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ON THE EXTENSOR STRUCTURE OF A FORMULATION GIVEN BY W. ŚLEBODZIŃSKI

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The formulation is

$$X(A^{l_1\ldots l_t}_{k_1\ldots k_s})+\sum_{i=1}^{s} A^{l_1\ldots l_t}_{k_1\ldots k_{i-1}rk_{i+1}\ldots k_s}\partial_{k_i}X^r-\sum_{j=1}^{t} A^{l_1\ldots l_{j-1}rl_{j+1}\ldots l_t}_{k_1\ldots k_s}\partial_r X^{l_j} \tag{I}$$

(with $X = X^r\partial/\partial x^r$, $\partial_r = \partial/\partial x^r$) and is the right-hand member of equation (3) in Ślebodziński's celebrated paper [10]. This expression is generally referred to as the Lie derivative of the tensor A relative to the vector X^a. In what follows we shall show that it has the structure of a generalized intrinsic derivative which is compounded as an extensor contraction of an extensor E (derived from the given tensor) with certain extensors of the types $g^{a}_{\beta b}$ and $g^{\beta b}_{a}$ which are derived from the components of the given vector field. These components are denoted by V^a in the present paper and by X^r in (I).

1. Notational and extensor preliminaries. We assume that we have given an N-dimensional space S_N which bears a coordinate system x with coordinates $x^1, x^2, \ldots, x^N$ and the collection of class C^M coordinate transformations ($M \geqslant 1$)

$$\bar{x}^r = \bar{x}^r(x) = \bar{x}^r(x^1, x^2, \ldots, x^N), \qquad x^a = x^a(\bar{x}), \tag{1.1}$$

of tensor analysis. Relative to the set of parametrized arcs $x^a = x^a(t)$ in S_N which are of class C^M, we have the extended coordinate transformation

$$\begin{cases} x^a = x^a(\bar{x}), & \bar{x}^r = \bar{x}^r(x), \\ x'^a = X^a_r\bar{x}'^r, & \bar{x}'^r = X^r_a x'^a, \\ x''^a = X^a_r\bar{x}''^r+X^a_{rs}\bar{x}'^r\bar{x}'^s, & \bar{x}''^r = X^r_a x''^a+X^r_{ab}x'^a x'^b, \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ x^{(M)a} = X^a_r\bar{x}^{(M)r}+\ldots, & \bar{x}^{(M)r} = X^r_a x^{(M)a}+\ldots \end{cases} \tag{1.2}$$

with

$$' \overset{\text{df}}{=} d/dt, \qquad X^a_r \overset{\text{df}}{=} \partial x^a/\partial \bar{x}^r, \qquad X^r_a \overset{\text{df}}{=} \partial \bar{x}^r/\partial x^a, \qquad X^a_{rs} \overset{\text{df}}{=} \partial X^a_r/\partial \bar{x}^s,$$

$$x^{(M)a} \overset{\text{df}}{=} d^M x^a/dt^M.$$

Because of the particular polynomial structure of (1.2) in the x-primes (x'^a, x''^a, etc.) and the $\bar{x}$-primes, the quantities $X^{\varrho r}_{\alpha a}$ and $X^{\alpha a}_{\varrho r}$ defined by

$$X^{\varrho r}_{\alpha a} \overset{\text{df}}{=} \partial \bar{x}^{(\varrho)r}/\partial \bar{x}^{(\alpha)a} \quad \text{and} \quad X^{\alpha a}_{\varrho r} \overset{\text{df}}{=} \partial x^{\alpha\, a}/\partial \bar{x}^{(\varrho)r},$$

respectively, of course exist and, in addition, satisfy the formulas

$$(1.3) \qquad X^{\varrho r}_{\alpha a} = \binom{\varrho r}{\alpha} X^{r(\varrho-\alpha)}_a, \quad \varrho \geqslant \alpha; \qquad X^{\alpha a}_{\varrho r} = \binom{\alpha}{\varrho} X^{a(\alpha-\varrho)}_r, \quad \alpha \geqslant \varrho.$$

Here $\binom{\varrho}{\alpha}$ is a binomial coefficient and there is no summation in (1.3). Also it should be noted that $X^{\varrho r}_{\alpha a} = 0$ if $\alpha > \varrho$. In the computation of $X^{\varrho r}_{\alpha a}$ all variables in the set $x, x', \ldots, x^{(M)}$ are held fixed except the differentiation variable $x^{(\alpha)a}$ and the analogous statement holds for $X^{\alpha a}_{\varrho r}$ of course. For further details see [2], p. 215, and [4], p. 65-67 and 92-94.

The capitals X's bearing doublet indices such as αa and ϱr are the multipliers of the components in the extensor transformation law. The general pattern of this law may be inferred from the special cases

$$(1.4) \qquad \bar{g}^r_{\sigma s} = g^a_{\beta b} X^r_a X^{\beta b}_{\sigma s}, \qquad g^a_{\beta b} = \bar{g}^r_{\sigma s} X^a_r X^{\sigma s}_{\beta b},$$

$$(1.5) \qquad \bar{g}^{\sigma s}_r = g^{\beta b}_a X^a_r X^{\sigma s}_{\beta b}, \qquad g^{\beta b}_a = \bar{g}^{\sigma s}_r X^r_a X^{\beta b}_{\sigma s},$$

$$(1.6) \qquad \bar{E}^{\varrho r \cdot \sigma s}_{\tau t \cdot u} = J^w(x/\bar{x}) E^{\alpha a \cdot \beta b}_{\gamma c \cdot d} X^{\varrho r}_{\alpha a} X^{\sigma s}_{\beta b} X^{\gamma c}_{\tau t} X^d_u$$

with repeated Greek and Latin indices summed over their ranges from 0 to M for the Greek and 1 to N for the Latin. The symbol $J(x/\bar{x})$ denotes the Jacobian determinant.

Here it may be noted that when the Greek letter of a doublet superscript on a component symbol is assigned the minimum value zero, the superscript becomes a tensor index or, in other words, the zero superscript provides a tensor rank. Similarly, a tensor rank is obtained by assigning the Greek letter of a doublet subscript the maximum value M. To illustrate, from (1.5) we have

$$\bar{g}^{0s}_r = g^{\beta b}_a X^a_r X^{0s}_{\beta b} = g^{0b}_a X^a_r X^s_b$$

(since $X^{0s}_{\beta b} = 0$ for $\beta > 0$ and $X^{0s}_{0b} = X^s_b$) while, according to (1.4),

$$\bar{g}^r_{Ms} = g^a_{\beta b} X^r_a X^{\beta b}_{Ms} = g^a_{Mb} X^r_a X^{Mb}_{Ms} = g^a_{Mb} X^r_a X^b_s.$$

For more details on the extensor transformation law see [2], p. 260-275, [4], p. 70-88, and [1].

2. A generalized intrinsic derivative of a tensor. It follows from certain general formulas for the construction of extensors from tensors by differentiation with respect to a curve parameter t (or it may be established directly), that if (1) $T^{a\dots}_{b\dots}$ is a tensor of weight zero and of class C^1 along an arc $x^a = x^a(t)$ of class C^1, (2) $M = 1$, and (3) $E^{\alpha a\dots}_{\beta b\dots} = 0$ if E has more than one tensor index, $E^{\alpha a\dots}_{\beta b\dots} = T^{a\dots}_{b\dots}$ if E has only one tensor index, and $E^{\alpha a\dots}_{\beta b\dots} = T^{a'}_b$ if E does not have any tensor indices. To illustrate, if T^a_b is of the type contravariant order one, covariant order one and of weight zero, and if in all admissible coordinate systems $E^{1a}_{0b} = T^{a'}_b$, $E^{0a}_{0b} = T^a_b$, $E^{1a}_{1b} = T^a_b$, $E^{0a}_{1b} = 0$, then $E^{\alpha a}_{\beta b}$ is an extensor. For proofs see [2], p. 276-279, [3], p. 332-336, and, for the general case, [9].

A slightly generalized intrinsic derivative IT of a tensor $T^{a\dots}_{b\dots}$ can be obtained by the contraction of the associated derived extensor $E^{\gamma c\dots}_{\delta d\dots}$ with extensors of the types $g^a_{\gamma c}$, $g^{\delta d}_b$, $M = 1$. For example, in the case of the tensor T^a_b in the preceding section, we have

$$IT^a_b = E^{\gamma c}_{\delta d} g^a_{\gamma c} g^{\delta d}_b = E^{1c}_{0d} g^a_{1c} g^{0d}_b + E^{0c}_{0d} g^a_{0c} g^{0d}_b + E^{1c}_{1d} g^a_{1c} g^{1d}_b \tag{2.1}$$
$$= T^{c'}_d g^a_{1c} g^{0d}_b + T^c_d g^a_{0c} g^{0d}_b + T^c_a g^a_{1c} g^{1d}_b.$$

In particular, if

$$g^a_{1b} = \delta^a_b, \quad g^a_{0b} = \left\{ {a \atop bc} \right\} x'^c, \quad g^{0a}_b = \delta^a_b, \quad g^{1a}_b = -\left\{ {a \atop bc} \right\} x'^c,$$

then

$$IT^a_b = T^{a'}_b + T^c_b \left\{ {a \atop cd} \right\} x'^d - T^a_d \left\{ {d \atop be} \right\} x'^e,$$

the ordinary intrinsic derivative.

In the case of the ordinary intrinsic derivative the product rule is usually established by resort to geodesic coordinates. This rule, however, holds in the more general case where the tensor g's are Kronecker deltas but the remaining g's are not necessarily the two-index Christoffel symbols $\left\{ {a \atop bc} \right\} x'^c$.. This fact becomes apparent on examination of the expansions for some special cases.

These expansions may be regarded as consisting of two sets of terms. The first set is obtained by assigning all of the Greek indices on E non--tensor values (1 for superscripts, 0 for subscripts) and produces the term $T^{\dots'}_{\dots}$ which in the case $T = UV\dots$ is of course equivalent to $U'V\dots+ + UV'\dots+$. The second set consists of the sum of all of the terms obtainable by assigning one (and only one) of the indices on E the tensor value. Corresponding to $U'V\dots$ in the first set, the additional terms which are needed to produce $(IU)V\dots$ will appear in the second set. For example, if $T^c_d = U^c V_d$, then we have

$$IT^a_b = E^{\gamma c}_{\delta d} g^a_{\gamma c} g^{\delta d}_b = (U^{c'} V_d + U^c V'_d) g^a_{1c} g^{0d}_b + U^c V_d g^a_{0c} g^{0d}_b + U^c V_d g^a_{1c} g^{1d}_b$$
$$= (U'^a + U^c g^a_{0c}) V_b + U^a (V'_b + V_d g^{1d}_b) = (IU^a) V_b + U^a I V_b.$$

Also, for $T_f^{de} = U_f^d V^e$, it follows that

$$IT_f^{de} = E_{\lambda f}^{\delta d\varepsilon e} g_{\delta d}^a g_{\varepsilon e}^b g_c^{\lambda f}$$
$$= U_c^{a'} V^b + U_c^a V^{b'} + (U_c^d g_{0d}^a + U_f^a g_c^{1f}) V^b + U_c^a V^e g_{0e}^b = (IU_c^a) V^b + U_c^a I V^b.$$

3. Basic extensors in the Ślebodziński formulation. In the establishment of the extensor character of the basic quantities associated with the formulation introduced by Ślebodziński, we shall necessarily have to consider two coordinate systems. These will be denoted by x and $\bar{x}$ and we shall associate index letters at the first of the alphabet with system x and reserve letters at the last of the alphabet for system $\bar{x}$. In addition, we shall denote partial derivatives by means of subscripts preceded, by a semicolon (;), in particular, $\bar{V}_{;v}^u = \partial \bar{V}^u / \partial \bar{x}^v$, and $V_{;f}^d = \partial V^d / \partial x^f$.

Two of the three extensors involved in the Ślebodziński formulation differ remarkably in structure from those previously encountered in differential geometry and mathematical physics. The formulation is given by the following proposition:

THEOREM 3.1. *Suppose that* (1) R *is a region of an* N*-dimensional space which bears a coordinate system* x *and that* $P_0(x_0^a)$ *is a point in* R; (2) $V^a(x)$ *is a contravariant vector field of weight zero and of class* C^1 *in* R; (3) C_0 *is a parametrized arc,* $x^a = x^a(t)$, *which passes through* P_0, *is of class* C^1 *along the part in* R *and is such that* $dx^a/dt(P_0, C_0) = V^a(P_0)$; *and* (4) *the coordinate transformations to be admitted are of class* C^2. *It then follows that the quantities* $h_a^{\beta b}$ *and* $h_{\beta b}^a$, *defined by*

$$h_a^{0b} \overset{\text{df}}{=} \delta_a^b, \quad h_a^{1b} \overset{\text{df}}{=} V_{;a}^b; \quad h_{1b}^a \overset{\text{df}}{=} \delta_b^a, \quad h_{0b}^a \overset{\text{df}}{=} -V_{;b}^a \tag{3.1}$$

with similar definitions in the other coordinate systems, are extensor components for P_0, C_0.

Proof. We have given that in R, $\bar{V}^u = V^d X_d^u$ and, therefore,

$$\bar{V}_{;v}^u = V_{;e}^d X_d^u X_v^e + V^d X_{de}^u X_v^e. \tag{3.2}$$

For P_0, C_0, $V^d = x'^d$ and, therefore, for P_0, C_0

$$V^d X_{de}^u = V^d X_{ed}^u = X_e^{u'} = X_{0e}^{1u},$$

since

$$X_{0e}^{1u} = \left.\frac{\partial \bar{x}'^u}{\partial x^e}\right|_{x'\text{s fixed}} = \left.\frac{\partial (V^d X_d^u)}{\partial x^e}\right|_{V \text{ fixed}}.$$

Consequently, we may write

$$\bar{h}_v^{1u} = \bar{V}_{;v}^u = h_e^{1d} X_{1d}^{1u} X_v^e + h_e^{0d} X_{0d}^{1u} X_v^e,$$

and of course

$$\bar{h}_v^{0u} = h_e^{0d} X_{0d}^{0u} X_v^e.$$

Thus we have the extensor transformation equation

$$\bar{h}_{v}^{\mu u} = h_{e}^{\delta d} X_{\delta d}^{\mu u} X_{v}^{e}.$$

To establish the remainder of the theorem, we employ the identity $X_{de}^{u} X_{v}^{e} = -X_{e}^{u} X_{vw}^{e} X_{d}^{w}$ (which is a consequence of the relation $X_{e}^{u} X_{v}^{e} = \delta_{v}^{u}$) to convert (3.2) into

$$-\bar{V}_{;v}^{u} = -V_{;e}^{d} X_{d}^{u} X_{v}^{e} + X_{vw}^{e} \bar{V}^{w} X_{e}^{u}.$$

This equation may be rewritten as follows:

$$\bar{h}_{0v}^{u} = -\bar{V}_{;v}^{u} = h_{0e}^{d} X_{d}^{u} X_{v}^{e} + X_{0v}^{1e} \delta_{e}^{d} X_{d}^{u} = h_{0e}^{d} X_{d}^{u} X_{0v}^{0e} + h_{1e}^{d} X_{d}^{u} X_{0v}^{1e}.$$

Accordingly, $\bar{h}_{\nu v}^{u} = h_{\varepsilon e}^{d} X_{d}^{u} X_{\nu v}^{\varepsilon e}$ and the proof is completed.

Examination of formula (I) given by Ślebodziński shows that it is the complete extensor contraction of the extensors h of theorem 3.1 with the extensor E derived from A by parameter differentiation in accordance with the procedure given in section 2. The curve C_0 associated with E must of course meet the requirement $dx^a/dt(P_0, C_0) = V^a(P_0)$ with P_0 the point at which expression (I) is evaluated.

The essential points in the comparison of the structure of (I) with the extensor contraction of the E (derived from the tensor A) with the extensors h are revealed by the typical special case where the tensor A is contravariant of order two and covariant of order two. The expansion of this contraction is as follows:

$$E_{\gamma c.\delta d}^{\alpha a.\beta b} h_{\alpha a}^{e} h_{\beta b}^{f} h_{g}^{\gamma c} h_{h}^{\delta d} = A_{gh}^{ef'} + E_{0c.0d}^{0a.1b} h_{0a}^{e} h_{1b}^{f} h_{g}^{0c} h_{h}^{0d} +$$
$$+ E_{0c.0d}^{1a.0b} h_{1a}^{e} h_{0b}^{f} h_{g}^{0c} h_{h}^{0d} + E_{1c.0d}^{1a.1b} h_{1a}^{e} h_{1b}^{f} h_{g}^{1c} h_{h}^{0d} + E_{0c.1d}^{1a.1b} h_{1a}^{e} h_{1b}^{f} h_{g}^{0c} h_{h}^{1d}.$$

Here the term $A_{gh}^{ef'}$ is obtained by noting that $E_{0c.0d}^{1a.1b} = A_{cd}^{ab'}$ and that this particular E is contracted with Kronecker deltas only. In the remaining terms, $h_{0a}^{e} = -V_{;a}^{e}$, $h_{0b}^{f} = -V_{;b}^{f}$, $h_{g}^{1c} = V_{;g}^{c}$ and $h_{h}^{1d} = V_{;h}^{d}$ with all the other h-symbols Kronecker deltas. Thus the contraction at the locality P_0, C_0 is given by

$$E_{\gamma c.\delta d}^{\alpha a.\beta b} h_{\alpha a}^{e} h_{\beta b}^{f} h_{g}^{\gamma c} h_{h}^{\delta d} = A_{gh;i}^{ef} V^{i} - A_{gh}^{af} V_{;a}^{e} - A_{gh}^{eb} V_{;b}^{f} + A_{ch}^{ef} V_{;g}^{c} + A_{gd}^{ef} V_{;h}^{d}, \tag{3.3}$$

which is in complete agreement with formulation (I) of Ślebodziński. Because the only free indices in the extensor contraction are tensor indices, it follows that formula (I) produces a tensor from the given tensor A. Furthermore, since (I) is essentially a generalized intrinsic derivative with $g_{a}^{0b} = h_{a}^{0b} = \delta_{a}^{b}$ and $g_{1b}^{a} = h_{1b}^{a} = \delta_{b}^{a}$, it follows that the product rule holds.

For different developments involving extensors and Ślebodziński type formulations, particularly the Lie derivatives of extensors, see [5]-[7].

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