonical form exists for algebraic operations in sums of direct systems of Boolean algebras.

Let us consider the sum of a direct system of Boolean algebras. Let us call a *non-zero atom* every operation of the form

$$p^*(x_1, \ldots, x_n) = x_1^{\delta_1}x_2^{\delta_2}\ldots x_n^{\delta_n},$$

where δ_i are equal to ± 1, and $x^1 = x$, $x^{-1} = x'$.

Every operation of the form

$$z^{(n)}(x_1, \ldots, x_n) = x_1\cdot x_1'\ldots x_n\cdot x_n'$$

will be called an *n-ary zero-atom*. Using 0(i) we get now the following equalities:

(i) $z^{(n)}(x_1, \ldots, x_n) = x_1\cdot x_1'\cdot x_2\cdot x_3\ldots\cdot x_n = x_1\cdot x_2\cdot x_2'\cdot x_3\cdot x_4\ldots\cdot x_n = \ldots;$

(ii) $p^*(x_1, \ldots, x_n)+z^{(n)}(x_1, \ldots, x_n) = p^*(x_1, \ldots, x_n)$, *where p^* is an arbitrary non-zero atom.*

Let us add that if the system $\mathscr{A}$ is proper, then all zero-atoms $z^{(n)}$ are essentially different, and that no zero-atom is equal to any non-zero atom. More precisely:

(iii) *If, for a certain n, we have*

$$z^{(n)}(x_1, \ldots, x_{n-1}, x_n) = z^{(n-1)}(x_1, \ldots, x_{n-1})$$

identically, then the sum $S(\mathscr{A})$ is a Boolean algebra (i.e. $\mathscr{A}$ contains only one algebra).

It suffices to put $x_1 = \ldots = x_{n-1} = x$ and $x_n = y$.

(iv) *If, for a certain n-ary non-zero atom p^*, we have*

$$p^*(x_1, \ldots, x_n) = z^{(n)}(x_1, \ldots, x_n)$$

identically, then $S(\mathscr{A})$ is a semilattice (i.e. every algebra belonging to $\mathscr{A}$ is a one-point algebra).

In fact, we have then

$$x_1^{\delta_1} \ldots x_n^{\delta_n} = x_1 x_1' \ldots x_n x_n'.$$

Put now here $x_k = x'$ if $\delta_k = 1$ and $x_k = x$ if $\delta_k = -1$. Then clearly $x = xx'$, thus $x' = xx'$, whence $x = x'$ and $x + y = xy$.

Now we prove

THEOREM II. *If $\mathscr{A}$ is a direct system of Boolean algebras, then every algebraic n-ary operation p in $S(\mathscr{A})$ depending on all its variables is either equal to an n-ary zero-atom or can be represented in the form*

$$p(x_1, \ldots, x_n) = \sum_{j=1}^{r} p_j^*(x_1, \ldots, x_n),$$

where p_j^* are non-zero atoms.

Proof. From (1.1)-(1.7) it is obvious that every n-ary algebraic operation can be represented as a sum of atoms $q_1^*, \ldots, q_r^*$, and the main thing to be verified is that one can obtain such a representation in which in every atom there occur all variables $x_1, \ldots, x_n$.

Denote by p_j^* the atom q_j^* multiplied by all sums $(x_k + x_k')$, where x_k are variables not occurring in q_j^*, and consider the equation

$$\text{(E)} \qquad \sum_{j=1}^{r} p_j^* = \sum_{j=1}^{r} q_j^*.$$

It is regular, since p depends on each of the variables $x_1, \ldots, x_n$, and it is trivially true in Boolean algebras. Consequently, it remains true in their sum, by 0(i). Finally, note that its left-hand side consists of n-ary atoms.

If in a product obtained in this way there exists a factor $x_i \cdot x_i'$, then by 2(i) this product is a zero-atom. In view of 2(ii) we may cancel it, save in the case when our sum does not contain non-zero atoms at all, in which case we obtain the first part of the theorem. So we arrive at the asserted form.

3. Independence in sums of Boolean algebras. Note that if in a sum of Boolean algebras the equality $x' = x$ is true, then $x+y = xy$ follows, and so we get a semi-lattice in which the independent sets were described by Szász [5]. If, however, (1.8) is satisfied, then we obtain a Boolean algebra in which the independent sets were described by E. Marczewski (see [2]).

Now we prove

THEOREM III. *If $\mathfrak{A}$ is a sum of a proper direct system of Boolean algebras, then a set D is dependent in $\mathfrak{A}$ if and only if for a certain finite system of its distinct elements $d_1, \ldots, d_n$ either*

(3.1) $$n > 1 \text{ and } z^{(n)}(d_1, \ldots, d_n) = z^{(n-1)}(d_1, \ldots, d_{n-1})$$

or there exists a non-zero atom p^* such that

(3.2) $$p^*(d_1, \ldots, d_n) = z^{(n)}(d_1, \ldots, d_n).$$

Proof. We shall use the following lemma of E. Marczewski, which characterizes independence by means of operations dependent on each of their variables (any "nullary" operation, i.e. a constant, is treated here as an operation depending on each of its variables):

(M) A set D is dependent in an algebra if and only if there are: a sequence $a_1, \ldots, a_m$ $(m \geqslant 0)$ of distinct elements of D, a sequence $a_{m+1}, \ldots, a_{m+n} (n \geqslant 0)$ of distinct elements of D and two operations $p_1 \epsilon A^{(m)}$ and $p_2 \epsilon A^{(n)}$ dependent on each of their variables such that

(a) $$p_1(a_1, \ldots, a_m) = p_2(a_{m+1}, \ldots, a_{m+n})$$

and either

(b_1) the sets $\{a_1, \ldots, a_m\}$ and $\{a_{m+1}, \ldots, a_{m+n}\}$ are distinct

or

(b_2) $$m = n,\ a_j = a_{m+j} \text{ for } 1 \leqslant j \leqslant m, \text{ and } p_1 \neq p_2.$$

Assume that D is a dependent set in $\mathfrak{A}$. We apply (M) and suppose (b_1) which can be precised e. g. as follows:

(b_1) $$a_k \notin \{a_{m+1}, \ldots, a_{m+n}\}$$

for a certain $k \leqslant m$. Since there is in $\mathfrak{A}$ no algebraic constant (see 0(i)), we have $m, n > 0$.

Using the dependence of p_1 and p_2 on each variable, we apply Theorem II to both sides of (a). Next, by multiplying the so modified formula (a) by all products $a_i a_i'$ with $k \neq i \leqslant m+n$, we obtain on both sides zero-atoms, namely, in view of 2(i),

$$z^{(m+n)}(a_1, \ldots, a_{m+n}) = z^{(m+n-1)}(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{m+n}),$$

and by applying 2(i) once more, a formula of the form (3.1).

Let us now consider the case (b_2). Formula (a) gives now

$$p_1(a_1, \ldots, a_m) = p_2(a_1, \ldots, a_m). \tag{a*}$$

We apply Theorem II again, and use the dependence of p_1 and p_2 on each of their variables. Let p^* be a non-zero atom which occurs in the canonical representation of only one of the operations p_1 and p_2. Now, by multiplying (a*) by $p^*(a_1, \ldots, a_m)$ we obtain, using 2(i) and 2(ii), an equality of the form (3.2).

The necessity is thus proved and the sufficiency follows from 2(ii) and 2(iv) by using the assumption that $\mathscr{A}$ is proper.

REFERENCES

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