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ON FACTORIZATION OF MAPS THROUGH τX

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All spaces are Hausdorff and all maps continuous. Let $\tau_X : X \subset \tau X$ denote H-closed Katětov extension of X (Katětov [4]). A map $f: X \to Y$ will be said to be *τ-proper* iff there exists (in consequence, unique) a map $\tau f: \tau X \to \tau Y$ completing the diagram

$$
\begin{array}{ccc}
X & \subset & \tau X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tau f} \\
Y & \subset & Y
\end{array}
\tag{1}
$$

If, in addition, f maps the remainder into the remainder, i.e. if $\tau f(\tau X - X) \subset \tau Y - Y$, then f will be said to be *τ-perfect*, in analogy to those maps in the diagram of the Čech-Stone compactification which behave in the same way and which are usually called *perfect* (or *β-perfect*). It is known from Henriksen and Isbell [2] that f is perfect iff f is closed and $f^{-1}(y)$ is compact for each $y \in Y$. By the definition, τ-perfect maps are precisely those which preserve H-closedness under counter-images, i.e. such that X is H-closed whenever $f: X \to Y$ is onto and Y is H-closed.

The aim of this paper is to give topological (non-diagrammatical) characterizations of τ-proper and τ-perfect maps. A necessary condition for a map to be τ-proper was given by one of the authors and L. Rudolf in [5]: τ-proper maps are proper in a sense defined in [5] (see also Section 1).

And a sufficient condition was given by Herrlich and Strecker [8]: semi-open maps in a sense of [3] (see also Section 1) are τ-proper.

Clearly, τ-perfect maps are τ-proper, but the converse is not true: each map $f: X \to Y$ into a compact space Y is τ-proper (the fact known since Katětov's paper [4]), but such a map is not τ-perfect unless X is H-closed.

One of characterizations of τ-perfect maps given below is similar to that of [2] for perfect maps, another is expressed in terms of ultrafilters and is more convenient for technical reasons. Also for τ-proper maps two characterizations are given, one similar to that of [5] and another expressed in terms of ultrafilters.

It is an easy consequence of the definitions that τ-proper as well as τ-perfect maps form subcategories of the category H of all Hausdorff spaces and their continuous maps. If we restrict ourselves to the category H' of all τ-proper maps of Hausdorff spaces, then the Katětov extension leads to a functor $\tau\colon H' \to HCL$ from H' into the full subcategory of H' including all H-closed spaces and τ is a reflection, i.e. τ is an adjoint functor to the embedding $\iota\colon HCL \subset H'$. Moreover, H' is the greatest subcategory of H having this property with respect to the Katětov extension (see a note in [5], p. 22). Herrlich and Strecker proved in [3] that in the subcategory of H consisting of all semi-open maps the Katětov extension also leads to a reflection.

1. τ-proper maps. In [5] were studied maps $f\colon X \to Y$, called there *proper*, enjoying the property

(2) if $y \in V \subset Y$ and V is open in Y, then there exists V', open in Y, such that $y \in V'$ and $\operatorname{Int} f^{-1}(\operatorname{Cl} V') \subset \operatorname{Cl} f^{-1}(V)$.

It was proved in [5] that, in the case of Y being H-closed, proper maps $f\colon X \to Y$ coincide with those for which the diagram

(3)
$$\begin{tikzcd} X \arrow[r, phantom, "\subset"] \arrow[d, "f"'] & \tau X \arrow[dl, dashed, "f_*"] \\ Y & \end{tikzcd}$$

can be completed. This means that for maps into H-closed spaces τ-proper maps coincide with the proper ones.

For arbitrary Y only one implication holds; namely, each τ-proper map is proper (see [5], 4.4 on p. 21, and 3.1 on p. 18). The converse is not true: an example of [5], p. 22, shows that there exists a proper embedding $X \subset Y$ such that the composition $X \subset Y \subset \tau Y$ is not proper; in consequence, $X \subset Y$ is not τ-proper. This example shows also that proper maps do not form a category (dense embeddings are always proper). It can be modified in a way that $X \to Y$ becomes onto.

Example. Let X be the rectangle $\{(x, y)\colon -1 \leqslant x \leqslant 1 \text{ and } 0 < y \leqslant 1\}$, and let Y be the "left half" of X consisting of points with $x \leqslant 0$. If $f\colon X \to Y$ is retraction $(x, y) \to (0, y)$ for $x \geqslant 0$, then f is proper, because Y is regular.

We proceed to show that composition $g\colon X \xrightarrow{f} Y \subset \tau Y$ is not proper. To do this, let $\eta \in \tau Y - Y$ be an ultrafilter in Y which is without adherence points(1) and is an extension of the filter consisting of all regularly open subsets of Y containing open segments $(0, a)$ of y-axis with $a > 0$. If $U' \in \eta$, then $\operatorname{Cl} U'$ meets the y-axis in a set having non-empty interior

(1) *Ultrafilter* means always ultrafilter in the family of all open subsets of the space; η is an *ultrafilter without adherence points if* $\bigcap\{\operatorname{Cl} V\colon V \in \eta\} = \emptyset$.

$J(U')$ relatively to the y-axis, because in the other case the set $Y - \mathrm{Cl}\, U'$ would be in η, and this is impossible, $Y - \mathrm{Cl}\, U'$ being disjoint with U'. Hence, if $U' \in \eta$, then the set $f^{-1}(\mathrm{Cl}\, U')$, being equal to the set $g^{-1}(\mathrm{Cl}_Y(U' \cup \cup \{\eta\}))$, meets the "right half" of X in a set having a non-empty interior $(0, 1] \times J(U')$ in X. The set $U = \{(x, y) \in Y: x^2 + y^2 < r^2, x < 0 \text{ and } y > 0\}$, where r is a positive real number, is a member of η. The set $\mathrm{Cl}\, g^{-1}(U \cup \{\eta\})$ contains no points with $x > 0$, Thus $U \cup \{\eta\}$ is an open neighbourhood V of η such that there does not exist a neighbourhood V' of η for which formula (2) holds. This means that g is not proper.

Note. The map f in the example is perfect, because it is closed and its counter-images of points consist of closed intervals $[0, 1]$ or points. Thus f is perfect and onto but not τ-proper. The existence of such a map contradicts a result of Veličko (see [6], Theorem 1) who claimed that if f is perfect and onto, then it is τ-perfect(²). In Section 2 we shall give yet another example to this effect. See also our theorems below.

THEOREM 1. *A map* $f: X \to Y$ *is τ-proper iff it is proper and*

(4) *for each ultrafilter η in Y without adherence points and each $V \in \eta$ there exists $V' \in \eta$ such that* $\mathrm{Int} f^{-1}(\mathrm{Cl}\, V') \subset \mathrm{Cl} F^{-1}(V)$.

Note. Of course, if Y is H-closed, (4) is superfluous.

Proof (of the lacked implication). According to a theorem of [5] it is sufficient to prove that if a map $f: X \to Y$ is proper and satisfies condition (4), then the composition $g: X \xrightarrow{f} Y \subset \tau Y$ is proper.

To do this take $\eta \in W \subset \tau Y$, where W is open in τY.

If $\eta \in Y$, the set $V = Y \cap W$ is an open neighbourhood of η in Y. Since f is proper, there exists V', open in Y, such that $\eta \in V'$ and $\mathrm{Int} f^{-1}(\mathrm{Cl}\, V') \subset \mathrm{Cl} f^{-1}(V)$. Since Y is open in τY, V' is open in τY. Therefore (2) is satisfied at η by the composition $g: X \xrightarrow{f} Y \subset \tau Y$, and this means that g is proper.

And if $\eta \in \tau Y - Y$, the sets $V = Y \cap W$, where W run over all open neighbourhoods of η in τY, form an ultrafilter in Y without adherence points (this ultrafilter is, in fact, equal to η itself). By (4), for a given $V \in \eta$, there exists $V' \in \eta$ such that $\mathrm{Int}\, f^{-1}(\mathrm{Cl}\, V') \subset \mathrm{Cl}\, f^{-1}(V)$. Define $W' = \{\eta\} \cup V'$. Then we have $Y \cap W' = V'$ and, therefore, $\mathrm{Cl}_Y V' = Y \cap \cap \mathrm{Cl}_Y W'$. Now proceed as in the preceding case.

THEOREM 2. *A map* $f: X \to Y$ *is τ-proper iff for each ultrafilter ξ in X there is*

$$\bigcap \{\mathrm{Cl}_Y f(U): U \in \xi\} \neq \emptyset$$

or if there exists an ultrafilter η in Y without adherence points such that $V \cap f(U) \neq \emptyset$ for each $V \in \eta$ and each $U \in \xi$.

(²) On p. 1255 of [6] there is an erroneous proposition "the family $t = \dots$ is a centered one"

Note. Condition on f can be expressed shortly in terms of Katětov extensions as follows: $\bigcap\{\mathrm{Cl}_{\tau Y} f(U) : U \in \xi\} \neq \emptyset$ for each ultrafilter ξ in X. Therefore Theorem 2 follows from the following.

THEOREM 2′. *The triangle diagram* (3) *can be completed for a map* $f: X \to Y$ (*we do not assume here that* Y *is* H-*closed*) *iff for each ultrafilter* ξ *in* X *there is*

$$\bigcap\{\mathrm{Cl}_Y f(U) : U \in \xi\} \neq \emptyset. \tag{5}$$

Proof of Theorem 2′. I. If there exists a map f_* such that $f_* \circ \tau_X = f$, then let ξ be an ultrafilter in X. Then

$$\begin{aligned}\bigcap\{\mathrm{Cl}_Y f(U) : U \in \xi\} &= \bigcap\{\mathrm{Cl}_Y f_*(U) : U \in \xi\} \\ &\supset \bigcap\{f_*(\mathrm{Cl}_{\tau X} U) : U \in \xi\} \supset f_*(\bigcap\{\mathrm{Cl}_{\tau X} U : U \in \xi\}),\end{aligned}$$

the last set in parentheses being non-empty.

II. Assume that (5) holds for each ultrafilter ξ in X. Define $f_*(x) = x$ for $x \in X$. It remains to define $f_*(\xi)$ for $\xi \in \tau X - X$, i.e. for ultrafilters ξ in X without adherence points.

The set $[\xi] = \bigcap\{\mathrm{Cl}_Y f(U) : U \in \xi\}$ is non-empty by the assumption. If $y \in [\xi]$ and if V is an open neighbourhood of y in Y, then for each $U \in \xi$ there is $U \cap f^{-1}(V) \neq \emptyset$ (because of $y \in \mathrm{Cl}_Y f(U)$), which implies $f^{-1}(V) \in \xi$. In consequence, $[\xi]$ is a one-point set: in fact, if y_1 and y_2 are in $[\xi]$ and $y_1 \neq y_2$, let V_1 and V_2 be disjoint open neighbourhoods of these points; then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are both in ξ, but $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$; a contradiction.

Define $f_*(\xi)$ to be the point $[\xi]$.

Clearly, $f_* \circ \tau_X = f$.

To prove that f_* is continuous, let V be open in Y. We have $f_*^{-1}(V) = f^{-1}(V) \cup R$, where $R = \{\xi \in \tau X - X : f_*(\xi) \in V\}$. The set $f^{-1}(V)$ is open in X and $f^{-1}(V) \in \xi$ for each $\xi \in R$. Thus $f^{-1}(V) \cup R$ is open in the topology of the Katětov extension τX (recall that the sets $\{\xi\} \cup U$, where $U \in \xi$, form an open base in τX at ξ, $\xi \in \tau X - X$).

Note. Theorem 2 implies that semi-open maps, i.e. maps $f: X \to Y$ such that $\mathrm{Int} f(U) \neq \emptyset$ for open U, are τ-proper — a fact known from [3]. On the other hand, however, there exist τ-proper maps which are not semi-open, e.g., each not semi-open map into a compact space.

2. τ-perfect maps. A (closed) subset A of X will be said to be *far from the remainder*, shortly f.f.r., if for each ultrafilter ξ in X without adherence points there exists $U \in \xi$ such that $A \cap \mathrm{Cl}_X U = \emptyset$.

Obviously, each compact subset A of X is always f.f.r. But there exist H-closed subspaces which are not f.f.r.

Example. In the known example of a (countable) minimal Hausdorff space due to Urysohn (and often quoted in the literature; see e.g.

[5], p. 25) let us remove one of two non-regular points. The H-closed "half" containing the other non-regular point is not f.f.r.

THEOREM 3. *A map $f\colon X \to Y$ is τ-perfect iff it is τ-proper and*

(6) $f^{-1}(y)$ is f.f.r. *for each* $y \in Y$,

(7) $f(A)$ *is closed for each regularly closed subset of* X.

Proof. I. Suppose that f is τ-proper and satisfies (6) and (7). Let $\tau f\colon \tau X \to \tau Y$ be a map completing diagram (1). Let $\xi \in \tau X - X$ and suppose that $\tau f(\xi) = y \in Y$. Then $U \cap f^{-1}(V) \neq \emptyset$ for each open neighbourhood V of y in Y and for each $U \in \xi$. Since $f^{-1}(y)$ is f.f.r., there exists $U \in \xi$ such that

(8) $$f^{-1}(y) \cap \mathrm{Cl}_X U = \emptyset.$$

We have $V \cap f(U) \neq \emptyset$ for each V, whence $y \in \mathrm{Cl}_Y f(U)$. But $\mathrm{Cl}_Y f(U) = f(\mathrm{Cl}_X U)$, since $\mathrm{Cl}_X U$ is regularly closed. Thus $y \in f(\mathrm{Cl}_X U)$, contrary to (8).

II. Let f be τ-perfect. To prove (6), let $y \in Y$ and $\xi \in \tau X - X$. Let $\eta = \tau f(\xi)$. By the assumption, $\eta \in \tau Y - Y$. Consequently, there exists W, an open neighbourhood of η in Y, such that $y \notin \mathrm{Cl}_{\tau Y} W$. Hence $f^{-1}(y) \cap \cap (\tau f)^{-1}(\mathrm{Cl}_{\tau Y} W) = \emptyset$, which implies that $f^{-1}(y) \cap \mathrm{Cl}_{\tau X} (\tau f)^{-1}(W) = \emptyset$ and, a fortiori $f^{-1}(y) \cap \mathrm{Cl}_X \big((\tau f)^{-1}(W) \cap X\big) = \emptyset$. But $(\tau f)^{-1}(W) \cap X \in \xi$, and this means that $f^{-1}(y)$ is f.f.r.

To prove (7), take a regularly closed subset A of X. Then $\mathrm{Cl}_{\tau X} A$ is H-closed. We have $f(A) = (\tau f)(\mathrm{Cl}_{\tau X} A) \cap Y$ and the last set is closed in Y, since $(\tau f)(\mathrm{Cl}_{\tau X} A)$ is closed in τY as an image of H-closed set.

Note. There is a difference in the conditions for τ-perfect and β-perfect maps. In order for a map to be β-perfect conditions like (6) and (7) are sufficient, but in the case of τ-perfect maps the additional condition that f is τ-proper cannot be omitted. Recall that f is τ-proper iff it is proper and satisfies (4). We shall show that none of these conditions can be omitted.

The example in Section 1 shows that (4) cannot be omitted: the map constructed there is proper and satisfies (6) and (7), while being not τ-proper and, in consequence, not τ-perfect.

Also the condition that f is proper cannot be omitted.

Example. Let Y be the square $\{(x, y)\colon 0 \leqslant x \leqslant 1 \text{ and } 0 \leqslant y \leqslant 1\}$ in which the bottom side (with $y = 0$) is a discrete subspace, the rest of Y having the topology of the plane. The space Y is H-closed. Let X be the disjoint union of Y and of a discrete space consisting of points $(x, -1)$, where $0 \leqslant x \leqslant 1$, and let $f\colon X \to Y$ be a retraction $(x, -1) \to (x, 0)$.

Since Y is H-closed and X is not, the map is not τ-perfect. It is easy to verify that f has properties (6), (7) (it is even β-perfect and onto), and (4). Clearly, f is not proper.

Note. Unfortunately, it is impossible to improve the theorem with "iff". But perhaps it is worth to mention that one of the implications in Theorem 3 may be strengthened.

Let us call a (closed) subset A of X *conditionally H-closed* iff $\mathrm{Cl}_{\tau X} A$ is H-closed (the definition may be expressed in terms avoiding τX). Clearly, each regularly closed subset is conditionally H-closed. It is easy to see that τ-perfectness implies

(7') $f(A)$ is closed for each conditionally H-closed subset of X,

a condition which is stronger than (7).

THEOREM 4. *A map $f\colon X \to Y$ is τ-perfect iff for each ultrafilter ξ in X without adherence points there exists an ultrafilter η in Y without adherence points such that for each $U \in \xi$ and $V \in \eta$ there is*

$$f(U) \cap V \neq \emptyset. \tag{8}$$

Proof. I. If $\xi \in \tau X - X$, choose $\eta \in \tau Y - Y$ for this ξ according to the assumption. Define $\tau f\colon \tau X \to \tau Y$ by setting $\tau f(x) = f(x)$ for $x \in X$ and $\tau f(\xi) = \eta$ for $\xi \in \tau X - X$.

Clearly, $\tau f(\tau X - X) \subset \tau Y - Y$.

To prove the continuity of τf, let us note that $U \cap f^{-1}(V) \neq \emptyset$ for $U \in \xi$ and $V \in \eta$, which follows from (8). Hence, if $V \in \eta$, then $f^{-1}(V) \in \xi$. Let now $V \in \{\eta\}$ be a base neighbourhood of ξ in τY. Then $(\tau f)^{-1}(V \cup \{\eta\})$ is an open neighbourhood of ξ in τX, and this means that τf is continuous at $\xi \in \tau X - X$. The continuity at $x \in X$ follows from the continuity of f at x in virtue of the fact that X and Y are both open in their Katětov extensions.

II. Let f be τ-perfect and let τf exist. Take $\xi \in \tau X - X$. We have $\tau f(\xi) \in \tau Y - Y$. We shall prove that formula (8) holds for each $U \in \xi$ and each $V \in \eta = \tau f(\xi)$. In fact, $(\tau f)^{-1}(V \cup \{\eta\})$ is an open neighbourhood of ξ in τX. According to the known properties of the Katětov extensions, the set $X \cap (\tau f)^{-1}(V \cup \{\eta\})$ is a member of ξ. But, since f is τ-perfect, this set is equal to $f^{-1}(V)$. Therefore, $f^{-1}(V)$, being a member of ξ, intersects each $U \in \xi$. Hence, $V \cap f(U) \neq \emptyset$.

Note. In fact, it was proved that if f is τ-perfect, then η is uniquely determined by ξ, since $\eta = \tau f(\xi)$ and τf is uniquely determined by f.

3. Non-extendable maps. A map $f\colon X \to Y$ will be said to be *non-extendable* iff there does not exist an extension of f to a map $\tilde{f}\colon Z \to Y$ such that $\tilde{f}(Z) = f(X)$, where Z contains X as a dense subspace; such extensions $\tilde{f}$ will be called *image preserving*. Clearly, in the definition of non-extendable maps it suffices to consider only spaces Z of the form $X \cup \{x\}$, where $x \notin X$.

It is known from [1] that in the category of all completely regular spaces and their continuous maps the non-extendable maps coincide with the perfect ($=\beta$-perfect) ones. Here we shall prove that in the category H' of all τ-proper maps of Hausdorff spaces the non-extendable maps coincide with the τ-perfect ones.

LEMMA. *A map $f\colon X\to Y$ is non extendable iff*

(9) *there does not exist an image preserving extension of f on $X\cup\{\xi\}$, where $\xi\in\tau X-X$ and the topology on $X\cup\{\xi\}$ is induced from τX.*

Proof. Clearly, if f is non-extendable, then (9) holds. Now assume (9) and suppose that f is not non-extendable. Hence there exists an image preserving extension of f on a space $Z=X\cup\{x\}$, where $x\notin X$ and X is dense in Z. The family of intersections with X of all open neighbourhoods of x in Z forms a centered familly in X. Let ξ be an ultrafilter generated by this family. Clearly, it is an ultrafilter without adherence points. Hence there exists a map $X\cup\{\xi\}\xrightarrow{g}X\cup\{x\}$ which is equal to the identity on X and satisfies $g(\xi)=x$. Consequently, there exists an image preserving extension of f on $X\cup\{\xi\}$. A contradiction.

THEOREM 5. *If a map f is τ-proper, then it is non-extendable iff it is τ-perfect.*

Proof. I. Let $f\colon X\to Y$ be non-extendable. Since f is τ-proper, there exists $\tau f\colon\tau X\to\tau Y$ completing the diagram (1) for this f. Suppose that f is not τ-perfect. Hence there exists a point $\xi\in\tau X-X$ such that $\tau f(\xi)\in Y$, and the map $(\tau f)|X\cup\{\xi\}$ is an image preserving extension of f. A contradiction.

II. Let $f\colon X\to Y$ be τ-perfect. Suppose that f is not non-extendable. Then, by the lemma, f has an image preserving extension on $X\cup\{\xi\}$, where $\xi\in\tau X-X$ and the topology on $X\cup\{\xi\}$ is induced from τX. But f is τ-proper and therefore there exists τf completing the diagram (1) for this f. Since f is τ-perfect, there is $\tau f(\xi)\in\tau Y-Y$ and the map $(\tau f)|X\cup\{\xi\}$ is an image preserving extension of map $X\xrightarrow{f}Y\subset\tau Y$. Clearly, this extension is different from the extension of the same map given before according to the lemma. We get a contradiction, because there exists at most one extension for a map defined on a dense subset.

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