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DIRICHLET PROBLEMS FOR DISTRIBUTION BOUNDARY VALUES

BY

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1. Harmonic functions. Much work has gone into extending the classical Dirichlet problem to more general boundaries, spaces, and operators (cf. Bauer [1], Brelot [2] and [3], Stampacchia [19]).

In contrast we will consider the case of smooth boundaries with (Schwartz) distribution boundary values. The Dirichlet problem for harmonic functions and distribution boundary values can be treated with operator techniques (cf. Lions [15]). We will, however, rely on methods based on kernel representations, a technique which was widely used — for special domains and different purposes — in Bremermann [8] (cf. also Bremermann [7]).

Distribution boundary value problems occur in fact — if not in name — already in classical problems of mathematical physics: The potentials of a "point charge," a "doublet," and a "double layer."

We consider regions Ω in Euclidean space E^n whose boundary $\partial\Omega$ is a real analytic manifold. Classically, there exists a kernel $k(x, \xi)$ such that

$$h(x) = \int_{\partial\Omega} k(x, \xi) b(\xi) d\omega_\xi$$

is the solution of the Dirichlet problem (for continuous boundary values $b(\xi)$ on $\partial\Omega$, $d\omega_\xi$ the surface element of $\partial\Omega$ induced by the metric of E^n). For fixed $\xi \in \partial\Omega$ the kernel is harmonic in x.

2. Distributions on $\partial\Omega$. Let $\mathscr{D}(\partial\Omega)$ be the space of (C^∞) functions with convergence defined as follows: Let

$$D_t^p = \frac{\partial^{p_1+\ldots+p_n}}{\partial t_1^{p_1}\ldots\partial t_n^{p_n}},$$

$p_i \geqslant 0$, t local coordinates. A sequence $\langle\varphi_n\rangle$ of (C^∞) functions is said *to converge in the sense of* $\mathscr{D}$ if and only if the sequence of derivatives $\langle D_t^p \varphi_n\rangle$

converges uniformly on any compact set in any local coordinate patch. Let $\mathscr{D}'(\partial\Omega)$ denote the dual space. We call its elements *distributions* on $\partial\Omega$. Locally the distributions on $\partial\Omega$ coincide with Schwartz distributions $\mathscr{D}'(E^n)$.

For $x \in \Omega$, $\xi \in \partial\Omega$, the kernel $k(x, \xi)$ is a (C^∞) function (cf. Courant-Hilbert [9]). Given $T \in \mathscr{D}'(\partial\Omega)$ we form $T^*(x) = \langle T_\xi, k(x, \xi)\rangle$. $T^*(x)$ is a harmonic function in Ω. This can be seen by differentiating $T^*(x)$:

$$\frac{\partial}{\partial x_j} T^*(x) = \lim_{\Delta h_j \to 0} \frac{T^*(x+\overrightarrow{\Delta h_j}) - T^*(x)}{\Delta h_j}$$

$$= \lim_{\Delta h_j \to 0} \langle T_\xi, [k(x+\overrightarrow{\Delta h_j}, \xi) - k(x, \xi)]/\Delta h_j\rangle,$$

where $\overrightarrow{\Delta h_j} = (0, \ldots, \Delta h_j, \ldots, 0)$. The difference quotient for k converges in the sense of $\mathscr{D}'(\partial\Omega)$ to $\partial\big(k(x, \xi)\big)/\partial x_j$. Thus $\Box T^*(x) = \langle T_\xi, \Box_x k(x, \xi)\rangle = 0$, since k is harmonic.

The function $T^*(x)$ assumes the boundary distribution in the following sense: Approximate $\partial\Omega$ by homologous surfaces S_ν in Ω that are analogously parametrized: For every test function $\varphi \in \mathscr{D}(\partial\Omega)$ we have

$$\lim_{\nu\to\infty} \int_{S_\nu} T^*(x)\varphi(x)\, d\omega_x = \langle T, \varphi\rangle.$$

This follows by converting $\int_{S_\nu} T^*(x)\varphi(x)\, d\omega_x$ into $\langle T_\xi, \int_{S_\nu} \varphi(x) k(x, \xi)\, d\omega_x\rangle$ by an argument analogous to Bremermann [8], p. 49.

Now

$$\int_{S_\nu} \varphi(x) k(x, \xi)\, d\omega_x \to \varphi(\xi)$$

in the sense of $\mathscr{D}$ convergence. (The argument is analogous to [8], p. 46, taking into consideration that $k(x, \xi)$ is the normal derivative of the Green's function of Ω and the fact that the Green's function is symmetric in x and ξ.)

Thus

$$\langle T, \int_{S_\nu} \varphi(x) k(x, \xi)\, d\omega_x\rangle \to \langle T, \varphi\rangle.$$

Thus the restrictions of $T^*(x)$ to the homologous surfaces S_ν converge to T in the topology of $\mathscr{D}$.

Examples. For dimension 2 and Ω the upper complex z-plane, $z = x+iy$, we have the kernel $k(z, \zeta) = y/\pi|z-\zeta|^2$. As "homologous surfaces" we can take lines parallel to the x-axis.

For the unit disk, using polar coordinates, we have the Poisson kernel

$$\frac{1}{2\pi}\,\frac{1-r^2}{1-2\cos(\vartheta-\psi)+r^2}.$$

The S_ν may be taken as concentrix circles of smaller diameter.

3. Plurisubharmonic functions. In a previous paper [6] it was shown that the Perron-Carathéodory method when applied to continuous boundary values prescribed on the Šilov boundary of a pseudo-convex region leads to a plurisubharmonic solution. Górski [12] and Siciak [16] subsequently have carried the methods further. Górski [11] and Siciak [17], [18] also have applied Leja's method of extremal points to this problem (cf. Leja [13], [14]).

There remains the problem of the uniqueness of the solution of the generalized Dirichlet problem. For special cases some results are known (cf. Szmydt [20]).

The Dirichlet problem for plurisubharmonic functions and *distribution boundary values*, however, is largely unsolved. Consider a strictly pseudo-convex region Ω in C^n which, as a region in E^{2n}, satisfies the conditions of section 1. The Šilov boundary, in this case, coincides with the topological boundary of Ω. (Note that any region of holomorphy can be approximated by strictly pseudo-convex regions with real analytic boundary, Bremermann [4] and [5].) The extremal plurisubharmonic solution of the continuous boundary value problem, in general, cannot be represented by means of a kernel. Let $\Phi(b_1, z)$, $\Phi(b_2, z)$ and $\Phi(b_1+b_2, z)$ be the extremal plurisubharmonic solutions of the boundary values b_1, b_2, and b_1+b_2 respectively. $\Phi(b_1, z)+\Phi(b_2, z)$ is a *solution* of the boundary values b_1+b_2 but it is not necessarily *extremal*: $\Phi(b_1, z)+\Phi(b_2, z) \leqslant \Phi(b_1+b_2, z)$. It is easy to give examples where the inequality occurs. If Φ were represented by a kernel, it would be additive. It is not known whether the extremal solution of continuous boundary values is continuous ([1]).

While we have shown that there is no kernel for the extremal solution there could still be a kernel that would give some plurisubharmonic solution, not necessarily extremal. This is not the case. There is no kernel $k(z, \xi)$ on $\Omega\times\partial\Omega$ such that for arbitrary continuous boundary values $b(z)$ the function $\Phi(z) = \int_{\partial\Omega} k(z, \xi)b(\xi)d\omega_\xi$ is plurisubharmonic in Ω and assumes the boundary values $b(z)$. If this were the case, then also the function

$$-\Phi(z) = \int_{\partial\Omega} k(z, \xi)[-b(\xi)]d\omega_\xi$$

([1]) Mr. J. Walsh (Stanford, Calif.) has shown that the answer to this problem is affirmative (unpublished) (*Added in proof*).

would be plurisubharmonic and consequently $\Phi(z)$ would be pluriharmonic. This is a contradiction; it is well known that not all continuous functions $b(\xi)$ on $\partial\Omega$ are boundary values of a pluriharmonic function in Ω.

Since there is no kernel, the method of section 2 is not applicable in order to obtain the solution of a distribution boundary value problem. Instead one could apply the Perron-Carathéodory method: Consider the family of all plurisubharmonic functions that are (in a sense to be explained) less than or equal to the given distribution.

For this purpose we consider distributions that are bounded below in the following sense: There exists a constant C such that $\langle C, \varphi\rangle \leqslant \langle T, \varphi\rangle$ for all non-negative test functions. Then $T-C$ is a non-negative distribution, and non-negative distributions are non-negative measures (Friedman [10], p. 51). Thus we are in effect considering the Dirichlet problem for *measure boundary values*. Without loss of generality, we will consider non-negative measures.

Let $\mathscr{F}$ be the class of all functions that are plurisubharmonic in Ω and less than or equal to the *harmonic* solution $h(z)$ of the boundary value problem. Let $V(z)$ be the upper envelope:

$$V(z) = \limsup_{z'\to z} \big(\sup_{U\in\mathscr{F}} U(z')\big).$$

This function is plurisubharmonic since $\mathscr{F}$ is bounded from above by $h(z)$ and thus is bounded from above on every compact subset of Ω. Since the constant 0 belongs to $\mathscr{F}$, we have $V(z) \geqslant 0$. It is not known whether the function $V(z)$ assumes the boundary measure (in the same sense as $h(z)$.)

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